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Finite-dimensional H_∞ control of a parallel-flow heat exchange process

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Abstract. This paper concerns the H_∞ control problem of a coupled transport-diffusion system with Neumann boundary condition, related to parallel-flow heat exchange process. It is shown that, by using the previous approach for a single diffusion system, the H_∞ control problem can be solved by constructing a residual mode filter (RMF)-based controller which is of finite-dimension. A numerical simulation result is given to demonstrate the validity of the proposed method.

Key words: distributed parameter system, H_∞ control, residual mode filter, semigroup.

1. Introduction

Since the beginning of the 1980's, the design method of dynamic stabilizing controllers for distributed parameter systems has been proposed by many researchers (see e.g. [20, 26, 3, 17, 1, 8, 21, 2, 7, 11, 24], and the references therein). On the other hand, as for H_∞ controllers for distributed parameter systems, the research has been progressed since the beginning of the 1990's. The design method of infinite-dimensional H_∞ state feedback/output feedback controllers was first studied in [10]. However, the algorithm was not feasible, because one needed to solve two kinds of operator Riccati equations. In connection with infinite-dimensional H_∞ state feedback, it was shown in [9] that the solutions to the finite-dimensional Riccati equations of high order approximately solve the original H_∞ control problem. After that, the approach was extended to the case of H_∞ output feedback in [16]. On the other hand, in [22] the design method of finite-dimensional H_∞ controllers for a single diffusion system was given, in which the residual mode filter (RMF) (see [1]) was used in the output feedback controller design. Besides, in [4, 6] the frequency-domain approach was developed in the synthesis of robust controllers for distributed parameter systems. Also, for nonsmooth distributed parameter systems, the design method of H_∞ controllers using linear matrix inequalities (LMIs) was recently reported in [18].

The purpose of this paper is to show that the result of [22] is applicable to a coupled transport-diffusion system with Neumann boundary condition, related to parallel-flow heat exchange process. Under the boundary condition, the open-loop system is not exponentially stable. In this paper, we formulate the system as an abstract system in a Hilbert space and give the design method of finite-dimensional H_∞ controllers. Moreover, we show the validity of the proposed method through a numerical simulation. An advantage of our method is to make possible to construct H_∞ controllers without solving operator Riccati equations in an infinite-dimensional space or in a finite-dimensional space of higher order. That is, the problem results in the H_∞ design problem for finite-dimensional systems of lower order. The key point is that we use the RMF which can counteract control/observation spillover and is easy to construct. As a result, the use of the RMF of higher order assures the closed-loop stability as well as a given H_∞ norm bound.

This paper is organized as follows: In Section 2, we introduce the PDE describing parallel-flow heat exchange process and formulate it in a Hilbert space. In Section 3, we partition the abstract system into three parts and give the design method of finite-dimensional H_∞ controllers. In Section 4, a numerical simulation result is given, and finally, the paper is concluded in Section 5.

2. System description and formulation

2.1. System description. We shall consider the following coupled transport-diffusion system related to parallel-flow heat exchange process over interval [0,1]:

$$\begin{cases} \dfrac{\partial z_1}{\partial t}(t,x) = D\dfrac{\partial^2 z_1}{\partial x^2}(t,x) - \alpha\dfrac{\partial z_1}{\partial x}(t,x) \\ \qquad\qquad + h_1(z_2(t,x) - z_1(t,x)), \\ \dfrac{\partial z_2}{\partial t}(t,x) = D\dfrac{\partial^2 z_2}{\partial x^2}(t,x) - \alpha\dfrac{\partial z_2}{\partial x}(t,x) + h_2(z_1(t,x) \\ \qquad\qquad - z_2(t,x)) + b_1(x)w_1(t) + b_2(x)u(t), \\ \qquad\qquad (t,x) \in (0,\infty)\times(0,1), \\ z_1(0,x) = z_{10}(x), \quad z_2(0,x) = z_{20}(x), \quad x \in [0,1], \end{cases} \tag{1}$$

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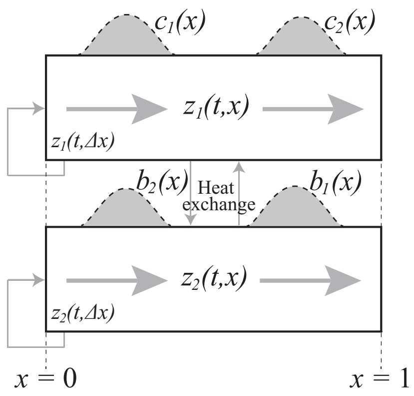

Fig. 1. Parallel-flow heat exchange process

where $z_1(t, x)$, $z_2(t, x)$ denote the temperatures of fluids at time t and at the point $x \in [0, 1]$, and $w_1(t) \in \mathbf{R}$ denotes the disturbance added through the influence function $b_1(x)$, $u(t) \in \mathbf{R}$ the control input added through the influence function $b_2(x)$. $D > 0$ is the heat diffusion coefficient, $\alpha > 0$ the fluid velocity, h_1, $h_2 > 0$ the heat exchange rates between two tubes. In general, the mathematical model of heat exchange process does not contain diffusive terms. However, the diffusive terms should be taken into account in the case of small flow rates of gaseous/liquid medium. For the relevant model, see e.g. [19]. For system (1), we consider the following Neumann boundary condition[1]:

$$\frac{\partial z_1}{\partial x}(t,0) = \frac{\partial z_1}{\partial x}(t,1) = \frac{\partial z_2}{\partial x}(t,0) = \frac{\partial z_2}{\partial x}(t,1) = 0. \quad (2)$$

Here, the Neumann boundary condition imposed at the inlet $x = 0$ means that the boundary feedback loop such as

$$z_1(t,0) = z_1(t,\Delta x), \quad z_2(t,0) = z_2(t,\Delta x)$$

is assumed at $x = 0$, where Δx is a sufficiently small positive constant (see Fig. 1). On the other hand, the Neumann boundary condition imposed at the outlet $x = 1$ is a general one (see e.g. [19]). Let us set the controlled output $z_c(t) \in \mathbf{R}^2$ and the measured output $y(t) \in \mathbf{R}$ as follows:

$$\begin{cases} z_c(t) = \left[\int_0^1 c_1(x) z_1(t,x)dx, u(t)\right]^T, \\ y(t) = \int_0^1 c_2(x) z_1(t,x)dx + w_2(t), \quad t > 0, \end{cases} \quad (3)$$

where $c_1(x)$, $c_2(x)$ are the influence functions, and $w_2(t) \in \mathbf{R}$ denotes the disturbance included to the measurement.

