The relationship between the observability of standard and fractional discrete-time and continuous-time linear systems are addressed. It is shown that the fractional discrete-time and continuous-time linear systems are observable if and only if the standard discrete-time and continuous-time linear systems are observable.

**Key words:** fractional, standard, linear, discrete-time, continuous-time, system, observability.

1. Introduction

The notion of controllability and observability of linear systems have been introduced by Kalman [14, 15]. Those notions are the basic concepts of the modern control theory [1, 6, 13, 16, 21, 24, 25]. They have been extended to positive and fractional linear and nonlinear systems [2, 4, 5, 7-11, 22, 23]. The mathematical fundamentals of fractional calculus are given in the monographs [18-20]. The positive fractional linear systems have been introduced in [8, 11].

In the paper [17] it has been shown that the fractional discrete-time and continuous-time linear systems are controllable if and only if the standard discrete-time and continuous-time systems are controllable.

In this paper it will be shown that the fractional discrete-time and continuous-time linear systems are observable if and only if the standard discrete-time and continuous-time linear systems are observable.

The paper is organized as follows. In section 2 the basic definitions and theorems concerning standard and fractional discrete-time and continuous-time linear systems are recalled. The relationship between the observability of the standard and fractional discrete-time linear systems is considered in section 3 and of continuous-time linear systems in section 4. Concluding remarks are given in section 5.
The following notation will be used: $\mathbb{R}^{n \times m}$ is the set of $n \times m$ real matrices and $\mathbb{R}^n = \mathbb{R}^{n \times 1}$, $\mathbb{Z}_+$ is the set of nonnegative integers, $I_n$ is the $n \times n$ identity matrix.

2. Preliminaries

Consider the standard discrete-time linear system

$$x_{i+1} = Ax_i + Bu_i, \quad i \in \mathbb{Z}_+ = \{0, 1, \ldots\}, \quad (1a)$$
$$y_i = Cx_i, \quad (1b)$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^p$ are state, input and output vectors and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$.

The solution to the equation $(1a)$ is given by

$$x_i = A^i x_0 + \sum_{j=0}^{i-1} A^{i-j-1} Bu_j. \quad (2)$$

Substituting (2) into (1b) we obtain

$$y_i = CA^i x_0 + \sum_{j=0}^{i-1} CA^{i-j-1} Bu_j. \quad (3)$$

Now let us consider the fractional discrete-time linear system

$$\Delta^\alpha x_{i+1} = Ax_i + Bu_i, \quad 0 < \alpha < 2, \quad (4a)$$
$$y_i = Cx_i, \quad (4b)$$

where

$$\Delta^\alpha x_i = \sum_{j=0}^{i} (-1)^j \left[ \begin{array}{c} \alpha \\ j \end{array} \right] x_{i-j}, \quad (4c)$$

$$\left[ \begin{array}{c} \alpha \\ j \end{array} \right] = \begin{cases} 1 & \text{for } j = 0 \\ \frac{\alpha(\alpha-1)\ldots(\alpha-j+1)}{j!} & \text{for } j = 1, 2, \ldots \end{cases} \quad (4d)$$

is the fractional $\alpha$-order difference of $x_i$ and $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^p$ are state, input and output vectors and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$.

Substitution of (4c) into (4a) yields

$$x_{i+1} = (A + I_n \alpha) x_i + \sum_{j=2}^{i+1} c_j x_{i-j+1} + Bu_i, \quad i \in \mathbb{Z}_+, \quad (5a)$$
where
\[ c_j = c_j(\alpha) = (-1)^{j+1} \begin{pmatrix} \alpha \\ j \end{pmatrix}, \quad j = 2, 3, \ldots \] (5b)

The solution to the equation (5a) has the form [11]
\[ x_{i+1} = (A + I_n \alpha)x_i + \sum_{j=2}^{i+1} c_j x_{i-j+1} + Bu_i, \quad i \in \mathbb{Z}_+, \] (6a)

where
\[ \Phi_{j+1} = \Phi_j(A + I_n \alpha) + \sum_{k=2}^{j+1} c_k \Phi_{j-k+1}, \quad \Phi_0 = I_n \] (6b)

and \( c_k \) is defined by (5b).

