The asymptotic stability of discrete-time and continuous-time linear systems described by the equations $x_{i+1} = \tilde{A}^k x_i$ and $\dot{x}(t) = A^k x(t)$ for $k$ being integers and rational numbers is addressed. Necessary and sufficient conditions for the asymptotic stability of the systems are established. It is shown that: 1) the asymptotic stability of discrete-time systems depends only on the modules of the eigenvalues of matrix $\tilde{A}^k$ and of the continuous-time systems depends only on phases of the eigenvalues of the matrix $A^k$, 2) the discrete-time systems are asymptotically stable for all admissible values of the discretization step if and only if the continuous-time systems are asymptotically stable, 3) the upper bound of the discretization step depends on the eigenvalues of the matrix $A$.

**Key words:** analysis, comparison, stability, discrete-time, continuous-time, linear system.

### 1. Introduction

The asymptotic stability is one of the basic notions of the theory of dynamical systems [1, 8, 10, 12]. It has been addressed in many books and papers [1, 3, 6, 10-12]. The approximation of positive standard and fractional stable continuous-time linear systems by suitable discrete-time systems has been analyzed in [3, 4]. Comparison of approximation methods of positive stable continuous-time linear systems by positive stable discrete-time systems has been presented in [5]. The influence of the value of discretization step on the stability of positive and fractional systems has been analyzed in [6]. Inverse systems of linear systems have been investigated in [7].

In this paper the asymptotic stability of discrete-time and continuous-time linear systems described by the equations $x_{i+1} = \tilde{A}^k x_i$ and $\dot{x}(t) = A^k x(t)$ for $k$ being integers and rational numbers will be investigated.

The paper is organized as follows. In section 2 the basic definitions and theorems concerning the asymptotic stability of continuous-time and discrete-time systems and theorem on the eigenvalues of the matrix function are recalled. The asymptotic stability
of the discrete-time linear systems for $k$ being integers and rational numbers are investigated in section 3. Similar problems for continuous-time linear systems are analyzed in section 4. Comparison of the stability of discrete-time and continuous-time linear systems is presented in section 5. Concluding remarks are given in section 6.

The following notation will be used: $\mathbb{R}$ — the set of real numbers, $\mathbb{R}^{n \times m}$ — the set of $n \times m$ real matrices, $I_n$ — the $n \times n$ identity matrix, $\mathbb{Z}_+$ — the set of nonnegative integers.

2. Preliminaries

Consider the autonomous continuous-time linear system

$$\dot{x}(t) = Ax(t), \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $A \in \mathbb{R}^{n \times n}$. The solution of (1) for the given initial condition has the form [1, 8, 10, 12]

$$x(t) = e^{At}x_0. \quad (2)$$

**Definition 1** The system (1) (or equivalently the matrix $A$) is called asymptotically stable if

$$\lim_{t \to \infty} x(t) = 0 \quad \text{for all } x_0 \in \mathbb{R}^n \quad (3)$$

**Theorem 3** [1, 8, 10, 12] The system (1) (the matrix $A$) is asymptotically stable if and only if

$$\Re s_l < 0 \Leftrightarrow \frac{\pi}{2} < \phi < \frac{3\pi}{2} \quad \text{for all } l = 1, \ldots, n, \quad (4)$$

where $s_l = |s_l|e^{j\phi_l}$, $l = 1, \ldots, n$ are the eigenvalues of the matrix $A$, i.e. the roots of the equation

$$\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0 = 0. \quad (5)$$

Similarly, let us consider the autonomous discrete-time linear system [1, 8, 10, 12]

$$x_{i+1} = \bar{A}x_i, \quad i \in \mathbb{Z}_+ = \{0, 1, \ldots\}, \quad (6)$$

where $x_i \in \mathbb{R}^n$ is the state vector and $\bar{A} \in \mathbb{R}^{n \times n}$. The solution of (6) for the given initial condition $x_0$ has the form [1, 8, 10, 12]

$$x_i = \bar{A}^i x_0, \quad i \in \mathbb{Z}_+. \quad (7)$$

**Definition 2** The system 6 (or equivalently the matrix $\bar{A}$) is called asymptotically stable if

$$\lim_{i \to \infty} x_i = 0 \quad \text{for all } x_0 \in \mathbb{R}^n. \quad (8)$$
Theorem 4 [1, 8, 10, 12] The system (6) (the matrix $\bar{A}$) is asymptotically stable if and only if
\[ |z_l| < 1 \text{ for all } l = 1, ..., n, \] (9)
where $z_l$, $l = 1, ..., n$ are the eigenvalues of the matrix $\bar{A}$, i.e. the roots of the equation
\[ \det[I_n z - \bar{A}] = z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0 = 0. \] (10)