[1] In [25], system (1) with the boundary condition $z_1(t, 0) = \partial z_1/\partial x(t, 1) = z_2(t, 0) = \partial z_2/\partial x(t, 1) = 0$ has been studied. In such case, the open-loop system is exponentially stable. On the other hand, system (1) with the boundary condition (2) is not exponentially stable (see Remark 2.2).

Remark 2.1. For the parallel-flow heat exchange process without diffusive terms, the exact transient solution was concretely given in [12]. Also, the reachability/observability results of the process were given in [23, 13]. Furthermore, the problem of regulating the outlet fluid temperature to a desired one was treated in [14]. Recently, in [15] the analytical solution was given for the parallel-flow three-fluid heat exchange process without diffusive terms.

2.2. Formulation of the system. By defining the differential operator $\mathscr{L}$ as

$$\mathscr{L}\varphi(x) = -D\frac{d^2\varphi(x)}{dx^2} + \alpha\frac{d\varphi(x)}{dx} + h_1\varphi(x), \quad x \in (0,1),$$

system (1) is written as

$$\begin{cases} \dfrac{\partial z_1}{\partial t}(t,x) = -\mathscr{L}z_1(t,x) + h_1 z_2(t,x), \\ \dfrac{\partial z_2}{\partial t}(t,x) = (-\mathscr{L} + h_1 - h_2)z_2(t,x) + h_2 z_1(t,x) \\ +b_1(x)w_1(t) + b_2(x)u(t),\ (t,x) \in (0,\infty)\times(0,1), \\ z_1(0,x) = z_{10}(x), \quad z_2(0,x) = z_{20}(x), \quad x \in [0,1]. \end{cases} \quad (4)$$

Here, by considering boundary condition (2), we define the unbounded operator A as

$$A\varphi = \mathscr{L}\varphi, \quad \varphi \in D(A),$$
$$D(A) = \{\varphi \in H^2(0,1);\, \varphi'(0) = \varphi'(1) = 0\}.$$

Then, A is expressed as an operator of Sturm-Liouville type as follows:

$$(A\varphi)(x) = \frac{1}{w(x)}\left(-\frac{d}{dx}(p(x)\frac{d\varphi(x)}{dx}) + q(x)\varphi(x)\right),$$
$$w(x) = e^{-\beta x}, \quad p(x) = De^{-\beta x}, \quad q(x) = h_1 e^{-\beta x},$$

where $\beta := \alpha/D\ (>0)$. Therefore, the operator A becomes self-adjoint in the weighted L^2-space $L^2_\beta(0, 1)$ whose inner product is defined by

$$\langle\varphi,\psi\rangle_\beta = \int_0^1 \varphi(x)\psi(x)e^{-\beta x}dx, \quad \varphi,\psi \in L^2_\beta(0,1).$$

A has a set of eigenpairs $\{\lambda_i, \varphi_i\}_{i=1}^\infty$ in $L^2_\beta(0, 1)$ such that $\{\varphi_i\}_{i=1}^\infty$ forms a complete orthonormal system in $L^2_\beta(0, 1)$. Hence, any $f \in L^2_\beta(0, 1)$ is expressed as

$$f = \sum_{i=1}^\infty \langle f,\varphi_i\rangle_\beta \varphi_i.$$

Concretely, the eigenvalues and eigenfunctions of A are calculated as follows (see e.g. [24]):

$$\begin{aligned}
&\lambda_1 = h_1, \;\; \lambda_{i+1} = i^2\pi^2 D + \frac{\beta^2}{4}D + h_1,\\
&\varphi_1(x) = \sqrt{\beta}(1-e^{-\beta})^{-\frac{1}{2}},\\
&\varphi_{i+1}(x) = \mu_i\left(e^{\frac{\beta}{2}x}\cos i\pi x - \frac{\beta}{2i\pi}e^{\frac{\beta}{2}x}\sin i\pi x\right),\\
&\mu_i := \sqrt{2}\left(1+\frac{\beta^2}{4i^2\pi^2}\right)^{-\frac{1}{2}} \;\; (\le \sqrt{2}), \quad i \ge 1.
\end{aligned}$$

Hereafter, for the initial condition and the influence functions, we assume that $z_{10}, z_{20}, b_1, b_2, c_1, c_2 \in L^2_\beta(0, 1)(= L^2(0, 1))$. Then, from (4) we have the following equation:

$$\begin{cases}
\dfrac{dz_1(t,\cdot)}{dt} = -Az_1(t,\cdot) + h_1 z_2(t,\cdot), \quad z_1(0,\cdot) = z_{10},\\
\dfrac{dz_2(t,\cdot)}{dt} = (-A + h_1 - h_2)z_2(t,\cdot) + h_2 z_1(t,\cdot)\\
\qquad\qquad + b_1 w_1(t) + b_2 u(t), \quad z_2(0,\cdot) = z_{20}.
\end{cases} \tag{5}$$

As for the output equation (3), we can formulate as follows:

$$\begin{cases}
z_c(t) = [\langle e^{\beta\cdot}c_1, z_1(t,\cdot)\rangle_\beta, u(t)]^T,\\
y(t) = \langle e^{\beta\cdot}c_2, z_1(t,\cdot)\rangle_\beta + w_2(t), \quad t > 0.
\end{cases} \tag{6}$$

Here, by defining the bounded operators $B_i : \mathbf{R} \to L^2_\beta(0, 1)$, $C_i : L^2_\beta(0, 1) \to \mathbf{R}$ $(i = 1, 2)$ as

$$\begin{aligned}
&B_i v = b_i v, \quad v \in \mathbf{R},\\
&C_i\varphi = \langle e^{\beta\cdot}c_i, \varphi\rangle_\beta, \quad \varphi \in L^2_\beta(0,1),
\end{aligned}$$

system (5, 6) is written as follows:

$$\begin{cases}
\dfrac{dz_1(t,\cdot)}{dt} = -Az_1(t,\cdot) + h_1 z_2(t,\cdot), \;\; z_1(0,\cdot) = z_{10},\\
\dfrac{dz_2(t,\cdot)}{dt} = (-A + h_1 - h_2)z_2(t,\cdot) + h_2 z_1(t,\cdot)\\
\qquad\qquad + B_1 w_1(t) + B_2 u(t), \;\; z_2(0,\cdot) = z_{20},\\
z_c(t) = \begin{bmatrix} C_1 z_1(t,\cdot) \\ u(t) \end{bmatrix},\\
y(t) = C_2 z_1(t,\cdot) + w_2(t).
\end{cases} \tag{7}$$