Substituting (6a) into (4b) we obtain
\[ y_i = C\Phi_i x_0 + \sum_{j=0}^{i-1} C\Phi_{i-j-1} Bu_j. \] (7)

Consider the standard continuous-time linear system
\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \quad \text{(8a)} \\
y(t) &= Cx(t), \quad \text{(8b)}
\end{align*}
\]
where \( x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p \) are state, input and output vectors and \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n} \).

The solution to the equation (8a) has the form
\[ x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau \] (9)

and
\[ y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)} Bu(\tau) d\tau. \] (10)

Now let us consider the fractional continuous-time linear system
\[
\begin{align*}
\frac{d^\alpha x(t)}{dt^\alpha} &= Ax(t) + Bu(t), \quad 0 < \alpha < 2 \quad \text{(11a)} \\
y(t) &= Cx(t), \quad \text{(11b)}
\end{align*}
\]
where
\[
\frac{d^\alpha x(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{x^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \quad x^{(n)}(\tau) = \frac{d^n x(\tau)}{d\tau^n} \quad \text{(12)}
\]
is the Caputo fractional derivative of order $n - 1 < \alpha < n$ ($n \in \mathbb{N}$) of $x(t)$, $\Gamma(x)$ is the Euler gamma function, $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^p$ are state, input and output vectors and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$.

The solution of the equation (11a) is given by [11]

$$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t - \tau)Bu(\tau)d\tau, \quad x_0 = x(0),$$

where

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha + 1)}, \quad (13b)$$

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k (k+1)^{\alpha-1}}{\Gamma((k+1)\alpha)} \quad (13c)$$

and

$$y(t) = C\Phi_0(t)x_0 + \int_0^t C\Phi(t - \tau)Bu(\tau)d\tau.$$  \hfill (14)

**Theorem 4** (Cayley–Hamilton) Let $A \in \mathbb{R}^{n \times n}$ and

$$\det[I_n \lambda - A] = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1 \lambda + a_0.$$  \hfill (15)

Then

$$A^n + a_{n-1}A^{n-1} + \ldots + a_1 A + a_0 I_n = 0.$$  \hfill (16)

**Proof** Proof is given in [3, 12].

**Theorem 5** (Kronecker-Capelli) The linear matrix equation

$$Ax = b, \quad A \in \mathbb{R}^{n \times n}, \quad b \in \mathbb{R}^n$$  \hfill (17)

has a solution $x \in \mathbb{R}^n$ if and only if

$$\text{rank}[A, b] = \text{rank}A.$$  \hfill (18)

**Proof** Proof is given in [12].

3. Observability of standard and fractional discrete-time linear systems

It is well-known [1, 2, 7] that the observability of the standard and fractional linear systems depends only of the pair $(A, C)$ and it is independent of the matrix $B$. 
**Definition 13** The standard linear discrete-time linear system (1) is called observable in the interval \([0, q]\) if knowing the output \(y_i\) for \(i = 0, 1, \ldots, q - 1, q \leq n\), it is possible to find the unique \(x_0\) of the system.

**Theorem 6** The standard linear discrete-time linear system (1) is observable if and only if

\[
\text{rank} \begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{bmatrix} = n. \quad (19)
\]

**Proof** Proof is given in [1, 6, 13].

**Definition 14** The fractional discrete-time linear system (4) is called observable in the interval \([0, q]\) if knowing the output \(y_i\) for \(i = 0, 1, \ldots, q - 1, q < n\), it is possible to find the unique \(x_0\) of the system.

We shall show that the fractional discrete-time linear system (4) is observable in the interval \([0, q]\) if and only if the standard linear discrete-time system (1) is observable in the same interval.