**Theorem 5** Let $s_l$, $l = 1, ..., n$ be the eigenvalues of the matrix $A \in \mathbb{R}^{n \times n}$ and $f(s_l)$ be well defined on the spectrum $\sigma_A = \{s_1, s_2, ..., s_n\}$ of the matrix $A$, i.e. $f(s_l)$ are finite for $l = 1, ..., n$. Then $f(s_l)$, $l = 1, ..., n$ are the eigenvalues of the matrix $f(A)$.

**Proof** The proof is given in [2, 9].

For example if $s_l$, $l = 1, ..., n$ are the nonzero eigenvalues (not necessary distinct) of the matrix $A \in \mathbb{R}^{n \times n}$ then $s_l^{-1}$, $l = 1, ..., n$ are the eigenvalues of the inverse matrix $A^{-1}$.

### 3. Discrete-time linear systems

In this section the asymptotic stability of the system
\[ x_{i+1} = \bar{A}^k x_i, \quad i \in \mathbb{Z}_+ \] (11)
will be investigated for $k$ being integers ($k = \pm 1, \pm 2, ...$) and rational numbers ($k = \frac{p}{q}$, $p, q$ – integers).

For $k = 1, 2, ...$ we have the following theorem.

**Theorem 6** The linear system (11) is asymptotically stable for $k = 1, 2, ...$ if and only if the linear system (6) is asymptotically stable.

**Proof** By Theorem 3 if $z_l$, $l = 1, ..., n$ are the eigenvalues of the matrix $\bar{A}$ then the eigenvalues of the matrix $\bar{A}^k$ are $z_l$, $l = 1, ..., n$. Note that $|z_l| < 1$ for and $k = 1, 2, ...$ if and only if the condition (9) is satisfied. Therefore, by Theorem 4 the system (11) is asymptotically stable if and only if the system (6) is asymptotically stable.

**Example 1** Consider the system (6) with
\[ \bar{A} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix} \] (12)

The characteristic polynomial of (12) has the form
\[ \det[I_3 z - \bar{A}] = \begin{vmatrix} z & -1 \\ 1 & z - \frac{1}{6} \end{vmatrix} = z^2 - \frac{1}{6} z - \frac{1}{6} \] (13)
and its zeros are $z_1 = \frac{1}{2}$ and $z_2 = -\frac{1}{3}$.

The eigenvalues of the matrix (12) satisfy the condition (9) and the system is asymptotically stable. By Theorem 6 the system (11) with (12) is also asymptotically stable for $k = 2, 3, \ldots$

For $k = -1, -2, \ldots$ we have the following theorem.

**Theorem 7** The linear system (11) is asymptotically stable for $k = -1, -2, \ldots$ if and only if the system (6) is unstable, i.e. the eigenvalues of the matrix $\tilde{A}$ satisfy the condition

$$|z_j| > 1 \quad \text{for} \quad j = 1, \ldots, n. \quad (14)$$

**Proof** By Theorem 5 if $z_j$, $j = 1, \ldots, n$ are the eigenvalues of the matrix $\tilde{A}$ then the eigenvalues of the matrix $\tilde{A}^k$ for $k = -1, -2, \ldots$ are $z_j^k$, $k = 1, 2, \ldots$. Note that $|z_j|^{-k} < 1$, $k = 1, 2, \ldots$ if and only if the condition (14) is satisfied. Therefore, by Theorem 4 the system (11) is asymptotically stable for $k = -1, -2, \ldots$ if and only if the system (6) is unstable. \hfill $\square$

**Example 2** (Continuation of Example 1) The inverse matrix of (12) has the form

$$\tilde{A}^{-1} = \begin{bmatrix} -1 & 6 \\ 1 & 0 \end{bmatrix} \quad (15)$$

and its eigenvalues are $\tilde{z}_1 = 2$, $\tilde{z}_2 = -3$. Therefore, the discrete-time linear system with the matrix (15) is unstable.