Moreover, by defining the unbounded operator $\mathscr{A} : [D(A)]^2 \subset \subset [L^2_\beta(0, 1)]^2 \to [L^2_\beta(0, 1)]^2$, the bounded operators $\mathscr{B}_1 : \mathbf{R}^2 \to \to [L^2_\beta(0, 1)]^2$, $\mathscr{B}_2 : \mathbf{R} \to [L^2_\beta(0, 1)]^2$, $\mathscr{C}_1 : [L^2_\beta(0, 1)]^2 \to \mathbf{R}^2$, $\mathscr{C}_2 : [L^2_\beta(0, 1)]^2 \to \mathbf{R}$ and the matrices D_{12}, D_{21} as

$$\mathscr{A} := \begin{bmatrix} -A & h_1 \\ h_2 & -A + h_1 - h_2 \end{bmatrix}, \quad \mathscr{B}_1 := \begin{bmatrix} 0 & 0 \\ B_1 & 0 \end{bmatrix},$$

$$\mathscr{B}_2 := \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \; \mathscr{C}_1 := \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}, \; \mathscr{C}_2 := \begin{bmatrix} C_2 & 0 \end{bmatrix},$$

$$D_{12} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_{21} := \begin{bmatrix} 0 & 1 \end{bmatrix},$$

system (7) is written as follows:

$$\begin{cases}
\dfrac{d}{dt}\begin{bmatrix} z_1(t,\cdot) \\ z_2(t,\cdot) \end{bmatrix} = \mathscr{A}\begin{bmatrix} z_1(t,\cdot) \\ z_2(t,\cdot) \end{bmatrix} + \mathscr{B}_1\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}\\
\qquad\qquad + \mathscr{B}_2 u(t),\\
\begin{bmatrix} z_1(0,\cdot) \\ z_2(0,\cdot) \end{bmatrix} = \begin{bmatrix} z_{10} \\ z_{20} \end{bmatrix},\\
z_c(t) = \mathscr{C}_1\begin{bmatrix} z_1(t,\cdot) \\ z_2(t,\cdot) \end{bmatrix} + D_{12}u(t),\\
y(t) = \mathscr{C}_2\begin{bmatrix} z_1(t,\cdot) \\ z_2(t,\cdot) \end{bmatrix} + D_{21}\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}.
\end{cases} \tag{8}$$

Remark 2.2. The system operator

$$\mathscr{A} = \begin{bmatrix} -A & h_1 \\ h_2 & -A + h_1 - h_2 \end{bmatrix},$$

which describes (8), generates an analytic semigroup $e^{t\mathscr{A}}$ on the Hilbert space $X := [L^2_\beta(0, 1)]^2$ whose growth bound is equal to $-\lambda_1 + h_1 = 0$. Therefore, system (8) is not exponentially stable. The purpose of this paper is to construct finite-dimensional H_∞ controllers for system (8). Here, note that the orthogonal condition

$$D^T_{12}\mathscr{C}_1 = 0, \quad D^T_{12}D_{12} = 1, \quad \mathscr{B}_1 D^T_{21} = 0, \quad D_{21}D^T_{21} = 1$$

is satisfied, where D^T_{12} denotes the transpose of D_{12}.

3. Finite-dimensional H_∞ controllers

3.1. Partitioned system. In order to derive a finite-dimensional model for system (7), we use the orthogonal projection P_k defined by

$$P_k f = \sum_{i=1}^{k} \langle f, \varphi_i\rangle_\beta \varphi_i.$$

First, we define the following constant.

$$\begin{aligned}
b := \max\Big\{ &1, \; \|C_1\|\,\|B_2\|\,\|D^T_{12}\|,\\
&\|C_1\|\,\|B_1\|, \; \|D^T_{21}\|\,\|C_2\|\,\|B_1\|,\\
&\|C_1\|\,\|B_2\|\,\|D^T_{12}\|\,\|D^T_{21}\|\,\|C_2\|\,\|B_1\| \Big\}.
\end{aligned} \tag{9}$$

Let $\delta > 0$ and $\theta \in (0, 1)$ be given positive numbers.

(S1) For a positive number ε satisfying

$$0 < \varepsilon < \frac{\theta\delta}{b},$$

we first choose the minimal integer $l(l \geq 1)$ such that

$$0 < \frac{(2+h_1+h_2)^2}{(h_1+h_2)(\lambda_{l+1}-h_1)} < \varepsilon.$$

(S2) Next, we choose another integer n larger than l. Using the operators P_l and $P_n(n > l)$, we decompose the state variables $z_1(t,\cdot)$ and $z_2(t,\cdot)$ as follows:

$$z_1(t,\cdot) = z_{1,1}(t) + z_{1,2}(t) + z_{1,3}(t),$$
$$z_2(t,\cdot) = z_{2,1}(t) + z_{2,2}(t) + z_{2,3}(t),$$

where

$$z_{1,1}(t) := P_l z_1(t,\cdot), \quad z_{1,2}(t) := (P_n - P_l) z_1(t,\cdot),$$
$$z_{1,3}(t) := (I - P_n) z_1(t,\cdot),$$

$$z_{2,1}(t) := P_l z_2(t,\cdot), \quad z_{2,2}(t) := (P_n - P_l) z_2(t,\cdot),$$
$$z_{2,3}(t) := (I - P_n) z_2(t,\cdot).$$

Also, the space $L^2_\beta(0, 1)$ is expressed as

$$L^2_\beta(0,1) = P_l L^2_\beta(0,1) \oplus (P_n - P_l) L^2_\beta(0,1) \oplus (I - P_n) L^2_\beta(0,1),$$

and their dimensions are given by

$$\dim P_l L^2_\beta(0,1) = l, \quad \dim (P_n - P_l) L^2_\beta(0,1) = n - l,$$
$$\dim (I - P_n) L^2_\beta(0,1) = \infty.$$