From (7) for \(B = 0\) and (6b) for \(i = 0, 1, \ldots, q - 1\) we have

\[
y_{0q} = \begin{bmatrix}
y_0 \\
y_1 \\
\vdots \\
y_{q-1}
\end{bmatrix} = \begin{bmatrix}
C\Phi_0 \\
C\Phi_1 \\
\vdots \\
C\Phi_{q-1}
\end{bmatrix} x_0 = O_{0q}x_0, \quad (20a)
\]

where

\[
O_{0q} = \begin{bmatrix}
C \\
C(A + I_n\alpha) \\
C[(A + I_n\alpha)^2 + c_2I_n] \\
\vdots \\
C[(A + I_n\alpha)^q - 1 + \ldots + (\alpha^{q-1} + \ldots + c_{q-1})I_n]
\end{bmatrix}. \quad (20b)
\]

By Kronecker-Capelli theorem the equation (20a) has a unique solution \(x_0\) for any given \(y_{0q}\) if and only if

\[
\text{rank} O_{0q} = n. \quad (20c)
\]

Therefore, the following theorem has been proved.

**Theorem 7** The fractional discrete-time linear system (4) or equivalently (5a), (4b), is observable in the interval \([0, q]\) if and only if the condition (20c) is satisfied.
It will be shown that the condition (20c) is equivalent to the condition (19). Note that

\[
O_{0q} = \begin{bmatrix}
C \\
C(A + I_n \alpha) \\
C[(A + I_n \alpha)^2 + c_2 I_n] \\
\vdots \\
C[(A + I_n \alpha)^{q-1} + \ldots + (\alpha^{q-1} + \ldots + c_{q-1})I_n]
\end{bmatrix}
\]

(21)

since

\[
(A + I_n \alpha)^k = A^k + k \alpha A^{k-1} + \ldots + \alpha^k I_n \quad \text{for} \quad k = 2, 3, \ldots, q - 1.
\]

(22)

From (21) it follows that

\[
\text{rank} \ O_{0q} = \text{rank}
\begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{q-1}
\end{bmatrix}
\]

(23)

since the matrix

\[
\begin{bmatrix}
I_n & 0 & 0 & \ldots & 0 \\
\alpha I_n & I_n & 0 & \ldots & 0 \\
(c_2 + \alpha^2)I_n & 2\alpha I_n & I_n & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(c_{q-1} + \ldots + \alpha^{q-1})I_n & \ldots & \ldots & \ldots & I_n
\end{bmatrix}
\]

(24)

is nonsingular for all values of \( \alpha \) and \( c_k, k = 1, 2, \ldots, q - 1 \). Therefore, the following theorem has been proved.

**Theorem 8** The fractional discrete-time linear system (4) is observable in the interval \([0, q], q \leq n\), if and only if the standard discrete-time linear system (1) is observable in the same interval \([0, q]\).

**Example 1** Consider the standard system (1) and the fractional system (4) for \( \alpha = 0.5 \)

with the same matrices

\[
A = \begin{bmatrix}
0 & 1 \\
-1 & -3
\end{bmatrix}, \quad C = [1 \ 1].
\]

(25)
Using (19) and (25) for $q = 2$ we obtain
\[
\text{rank} \begin{bmatrix} C \\ CA \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = 2 \tag{26}
\]
and by Theorem 6 the standard system is observable in the interval $[0, 2]$.

For the fractional system with (25) using (20b) we obtain
\[
\text{rank} \begin{bmatrix} C \\ C(A + \alpha I_2) \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 1 \\ -0.5 & -1.5 \end{bmatrix} = 2. \tag{27}
\]
By Theorem 7 the fractional system with (25) is also observable in the interval $[0, 2]$.

4. Observability of standard and fractional continuous-time linear systems

**Definition 15** The standard continuous-time linear system (8) is called observable in the interval $[0, t_f]$ if knowing the output $y(t)$ for $t \in [0, t_f]$ it is possible to find the unique $x_0$ of the system.

**Theorem 9** The standard continuous-time linear system (8) is observable if and only if
\[
\text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n. \tag{28}
\]

**Proof** Proof is given in [1, 6, 13].

**Definition 16** The fractional continuous-time linear system (11) is called observable in the interval $[0, t_f]$ if knowing the output $y(t)$ for $t \in [0, t_f]$ it is possible to find the unique $x_0$ of the system.