Note that for (15) we obtain the matrix

$$\tilde{A}^{-2} = (\tilde{A}^{-1})^2 = \begin{bmatrix} -1 & 6 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 7 & -6 \\ -1 & 6 \end{bmatrix} \quad (16)$$

and its eigenvalues are $\tilde{z}_1 = 4$, $\tilde{z}_2 = 9$. The linear system (11) for with (16) is unstable. Similar results can be obtained for $k = -3, -4, \ldots$

For $k = \pm \frac{p}{q}$, $p, q \in \{1, 2, \ldots\}$ we have the following theorem

**Theorem 8** The linear system (11) is asymptotically stable

1) for $k = \frac{p}{q}$, $p, q \in \{1, 2, \ldots\}$ if and only if the linear system (6) is asymptotically stable,

2) for $k = -\frac{p}{q}$, $p, q \in \{1, 2, \ldots\}$ if and only if the linear system is unstable.
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Proof By Theorem 5 if \( z_j, j = 1, \ldots, n, \) are the eigenvalues of the matrix \( \tilde{A} \) then the eigenvalues of the matrix \( \tilde{A}^{\pm \frac{p}{q}} \) are \( z_j^{\pm \frac{p}{q}} \) for \( j = 1, \ldots, n \) and

\[
\ln |z_j|^{\pm \frac{p}{q}} = \pm \frac{p}{q} \ln |z_j| \quad \text{for} \quad j = 1, \ldots, n.
\]

If \( \frac{p}{q} > 0 \) and \(|z_j| < 1, j = 1, \ldots, n\) then from (17) we have

\[
\frac{p}{q} \ln |z_j| < 0 \quad \text{and} \quad |z_j|^\frac{p}{q} < 1 \quad \text{for} \quad j = 1, \ldots, n.
\]

Therefore, the system (11) is asymptotically stable for \( k = \frac{p}{q} > 0 \) if and only if the system (6) is asymptotically stable. Proof in the case 2) is similar. \( \square \)

Example 3 (Continuation of Example 1) Consider the system (6) with (12) for \( p = 3, q = 2 \). Using (12) we obtain the matrix

\[
\tilde{A}^3 = \frac{1}{6^2} \begin{bmatrix} 1 & 7 \\ 7 & 13 \\ 6 & 6 \end{bmatrix}
\]

with the eigenvalues \( z_1 = \frac{1}{8}, z_2 = -\frac{1}{27} \). The eigenvalues of the matrix

\[
\tilde{A}^{\frac{3}{2}} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 6 & 6 \end{bmatrix}^{\frac{3}{2}}
\]

are \( \tilde{z}_1 = \left( \frac{1}{2} \right)^{\frac{3}{2}}, \tilde{z}_2 = \left( -\frac{1}{3} \right)^{\frac{3}{2}} \) and satisfy the condition (9). Therefore, by Theorem 7 the system (6) with (12) for \( p = 3, q = 2 \) is asymptotically stable.

Remark 1 The asymptotic stability of the discrete-time system (6) depends only on the modules of the eigenvalues of the matrix \( \tilde{A} \) and it is independent of the phases of the eigenvalues.

Remark 2 The matrix \( -\tilde{A} \in \mathbb{R}^{n \times n} \) is asymptotically stable if and only if the matrix \( A \in \mathbb{R}^{n \times n} \) is asymptotically stable since the eigenvalues of the matrices \( A \) and \( -\tilde{A} \) have the same modules.
4. Continuous-time linear systems

In this section the asymptotic stability of the continuous-time linear system
\[ \dot{x}(t) = A^k x(t), \quad A \in \mathbb{R}^{n \times n} \]  
will be investigated for \( k \) being integers (\( k = \pm 1, \pm 2, \ldots \)) and rational numbers (\( k = \frac{p}{q}, p, q \) – integers).

**Theorem 9** Let \( s_l = |s_l| e^{j\phi_l}, \ l = 1, \ldots, n \) be the \( l \)-th eigenvalue of the matrix \( A \). The system (6) is asymptotically stable if and only if
\[ \frac{\pi}{2} < k\phi_l < \frac{3\pi}{2} \text{ for } l = 1, \ldots, n. \]  

**Proof** By Theorem 5 if \( s_l \) is the \( l \)-th eigenvalue of the matrix \( A \) then \( s_l^k, l = 1, \ldots, n \) are the eigenvalues of the matrix \( A^k \) and by Theorem 3 the system (21) is asymptotically stable if and only if the condition (22) is satisfied. \( \square \)

From the condition (22) of Theorem 9 we have the following conclusion.

**Conclusion 1** The asymptotic stability of the system (21) for any \( k \) depends only on the phases of the eigenvalues \( s_l, l = 1, \ldots, n \) of the matrix \( A \) and it is independent of their modules.