Therefore, system (7) is equivalently expressed as

$$\begin{cases} \dfrac{dz_{1,1}(t)}{dt} = -A_1 z_{1,1}(t) + h_1 z_{2,1}(t), \quad z_{1,1}(0) = z^1_{10}, \\ \dfrac{dz_{1,2}(t)}{dt} = -A_2 z_{1,2}(t) + h_1 z_{2,2}(t), \quad z_{1,2}(0) = z^2_{10}, \\ \dfrac{dz_{1,3}(t)}{dt} = -A_3 z_{1,3}(t) + h_1 z_{2,3}(t), \quad z_{1,3}(0) = z^3_{10}, \\ \dfrac{dz_{2,1}(t)}{dt} = (-A_1 + h_1 - h_2) z_{2,1}(t) + h_2 z_{1,1}(t) \\ \qquad + B_{1,1} w_1(t) + B_{2,1} u(t), \quad z_{2,1}(0) = z^1_{20}, \\ \dfrac{dz_{2,2}(t)}{dt} = (-A_2 + h_1 - h_2) z_{2,2}(t) + h_2 z_{1,2}(t) \\ \qquad + B_{1,2} w_1(t) + B_{2,2} u(t), \quad z_{2,2}(0) = z^2_{20}, \\ \dfrac{dz_{2,3}(t)}{dt} = (-A_3 + h_1 - h_2) z_{2,3}(t) + h_2 z_{1,3}(t) \\ \qquad + B_{1,3} w_1(t) + B_{2,3} u(t), \quad z_{2,3}(0) = z^3_{20}, \\ z_c(t) = \begin{bmatrix} C_{1,1} z_{1,1}(t) + C_{1,2} z_{1,2}(t) + C_{1,3} z_{1,3}(t) \\ u(t) \end{bmatrix}, \\ y(t) = C_{2,1} z_{1,1}(t) + C_{2,2} z_{1,2}(t) + C_{2,3} z_{1,3}(t) + w_2(t), \end{cases} \tag{10}$$

where

$$\begin{cases} A_1 := P_l A P_l, \\ B_{1,1} := P_l B_1,\ B_{2,1} := P_l B_2, \\ C_{1,1} := C_1 P_l,\ C_{2,1} := C_2 P_l, \\ z^1_{10} := P_l z_{10},\ z^1_{20} := P_l z_{20}, \end{cases}$$

$$\begin{cases} A_2 := (P_n - P_l) A (P_n - P_l), \\ B_{1,2} := (P_n - P_l) B_1,\ B_{2,2} := (P_n - P_l) B_2, \\ C_{1,2} := C_1 (P_n - P_l),\ C_{2,2} := C_2 (P_n - P_l), \\ z^2_{10} := (P_n - P_l) z_{10},\ z^2_{20} := (P_n - P_l) z_{20}, \end{cases}$$

$$\begin{cases} A_3 := (I - P_n) A (I - P_n), \\ B_{1,3} := (I - P_n) B_1,\ B_{2,3} := (I - P_n) B_2, \\ C_{1,3} := C_1 (I - P_n),\ C_{2,3} := C_2 (I - P_n), \\ z^3_{10} := (I - P_n) z_{10},\ z^3_{20} := (I - P_n) z_{20}. \end{cases}$$

In the above, the operator A_3 is unbounded, whereas all the other operators are bounded.

Hereafter, we identify the finite-dimensional Hilbert space $P_l L^2_\beta(0, 1)$ with the Euclidean space $\mathbf{R}^l$ with respect to the basis $\{\varphi_1, \varphi_2, \ldots, \varphi_l\}$. In this way, each element in $P_l L^2_\beta(0, 1)$ is identified with an l-dimensional vector, and the operators A_1, $B_{1,1}$, $B_{2,1}$, $C_{1,1}$ and $C_{2,1}$ are identified with matrices with appropriate size. Similarly, each element in $(P_n - P_l) L^2_\beta(0, 1)$ is identified with an $(n - l)$-dimensional vector, and the operators A_2, $B_{1,2}$, $B_{2,2}$, $C_{1,2}$ and $C_{2,2}$ are identified with matrices with appropriate size.

Combining the first and fourth equations, and the second and fifth equations, and further the third and sixth equations for system (10), we have the following equation:

$$\begin{cases} \dfrac{d\bar{x}_1(t)}{dt} = \bar{A}_1 \bar{x}_1(t) + \bar{B}_{1,1} w(t) + \bar{B}_{2,1} u(t), \quad \bar{x}_1(0) = \bar{x}_{10}, \\ \dfrac{d\bar{x}_2(t)}{dt} = \bar{A}_2 \bar{x}_2(t) + \bar{B}_{1,2} w(t) + \bar{B}_{2,2} u(t), \quad \bar{x}_2(0) = \bar{x}_{20}, \\ \dfrac{d\bar{x}_3(t)}{dt} = \bar{A}_3 \bar{x}_3(t) + \bar{B}_{1,3} w(t) + \bar{B}_{2,3} u(t), \quad \bar{x}_3(0) = \bar{x}_{30}, \\ z_c(t) = \bar{C}_{1,1} \bar{x}_1(t) + \bar{C}_{1,2} \bar{x}_2(t) + \bar{C}_{1,3} \bar{x}_3(t) + D_{12} u(t), \\ y(t) = \bar{C}_{2,1} \bar{x}_1(t) + \bar{C}_{2,2} \bar{x}_2(t) + \bar{C}_{2,3} \bar{x}_3(t) + D_{21} w(t), \end{cases} \tag{11}$$

where the state variables $\bar{x}_i(t) (i = 1, 2, 3)$ and the disturbance $w(t)$ are

$$\bar{x}_1(t) := \begin{bmatrix} z_{1,1}(t) \\ z_{2,1}(t) \end{bmatrix} \in \mathbf{R}^{2l},$$

$$\bar{x}_2(t) := \begin{bmatrix} z_{1,2}(t) \\ z_{2,2}(t) \end{bmatrix} \in \mathbf{R}^{2(n-l)},$$

$$\bar{x}_3(t) := \begin{bmatrix} z_{1,3}(t) \\ z_{2,3}(t) \end{bmatrix} \in [(I - P_n) L^2_\beta(0,1)]^2,$$

$$w(t) := \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \in \mathbf{R}^2,$$

and the matrices and operators are as follows:

$$\overline{A}_1 := \begin{bmatrix} -A_1 & h_1 I_l \\ h_2 I_l & -A_1 + (h_1 - h_2) I_l \end{bmatrix},$$

$$\overline{B}_{1,1} := \begin{bmatrix} 0 & 0 \\ B_{1,1} & 0 \end{bmatrix}, \quad \overline{B}_{2,1} := \begin{bmatrix} 0 \\ B_{2,1} \end{bmatrix},$$

$$\overline{C}_{1,1} := \begin{bmatrix} C_{1,1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \overline{C}_{2,1} := \begin{bmatrix} C_{2,1} & 0 \end{bmatrix},$$

$$\overline{A}_2 := \begin{bmatrix} -A_2 & h_1 I_{n-l} \\ h_2 I_{n-l} & -A_2 + (h_1 - h_2) I_{n-l} \end{bmatrix},$$