We shall show that the fractional continuous-time linear system (11) is observable in the interval $[0, t_f]$ if and only if the standard continuous-time linear system (8) is observable in the same interval.

Using the Cayley-Hamilton theorem (the equality (10)) it is possible to eliminate the powers $k = n, n + 1, \ldots$ of the matrix $A^k$ in (13b) and we obtain
\[
\Phi_0(t) = \sum_{k=0}^{n-1} c_k(t)A^k. \tag{29}
\]

The coefficients $c_k$ in (29) can be computed as follows.
To simplify the calculations it is assumed the eigenvalues $\lambda_k$ of the matrix $A$ are distinct, i.e. $\lambda_i \neq \lambda_j$ for $i \neq j$. In this case using (29) we obtain

$$
\begin{bmatrix}
\Phi_0(\lambda_1) \\
\Phi_0(\lambda_2) \\
\vdots \\
\Phi_0(\lambda_n)
\end{bmatrix}
= H
\begin{bmatrix}
c_0(t) \\
c_1(t) \\
\vdots \\
c_{n-1}(t)
\end{bmatrix},
$$

(30)

where

$$
H =
\begin{bmatrix}
1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\
1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \lambda_n & \cdots & \lambda_n^{n-1}
\end{bmatrix}.
$$

(31)

If the eigenvalues are distinct, then the matrix (31) is nonsingular and from (30) we have

$$
\begin{bmatrix}
c_0(t) \\
c_1(t) \\
\vdots \\
c_{n-1}(t)
\end{bmatrix}
= H^{-1}
\begin{bmatrix}
\Phi_0(\lambda_1) \\
\Phi_0(\lambda_2) \\
\vdots \\
\Phi_0(\lambda_n)
\end{bmatrix}.
$$

(32)

The coefficients $c_k(t)$, $k = 0, 1, \ldots, n - 1$ can be also found using the well-known Lagrange-Sylvester formula [3, 12].

Substitution of (29) into (14) for $B = 0$ yields

$$
y(t) = C\Phi_0(t)x_0 = \sum_{k=0}^{n-1} c_k(t)CA^k =
\begin{bmatrix}
c_0(t) & c_1(t) & \cdots & c_{n-1}(t)
\end{bmatrix}
\begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{bmatrix}x_0.
$$

(33)

From (33) it follows that it is possible to find $y(t)$ for given $t \in [0,t_f]$, if and only if

$$
\text{rank}
\begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{bmatrix} = n
$$

(34)

since $c_k(t) \neq 0$ for $t \in [0,t_f]$. Therefore, the following theorem has been proved.
Theorem 10 The fractional continuous-time linear system (11) is observable in the interval \([0,t_f]\) if and only if the standard continuous-time linear system (8) is observable in the same interval.

Example 2 Consider the standard system (8) and the fractional system (11) with the same matrices
\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.
\] (35)

Using (28) and (35) we obtain
\[
\text{rank} \begin{bmatrix} C \\ CA \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2 \quad (36)
\]
and by Theorem 10 the standard system is observable. In this case for the fractional system (11) with (35) we obtain
\[
\Phi_0(t) = I_2 + \frac{At^\alpha}{\Gamma(\alpha+1)} = I_2 + \frac{At^\alpha}{\alpha} = \begin{bmatrix} 1 & \frac{t^\alpha}{\alpha} \\ 0 & 1 \end{bmatrix} = c_0(t)I_2 + c_1(t)A, \quad (37)
\]
where
\[
c_0(t) = 1, \quad c_1(t) = \frac{t^\alpha}{\alpha}. \quad (38)
\]
By Theorem 10 the fractional system is also observable.

5. Concluding remarks

The relationship between the observability of the standard and fractional discrete-time and continuous-time linear systems has been addressed. It has been shown that: 1) the fractional discrete-time linear systems are observable if and only if the standard discrete-time linear systems are observable (Theorem 8); 2) the fractional continuous-time linear systems are observable if and only if the standard continuous-time linear systems are observable (Theorem 10). The considerations have been illustrated by numerical examples. The considerations can be extended to the standard and fractional time-varying linear systems.

References