**Example 4** Consider the asymptotic stability of the continuous-time linear system (21) with the matrix
\[ A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \]  
for \( k = 2, 3 \) and \( k = -1, -2, -3 \). The characteristic polynomial of the matrix (4.3) has the form
\[ \det[I_2 s - A] = \begin{vmatrix} s & -1 \\ 1 & s + 1 \end{vmatrix} = s^2 + s + 1 \]  
and its zeros are
\[ s_1 = -\frac{1}{2} + j\frac{\sqrt{3}}{2} = e^{j\frac{2\pi}{3}}, \quad s_1 = -\frac{1}{2} - j\frac{\sqrt{3}}{2} = e^{-j\frac{2\pi}{3}}. \]  
Therefore, the system (21) with (23) for \( k = 1 \) is asymptotically stable since (25) satisfy the condition (22).

It is easy to verify that for (23)
\[ A^2 = A^{-1} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \quad A^{-2} = A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \]
and the matrices have the same characteristic polynomial (24) and are asymptotically stable. Note that
\[ A^3 = A^{-3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \] (27)
and the system (21) with (27) is unstable. The same result follows for (27) from the condition (22) since for (25) with \( k = \pm 3 \) we have the phases \( \pm \frac{2\pi}{3} = \pm 2\pi \).

**Example 5.** Consider the asymptotic stability of the system (21) with the matrix
\[ A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \] (28)
for \( k = -1, -2, -3, 2, 3, \frac{1}{2} \). The characteristic polynomial of the matrix (27) has the form
\[ \det[I_2s - A] = \begin{vmatrix} s & -1 \\ 2 & s + 3 \end{vmatrix} = s^2 + 3s + 2 \] (29)
and its zeros are \( s_1 = -1, s_2 = -2 \). Thus, the system for \( k = 1 \) is asymptotically stable. For \( k = -1 \) we have
\[ A^{-1} = \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} \\ 1 & 0 \end{bmatrix} \] (30)
and
\[ \det[I_2s - A^{-1}] = \begin{vmatrix} s + \frac{3}{2} & \frac{1}{2} \\ -1 & s \end{vmatrix} = s^2 + \frac{3}{2}s + \frac{1}{2} \] (31)
and the eigenvalues of (30) are \( s_1 = -1 = e^{j180^\circ}, s_2 = -\frac{1}{2} = \frac{1}{2}e^{j180^\circ} \). The system (21) with (30) for \( k = 1 \) is asymptotically stable (the condition (22) is satisfied). For \( k = -2 \) we obtain the matrix
\[ A^{-2} = \begin{bmatrix} 7 & 3 \\ 4 & 4 \\ 3 & 1 \\ 2 & 2 \end{bmatrix} \] (32)
with the eigenvalues \( s_1 = 1 = e^{j0^\circ}, s_2 = \frac{1}{4} = \frac{1}{4}e^{j0^\circ} \). Therefore, the system (21) with (32) is unstable.
The same result follows from (22) since \( k\phi = -2 \cdot 180^\circ = 0^\circ \). For \( k = -3 \) we obtain the matrix
\[
A^{-3} = \begin{bmatrix}
-15 & 7 \\
8 & -8 \\
7 & 3 \\
4 & 4
\end{bmatrix}
\] (33)
with the eigenvalues \( s_1 = -1 = e^{j180^\circ}, s_2 = -\frac{1}{8} = \frac{1}{8}e^{j180^\circ} \). Therefore, the system is asymptotically stable. The same result follows from (22). For \( k = 2 \) we obtain the matrix
\[
A^2 = \begin{bmatrix}
-2 & -3 \\
6 & 7
\end{bmatrix}
\] (34)
with the eigenvalues \( s_1 = 1 = e^{j0^\circ}, s_2 = 4 = 4e^{j0^\circ} \). The system (21) with (34) is unstable. For \( k = 3 \) we have the matrix
\[
A^3 = \begin{bmatrix}
6 & 7 \\
-14 & -15
\end{bmatrix}
\] (35)
with the eigenvalues \( s_1 = -1 = e^{j180^\circ}, s_2 = -8 = 8e^{j180^\circ} \). Therefore, the system for \( k = 3 \) is asymptotically stable. In general case we obtain that the system (21) with (28) is asymptotically stable for \( k = \pm (1 + 2l), l = 0, 1, \ldots \) and unstable for \( k = \pm 2l, l = 1, 2, \ldots \).