$$\overline{B}_{1,2} := \begin{bmatrix} 0 & 0 \\ B_{1,2} & 0 \end{bmatrix}, \quad \overline{B}_{2,2} := \begin{bmatrix} 0 \\ B_{2,2} \end{bmatrix},$$

$$\overline{C}_{1,2} := \begin{bmatrix} C_{1,2} & 0 \\ 0 & 0 \end{bmatrix}, \quad \overline{C}_{2,2} := \begin{bmatrix} C_{2,2} & 0 \end{bmatrix},$$

$$\overline{A}_3 := \begin{bmatrix} -A_3 & h_1 I \\ h_2 I & -A_3 + (h_1 - h_2) I \end{bmatrix},$$

$$\overline{B}_{1,3} := \begin{bmatrix} 0 & 0 \\ B_{1,3} & 0 \end{bmatrix}, \quad \overline{B}_{2,3} := \begin{bmatrix} 0 \\ B_{2,3} \end{bmatrix},$$

$$\overline{C}_{1,3} := \begin{bmatrix} C_{1,3} & 0 \\ 0 & 0 \end{bmatrix}, \quad \overline{C}_{2,3} := \begin{bmatrix} C_{2,3} & 0 \end{bmatrix},$$

$$D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_{21} := \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

3.2. Finite-dimensional H_∞ controllers using RMFs. By the partitioned system (11), we consider the finite-dimensional system

$$\begin{cases} \dfrac{d\overline{x}_1(t)}{dt} = \overline{A}_1 \overline{x}_1(t) + \overline{B}_{1,1} w(t) + \overline{B}_{2,1} u(t), \\ z_c(t) = \overline{C}_{1,1} \overline{x}_1(t) + D_{12} u(t), \\ y(t) = \overline{C}_{2,1} \overline{x}_1(t) + D_{21} w(t) \end{cases} \tag{12}$$

as a finite-dimensional model for system (8). Here, note that this model also satisfies the orthogonal condition

$$D_{12}^T \overline{C}_{1,1} = 0, \;\; D_{12}^T D_{12} = 1, \;\; \overline{B}_{1,1} D_{21}^T = 0, \;\; D_{21} D_{21}^T = 1.$$

Here, we set the following assumptions for the model (12):

(A1) The pair $(\overline{A}_1, \overline{B}_{1,1})$ is stabilizable, and the pair $(\overline{C}_{1,1}, \overline{A}_1)$ is detectable.

(A2) The pair $(\overline{A}_1, \overline{B}_{2,1})$ is stabilizable, and the pair $(\overline{C}_{2,1}, \overline{A}_1)$ is detectable.

For the finite-dimensional model (12), we denote the closed-loop transfer function from w to z_c by $T_{z_c w}(s)$. For the given positive numbers $\delta > 0$, $\theta \in (0, 1)$, and $\varepsilon \in (0, \frac{\theta\delta}{b})$, let us define the postive number γ by

$$\gamma := \frac{1}{(1+\varepsilon)^2} \left(\frac{\theta\delta}{b} - \varepsilon \right) \in (0, \delta). \tag{13}$$

Then, from the result given by Doyle *et al.*, under the assumptions (A1–A2), the necessary and sufficient condition for the controller $u = \tilde{K}(s) y$ such that

(i) it internally stabilizes the model (12), and

(ii) $\|T_{z_c w}(\cdot)\|_{\mathscr{H}_\infty(\mathscr{L}(\mathbf{C}^2, \mathbf{C}^2))} < \gamma$

to exist for the model (12) is given, by using the matrices H_∞ and J_∞ defined by

$$H_\infty := \begin{bmatrix} \overline{A}_1 & \gamma^{-2} \overline{B}_{1,1} \overline{B}_{1,1}^T - \overline{B}_{2,1} \overline{B}_{2,1}^T \\ -\overline{C}_{1,1}^T \overline{C}_{1,1} & -\overline{A}_1^T \end{bmatrix},$$

$$J_\infty := \begin{bmatrix} \overline{A}_1^T & \gamma^{-2} \overline{C}_{1,1}^T \overline{C}_{1,1} - \overline{C}_{2,1}^T \overline{C}_{2,1} \\ -\overline{B}_{1,1} \overline{B}_{1,1}^T & -\overline{A}_1 \end{bmatrix},$$

as the following (B1–B3):

(B1) $H_\infty \in \mathrm{dom}(\mathrm{Ric})$, $X_\infty := \mathrm{Ric}(H_\infty) \geq 0$,

(B2) $J_\infty \in \mathrm{dom}(\mathrm{Ric})$, $Y_\infty := \mathrm{Ric}(J_\infty) \geq 0$,

(B3) $\rho(X_\infty Y_\infty) < \gamma^2$,

where the notation $\rho(X_\infty Y_\infty)$ denotes the spectral radius of the matrix $X_\infty Y_\infty$ (see [5]).

Hereafter, it is supposed that the assumptions (B1–B3) are satisfied. Then, all stabilizing controllers for the model (12) that satisfy $\|T_{z_c w}(\cdot)\|_{\mathscr{H}_\infty(\mathscr{L}(\mathbf{C}^2, \mathbf{C}^2))} < \gamma$ are given as follows:

$$\begin{cases} \dfrac{dq(t)}{dt} = \hat{A}_\infty q(t) - Z_\infty L_\infty y(t) + Z_\infty \overline{B}_{2,1} v(t), \; q(0) = q_0, \\ u(t) = F_\infty q(t) + v(t), \\ r(t) = -\overline{C}_{2,1} q(t) + y(t), \end{cases} \tag{14}$$

$$\begin{cases} \dfrac{d\lambda(t)}{dt} = A_\Lambda \lambda(t) + B_\Lambda r(t), \quad \lambda(0) = \lambda_0, \\ v(t) = C_\Lambda \lambda(t) + D_\Lambda r(t), \end{cases} \tag{15}$$

where

$$\hat{A}_\infty := \overline{A}_1 + \gamma^{-2} \overline{B}_{1,1} \overline{B}_{1,1}^T X_\infty + \overline{B}_{2,1} F_\infty + Z_\infty L_\infty \overline{C}_{2,1},$$

$$F_\infty := -\overline{B}_{2,1}^T X_\infty, \quad L_\infty := -Y_\infty \overline{C}_{2,1}^T,$$

$$Z_\infty := (I - \gamma^{-2} Y_\infty X_\infty)^{-1}.$$

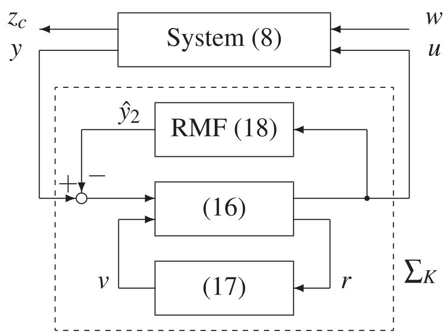

Fig. 2. Finite-dimensional H_∞ controller Σ_K

In this, (15) is a free parameter that satisfies $\mathrm{Re}\,\sigma(A_\Lambda) < 0$ and

$$\|C_\Lambda((\cdot)I - A_\Lambda)^{-1}B_\Lambda + D_\Lambda\|_{\mathscr{H}_\infty(\mathscr{L}(\mathbf{C},\mathbf{C}))} < \gamma,$$

where $\sigma(A_\Lambda)$ denotes the spectrum of A_Λ (see [5]).

In the above, we especially set $D_\Lambda = 0$ and consider a controller to which the residual mode filter (RMF)

$$\begin{cases} \dfrac{d\hat{x}_2(t)}{dt} = \overline{A}_2\hat{x}_2(t) + \overline{B}_{2,2}u(t), \quad \hat{x}_2(0) = \hat{x}_{20}, \\ \hat{y}_2(t) = \overline{C}_{2,2}\hat{x}_2(t) \end{cases}$$

is added. In this paper, we show that the proposed controller becomes a stabilizing controller that satisfies $\|G_{z_c w}(\cdot)\|_{\mathscr{H}_\infty(\mathscr{L}(\mathrm{C}^2,\mathrm{C}^2))} < \delta$ for system (8). Then, the whole controller Σ_K is described as follows (Fig. 2):

$$\begin{cases} \dfrac{dq(t)}{dt} = \hat{A}_\infty q(t) - Z_\infty L_\infty(y(t) - \hat{y}_2(t)) + Z_\infty \overline{B}_{2,1}v(t), \\ q(0) = q_0, \\ u(t) = F_\infty q(t) + v(t), \\ r(t) = -\overline{C}_{2,1}q(t) + y(t) - \hat{y}_2(t), \end{cases} \tag{16}$$

Σ_K:

$$\begin{cases} \dfrac{d\lambda(t)}{dt} = A_\Lambda\lambda(t) + B_\Lambda r(t), \quad \lambda(0) = \lambda_0, \\ v(t) = C_\Lambda\lambda(t), \end{cases} \tag{17}$$

$$\begin{cases} \dfrac{d\hat{x}_2(t)}{dt} = \overline{A}_2\hat{x}_2(t) + \overline{B}_{2,2}u(t), \quad \hat{x}_2(0) = \hat{x}_{20}, \\ \hat{y}_2(t) = \overline{C}_{2,2}\hat{x}_2(t). \end{cases} \tag{18}$$

Then, we have the following theorem.

Theorem 3.1. Let $\delta > 0$, $\theta \in (0, 1)$, and $\varepsilon \in (0, \frac{\theta\delta}{b})$ be given positive numbers, where the constant b is defined by (9), and choose the integers l and n according to the steps (S1–S2). Suppose that the assumptions (A1–A2) are satisfied, and further that the assumptions (B1–B3) are satisfied for the positive constant γ given by (13). Then, the controller Σ_K consisting of (16–18) becomes a finite-dimensional stabilizing controller that satisfies $\|G_{z_c w}(\cdot)\|_{\mathscr{H}_\infty(\mathscr{L}(\mathrm{C}^2,\mathrm{C}^2))} < \delta$ for system (8), if the integer n is chosen sufficiently large.

Proof. Introducing the variable $\bar{e}_2(t) := \bar{x}_2(t) - \hat{x}_2(t)$, the closed-loop system from w to z_c is written as follows:

$$\begin{cases} \dfrac{d\xi(t)}{dt} = (\mathscr{A} + \Delta\mathscr{A})\xi(t) + (\mathscr{B} + \Delta\mathscr{B})w(t), \; \xi(0) = \xi_0, \\ z(t) = (\mathscr{C} + \Delta\mathscr{C})\xi(t), \end{cases}$$

where

$$\xi(t) := [\bar{x}_1(t)^T, p(t)^T, \bar{x}_2(t)^T, \bar{x}_3(t), \bar{e}_2(t)^T]^T$$

(where $p(t) := [q(t)^T, \lambda(t)^T]^T$) takes value in the real Hilbert space $X := \mathbf{R}^{2l} \times (\mathbf{R}^{2l} \times \mathbf{R}^S) \times \mathbf{R}^{2(n-l)} \times [(I - P_n)L^2_\beta(0, 1)]^2 \times \mathbf{R}^{2(n-l)}$ (where the state space of the free parameter (17) is set to be $\mathbf{R}^S$), and the operators $\mathscr{A}, \Delta\mathscr{A}, \mathscr{B}, \Delta\mathscr{B}, \mathscr{C}$, and $\Delta\mathscr{C}$ are defined as follows:

$$\mathscr{A} := \begin{bmatrix} \overline{A}_1 & \overline{B}_{2,1}L & 0 & 0 & 0 \\ N\overline{C}_{2,1} & M & 0 & 0 & N\overline{C}_{2,2} \\ 0 & \overline{B}_{2,2}L & \overline{A}_2 & 0 & 0 \\ 0 & 0 & 0 & \overline{A}_3 & 0 \\ 0 & 0 & 0 & 0 & \overline{A}_2 \end{bmatrix},$$

$$\Delta\mathscr{A} := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & N\overline{C}_{2,3} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \overline{B}_{2,3}L & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathscr{B} := \begin{bmatrix} \overline{B}_{1,1} \\ ND_{21} \\ \overline{B}_{1,2} \\ 0 \\ \overline{B}_{1,2} \end{bmatrix}, \quad \Delta\mathscr{B} := \begin{bmatrix} 0 \\ 0 \\ 0 \\ \overline{B}_{1,3} \\ 0 \end{bmatrix},$$

$$\mathscr{C} := \begin{bmatrix} \overline{C}_{1,1} & D_{12}L & \overline{C}_{1,2} & 0 & 0 \end{bmatrix},$$

$$\Delta\mathscr{C} := \begin{bmatrix} 0 & 0 & 0 & \overline{C}_{1,3} & 0 \end{bmatrix},$$

$$M := \begin{bmatrix} \hat{A}_\infty & Z_\infty\overline{B}_{2,1}C_\Lambda \\ -B_\Lambda\overline{C}_{2,1} & A_\Lambda \end{bmatrix}, \quad N := \begin{bmatrix} -Z_\infty L_\infty \\ B_\Lambda \end{bmatrix},$$

$$L := \begin{bmatrix} F_\infty & C_\Lambda \end{bmatrix}.$$

In the above, the operator $\mathscr{A}$ is unbounded since it contains the unbounded operator $\overline{A}_3$.