**Theorem 10** If the matrix \( A \in \mathbb{R}^{n \times n} \) has at least one real positive eigenvalue then the system (21) is unstable for all values of \( k \) (integer and rational).

**Proof** By Theorem 5 if \( s_l, l = 1, \ldots, n \) are the real positive eigenvalues of the matrix \( A \) then \( s_{l^k}, l = 1, \ldots, n \), are the real positive eigenvalues of the matrix \( A^k \) and the system (21) is unstable. \( \square \)

**Example 6** Consider the system (21) with the matrix
\[
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & -2 & 2
\end{bmatrix}
\] (36)
for \( k = 2 \). The characteristic polynomial of the matrix (36) has the form
\[
\det[I_3s - A] = \begin{vmatrix}
s & -1 & 0 \\
0 & s & -1 \\
-1 & 2 & s - 2
\end{vmatrix} = s^3 - 2s^2 + 2s - 1
\] (37)
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and its zeros are: \( s_1 = 1 = e^{j0^\circ}, s_2 = \frac{1}{2} + j\frac{\sqrt{3}}{2} = e^{j60^\circ}, s_3 = \frac{1}{2} - j\frac{\sqrt{3}}{2} = e^{-j60^\circ} \). The system (21) with (36) for is unstable. Using (36) we obtain the matrix

\[
A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -2 & 2 \\ 2 & -3 & 2 \end{bmatrix}.
\]

Characteristic polynomial of (38) has the form

\[
\det[I_3s - A^2] = \begin{bmatrix} s & 0 & -1 \\ -1 & s+2 & -2 \\ -2 & 3 & s-2 \end{bmatrix}
\]

and its zeros are \( s_1 = 1 = e^{j0^\circ}, s_2 = -\frac{1}{2} + j\frac{\sqrt{3}}{2} = e^{j120^\circ}, s_3 = -\frac{1}{2} - j\frac{\sqrt{3}}{2} = e^{-j120^\circ} \).

The system (21) with (36) for \( k = 2 \) is unstable. By Theorem 10 it is unstable for any \( k \). The following example shows that the system (21) can be unstable for \( k = 1, 2 \) and asymptotically stable for \( k = 3l, l = 1, 2, \ldots \).

**Example 7** Consider the system (21) with the matrix

\[
A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}
\]

for \( k = 1, 2, 3, \ldots \). The characteristic polynomial of (40) has the form

\[
\det[I_3s - A] = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 0 & s \end{bmatrix} = s^3 + 1
\]

and its zeros are: \( s_1 = -1 = e^{j180^\circ}, s_2 = \frac{1}{2} + j\frac{\sqrt{3}}{2} = e^{j60^\circ}, s_3 = \frac{1}{2} - j\frac{\sqrt{3}}{2} = e^{-j60^\circ} \). For \( k = 1 \) the condition (22) is not satisfied and the system is unstable. For \( k = 2 \) we have the matrix

\[
A^2 = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}
\]

with the eigenvalues \( s_1 = 1 = e^{j0^\circ}, s_2 = -\frac{1}{2} + j\frac{\sqrt{3}}{2} = e^{j120^\circ}, s_3 = -\frac{1}{2} - j\frac{\sqrt{3}}{2} = e^{-j120^\circ} \) and the system is also unstable.
For \( k = 3 \) we obtain the matrix
\[
A^3 = \begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\] (43)

with the eigenvalues \( s_1 = s_2 = s_3 = -1 = e^{j180^\circ} \). Therefore, the system for \( k = 3 \) is asymptotically stable.

It is easy to prove that the system is asymptotically stable for \( k = 3l, l = 1, 2, \ldots \).  

5. Comparison of the stability of discrete-time and continuous-time linear systems

From the conditions (4) and (9), Remark 1 and Conclusion 1 it follows that the asymptotic stability of the discrete-time linear systems depends only on the modules of the eigenvalues of the matrix \( \tilde{A} \) and of the continuous-time linear systems only on the phases of the eigenvalues of the matrix \( A \).

To obtain to the continuous-time linear system (1) the corresponding discrete-time linear system (6) we apply the approximation
\[
\dot{x}(t) \approx \frac{x(t + h) - x(t)}{h} = \frac{x_{i+1} - x_i}{h} = Ax_i, \quad i \in \mathbb{Z}_+
\] (44)
where \( x_i = x(t), x_{i+1} = x(t + h), h = \Delta t > 0 \). From (44) we have
\[
x_{i+1} = \tilde{A}x_i
\] (45)
where
\[
\tilde{A} = I_n + hA
\] (46)

From Theorem 5 applied to (46) we obtain
\[
z_l = 1 + hs_l, \quad l = 1, \ldots, n
\] (47)
where \( z_l \) are the eigenvalues of the matrix \( \tilde{A} \) and \( s_l \) are the eigenvalues of the matrix \( A \).