By the same discussion as in [22], and further by using the inequality

$$\|((\cdot)I - \overline{A}_2)^{-1}\|_{\mathscr{H}_\infty(\mathscr{L}(\mathbf{C}^{2(n-l)}))} \le \frac{(2 + h_1 + h_2)^2}{(h_1 + h_2)(\lambda_{l+1} - h_1)},$$

we can prove that, for sufficiently large n, the C_0-semigroup $e^{t(\mathscr{A}+\Delta\mathscr{A})}$ generated by the operator $\mathscr{A}+\Delta\mathscr{A}$ becomes exponentially stable and the norm condition $\|G_{z_c w}(\cdot)\|_{\mathscr{H}_\infty(\mathscr{L}(\mathbf{C}^2,\mathbf{C}^2))} < \delta$ is satisfied, where $G_{z_c w}(s) := (\mathscr{C}+\Delta\mathscr{C})\big(sI-(\mathscr{A}+\Delta\mathscr{A})\big)^{-1}(\mathscr{B}+\Delta\mathscr{B})$. $\square$

Remark 3.1. When $\theta \to 1$ and $\varepsilon \to 0$, it is easy to see that $\gamma \to \frac{\delta}{b}$ and $n > l \to \infty$. Especially, when the influence functions $b_1(x)$, $b_2(x)$, $c_1(x)$, and $c_2(x)$ are chosen such that the L^2_β-norm satisfies

$$\|b_1\|_\beta = \|b_2\|_\beta = \|c_1\|_\beta = \|c_2\|_\beta = 1,$$

it follows that $b = 1$, which implies that $\gamma \to \delta$ as $\theta \to 1$ and $\varepsilon \to 0$. However, the restriction with respect to L^2_β-norm is somewhat strong when we consider the other control objectives. For example, for stability-enhancing control, the following choice will be more effective:

$$\|b_1\|_\beta = \|b_2\|_\beta = \|c_2\|_\beta = 1, \quad \|c_1\|_\beta = Q\ (\geq 1).$$

4. Numerical simulation

Let $D = 0.1$, $\alpha = 0.25$, $\beta = {}^{\alpha}/_{D} = 2.5$, $h_1 = 0.7$, $h_2 = 0.8$, $b_1(x) = e^{\frac{\beta x}{2}}/\sqrt{0.6}\,1_{[0.3,0.9]}(x)$, $b_2(x) = e^{\frac{\beta x}{2}}/\sqrt{0.1}\,1_{[0.1,0.2]}(x)$, $c_1(x) = Qe^{-\frac{\beta x}{2}}/\sqrt{0.6}\,1_{[0.1,0.7]}(x)$, and $c_2(x) = e^{-\frac{\beta x}{2}}/\sqrt{0.1}\,1_{[0.8,0.9]}(x)$, where $1_{[\cdot,\cdot]}(x)$ denotes the characteristic function, and Q is a constant larger than 1. Then, we have $\|b_1\|_\beta = \|b_2\|_\beta = \|c_2\|_\beta = 1$ and $\|c_1\|_\beta = Q$, as a result, we have $b = \max\{1, Q\} = Q$. Also, we set the initial conditions and the disturbances as follows: $z_1(0,x) \equiv 0$, $z_2(0,x) \equiv 0$, $q(0) = 0$, $\lambda(0) = 0$, $\hat{x}_2(0) = 0$, $w_1(t) = 7.5e^{-35(t-0.3)^2}$, and $w_2(t) = 0.02e^{-0.8t}\sin 40t$. In this section, let us set $z(t) = \int_0^1 c_1(x) z_1(t,x)dx$, which is the first element of the controlled output $z_c(t) \in \mathbf{R}^2$ (see (3)).

By setting $Q = 2$, we have $b = 2$. Let $\delta = 85$, $\theta = 0.5$, and $\varepsilon = 0.52$. Then, we can choose the integer l as $l = 4$. The assumptions (A1–A2) are satisfied, since

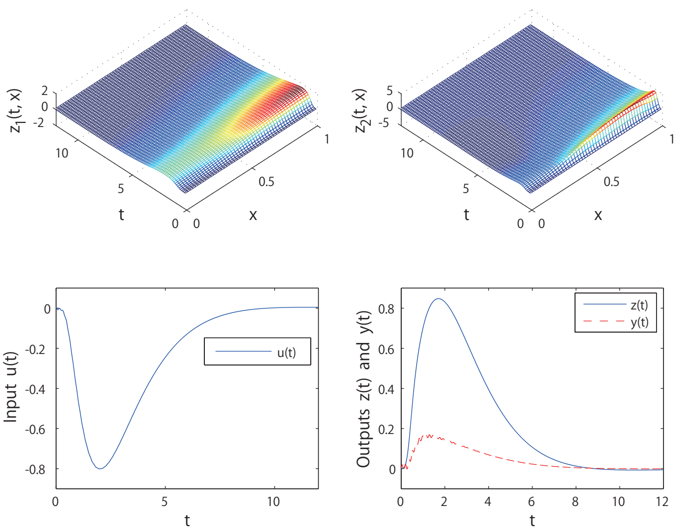

Fig. 3. Evolution of the states, input, and outputs

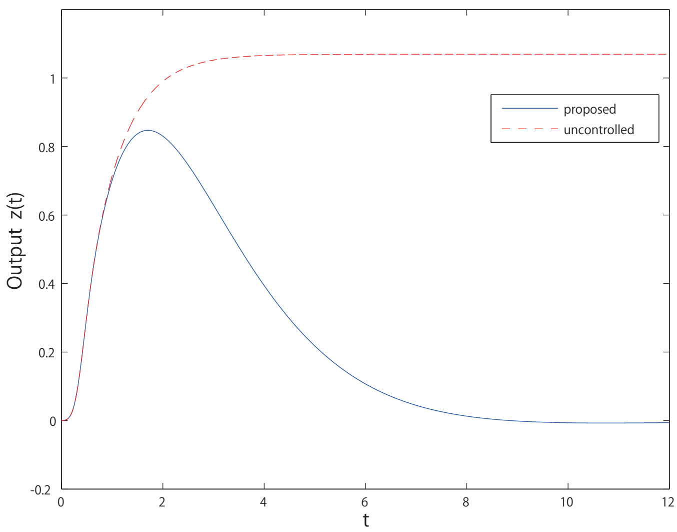

Fig. 4. Evolution of the output $z(t)$

- $(\overline{A}_1, \overline{B}_{1,1})$ is controllable and $(\overline{C}_{1,1}, \overline{A}_1)$ is observable,
- $(\overline{A}_1, \overline{B}_{2,1})$ is controllable and $(\overline{C}_{2,1}, \overline{A}_1)$ is observable.

Also, the assumptions (B1–B3) are satisfied when $\gamma = 8.97247$. In the simulation, we set $A_\Lambda = -(\frac{1}{\gamma} + 0.01) = -0.121452$, $B_\Lambda = 1$, $C_\Lambda = -1$, and $n = 10$. In this case, the open-loop system is not exponentially stable, since the spectra of the system contains zero eigenvalue (see Remark 2.2). Figs. 3 and 4 show that the system is stabilized by the proposed control law.

Finally, to compare with the other finite-dimensional controllers, we consider the case where the observer-based controller is used instead of the proposed one. The observer-based controller is described as follows:

$$\begin{cases} \dfrac{dq(t)}{dt} = (\overline{A}_1 - G_1\overline{C}_{2,1})q(t) + G_1(y(t) - \hat{y}_2(t)) \\ \qquad\qquad + \overline{B}_{2,1}u(t), \quad q(0) = q_0, \\ u(t) = -F_1 q(t), \end{cases} \tag{19}$$

$$\begin{cases} \dfrac{d\hat{x}_2(t)}{dt} = \overline{A}_2\hat{x}_2(t) + \overline{B}_{2,2}u(t), \quad \hat{x}_2(0) = \hat{x}_{20}, \\ \hat{y}_2(t) = \overline{C}_{2,2}\hat{x}_2(t). \end{cases} \tag{20}$$

If the assumption (A2) is satisfied, the closed-loop stability with system (8) is assured, if the integer n is chosen sufficiently large. The proof is the almost same as that of internal stability in Theorem 3.1. Let $l = 4$ and $n = 10$. From the setting of actuator and sensor influence functions, since $(\overline{A}_1, \overline{B}_{2,1})$ is controllable, we can choose a matrix F_1 such that a set of eigenvalues of $\overline{A}_1 - \overline{B}_{2,1}F_1$ is equal to $\{-0.5, -1, -1.5, -2, -2.5, -3, -3.5, -4\}$. Also, since $(\overline{C}_{2,1}, \overline{A}_1)$ is observable, we can choose a matrix G_1 such that a set of eigenvalues of $\overline{A}_1 - G_1\overline{C}_{2,1}$ is equal to $\{-5, -5.5, -6, -6.5, -7, -7.5, -8, -8.5\}$. The matrices F_1 and G_1 can be solved by using MATLAB Control System Toolbox.

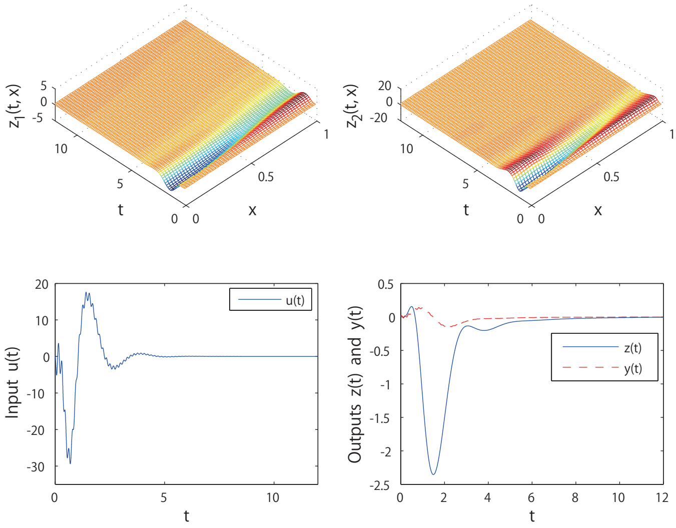

Fig. 5. Evolution of the states, input, and outputs

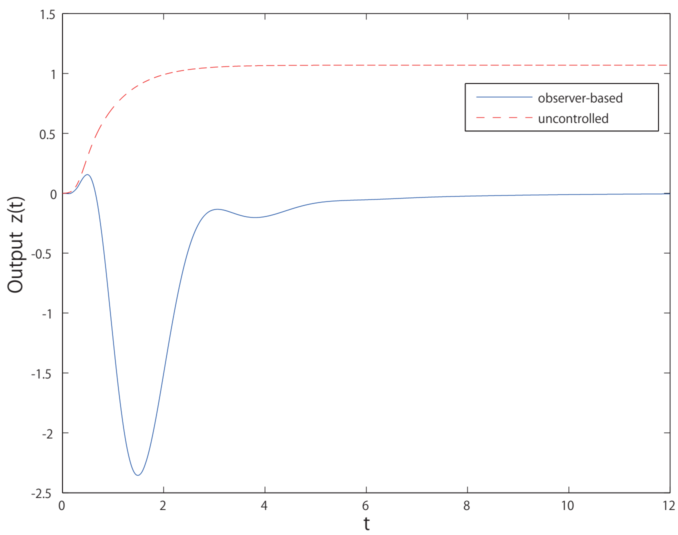

Fig. 6. Evolution of the output $z(t)$

Figs. 5 and 6 show that the stabilization of the system is fast accomplished, however, the input $u(t)$ uses much larger power than that of the proposed controller (see Fig. 3).

To solve the two linear transport-diffusion equations numerically, we used the finite difference method with mesh width $\Delta x = 0.02$, and the Runge-Kutta method of the fourth order with time step $\Delta t = 0.0001$ for its time integration. For the finite-dimensional controllers, we used the Runge-Kutta method of the fourth order with the same time step Δt.

5. Conclusions

In this paper, we treated a coupled transport-diffusion system related to parallel-flow heat exchange process, and provided a design method of finite-dimensional H_∞ controllers under distributed control and distributed observation. We could successfully apply the result of [22] to the coupled transport-diffusion system. As shown in the numerical simulation, H_∞ control is effective in the situation where the input to the system is restricted. Also, it is important to set the parameter Q contained in the influence function $c_1(x)$ suitably according to the control objective. From the practical point of view, the heat exchange processes of counter-flow type are more frequently used in industrial systems, since it is possible to make the output temperature of the heated medium much higher than that of the heating medium [27]. Also, time lag in the control input should be taken into account, since it is generally difficult to adjust heat quickly. In the future, we plan to study the similar problem for the heat exchange processes of counter-flow type as well as with input delay.

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