**Theorem 11** The discrete-time linear system (45) is asymptotically stable for all admissible values of \( h > 0 \) if and only if the continuous-time linear system (1) is asymptotically stable.

**Proof** From (47) we have
\[
s_l = \frac{z_l - 1}{h} = \frac{|z_l|e^{j\psi_l} - 1}{h}, \quad l = 1, \ldots, n
\] (48)
where $|z_l|$ and $\psi_l$ are the module and phase of $z_l$ and
\[
\text{Re} s_l = \frac{|z_l| \cos \psi_l - 1}{h}, \quad l = 1, \ldots, n.
\] (49)

From (49) it follows that $\text{Re} s_l < 0$ for any admissible $h > 0$ if and only if $|z_l| < 1$, i.e. the discrete-time system is asymptotically stable.

Similarly, from (47) for $s_l = |s_l| e^{j\phi_l}$ we have
\[
|z_l|^2 = |1 + h s_l|^2 = [1 + h |s_l| \cos \phi_l]^2 + [h |s_l| \sin \phi_l]^2 = 1 + 2h |s_l| \cos \phi + h^2 |s_l|^2 < 1
\] (50)
and $|z_l|^2 < 1$ if and only if $\cos \phi < 0$ or equivalently the condition (4) is satisfied.

Note that the admissible value of $h > 0$ should satisfy the condition (50).

**Theorem 12** The discretization step $h$ of the asymptotically stable systems satisfies the condition
\[
h < \min_{1 \leq l \leq n} \frac{2\alpha_l}{\alpha_l^2 + \beta_l^2},
\] (51)
where $s_l = -\alpha_l + j\beta_l$, $l = 1, \ldots, n$ are the eigenvalues of the matrix $A$.

**Proof** From (47) it follows that the discrete-time system (45) is asymptotically stable if and only if
\[
|z_l| = |hs_l + 1| = |1 - h\alpha_l + jh\beta_l| < 1 \quad \text{for} \quad l = 1, \ldots, n.
\] (52)
From (52) we have
\[
(1 - h\alpha_l)^2 + (h\beta_l)^2 < 1
\] (53)
and solving (53) with respect to $h$ we obtain (51).

**Example 8** Consider the continuous-time linear system (1) with the matrix
\[
A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}.
\] (54)

The characteristic polynomial of (54) has the form
\[
\det[I_2 s - A] = \begin{vmatrix} s & -1 \\ 2 & s + 3 \end{vmatrix} = s^2 + 3s + 2
\] (55)
and the eigenvalues of the matrix (54) are $s_1 = -1$, $s_2 = -2$. The system (1) with (54) is asymptotically stable. The eigenvalues of the corresponding matrix
\[
\tilde{A} = I_n + hA = \begin{bmatrix} 1 & h \\ -2h & 1 - 3h \end{bmatrix}
\] (56)
of discrete-time system are $z_1 = 1 - h$, $z_2 = 1 - 2h$. The discrete-time system (45) with (56) is asymptotically stable for all $0 < h < 1$. 

6. Concluding remarks

The asymptotic stability of discrete-time linear systems (11) and continuous-time linear systems (21) for \( k \) integers (\( k = \pm 1, \pm 2, \ldots \)) and rational \( \left( \frac{p}{q}, p, q - \text{integers} \right) \) has been investigated. Necessary and sufficient conditions for the asymptotic stability of the systems have been established (Theorems 6, 7, 8, 9, 10). It has been shown that:

1) The asymptotic stability of (11) depends only on the modules of the eigenvalues of the matrix \( \bar{A}^k \) and of (21) only on the phases of the eigenvalues of the matrix \( A^k \).

2) The discrete-time systems (11) are asymptotically stable for all admissible values of \( h \) if and only if the continuous-time systems (21) are asymptotically stable.

3) The upper bounds of \( h \) depends on the eigenvalues of the matrix \( A \).

The considerations have been illustrated by numerical examples of discrete-time and continuous-time linear systems.

The presented considerations can be extended to positive discrete-time and continuous-time linear systems. An open problem is an extension of the considerations to fractional linear systems.

References


