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Restriction Testing in Binary Choice Model with I(1) Regressors

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Abstract

This paper deals with the problem of nonstationarity of regressors in binary choice model. The limit distribution of the ML-estimator is mixed normal, but restriction testing shall not be based on standard t-statistic. The results of the conducted Monte Carlo experiment demonstrate that the true size of the restriction test is far from the significance level. Therefore, the *t*-Student statistic should be modified and this paper proposes its modification. The results of the Monte Carlo investigation point to the superiority of the new statistic.

Keywords: nonstationarity, maximum likelihood, restriction testing

JEL Classification: C12, C25.

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1 Introduction

Though binary choice models have been an important tool of microeconometrics for many years, there are many settings in which the macroeconomic outcome we seek to model is a discrete choice between two alternatives, rather than a continuous measure of some activity. For instance, a central bank decides whether or not to intervene in the foreign exchange market. Consequently, a currency intervention shall be considered a binary variable. Many papers treat currency crises and financial crises as a binary variable (crisis, tranquility period). Papers devoted to business cycle modeling distinguish between two states (recession, no recession). These macroeconomic time series often depend on time series integrated of order 1. Park and Phillips (2000) considered a binary choice model with I(1) regressors and developed asymptotic theory on such models. The authors proved that the limit distribution of the maximum likelihood estimator was mixed normal with mixing variates being dependent upon Brownian local time as well as Brownian motion. In the paper Park and Phillips (2000) it is shown that the ML estimator converges at a rate $T^{\frac{3}{4}}$ along its principal component, having a slower rate of $T^{\frac{1}{4}}$ convergence in all other directions. Guerre and Moon (2002) and Grabowski (2009) show that if all parameters in a binary choice model are equal to zero, the maximum likelihood estimator is T-consistent and asymptotically normal. The problem of nonstationary regressors in a binary choice model was considered by Grabowski (2007a, 2007b), who included I(2) time series in a binary choice model. It is well known that in the case of a binary choice model with stationary regressors the rate of convergence is $T^{\frac{1}{2}}$ and the variance of the ML estimator is equal to the inverted Fisher Information Matrix. Restriction testing in a binary choice model with stationary regressors may be based on the traditional t-statistic. If nonstationary regressors are included in the right-hand side of a binary choice model, the asymptotic distribution of the ML estimator is normal, but the standard statistic shall not be used for testing restriction $\beta_k = b$ if $b \neq 0$. This paper proposes an alternative, appropriate statistic. It shows that the true size of a restriction test differs from the theoretical size if I(1) variables are included in the binary choice model and the standard t-student statistic is used.

The paper is organized as follows. Section 2 presents asymptotic theory for binary choice models having only stationary or I(1) regressors. Section 3 discusses the results of the Monte Carlo investigation where the true size of the restriction test was compared with its theoretical size for restriction testing involving the traditional t-statistic. In Section 4, an alternative statistic is derived and its superiority is proved by means of another Monte Carlo experiment. The paper concludes with Section 5.

2 Asymptotics for the binary choice models

Consider the following binary choice model:

$$\begin{aligned} y_t^* &= x_t\beta + \xi_t, \quad \xi_t \sim F \quad t = 1, \dots, T, \\ y_t &= \mathbf{1}_{\{y_t^* \geq 0\}}, \end{aligned} \tag{1}$$

where x_t is a row vector of regressors, β is a column vector of parameters and F is a cumulative distribution function of disturbance ξ. We assume that $\beta \neq 0$. If x_t consists only of stationary variables, then the asymptotic distribution of the maximum likelihood estimator is as follows:

$$\widehat{\beta}^{ML} \sim N\left(\beta, I(\beta)^{-1}\right), \tag{2}$$

where $I(\beta) = E\left[-\frac{\partial^2 \log L(\beta|X,y)}{\partial\beta\partial\beta^T}\right]$. If $F(\cdot)$ is a cumulative distribution function of the normal distribution ($F = \Phi$), then:

$$\widehat{\beta}^{ML} \sim N\left(\beta, \left\{\sum_{t=1}^{T} \frac{\varphi(x_t\beta)^2}{(1-\Phi(x_t\beta))\,\Phi(x_t\beta)} x_t^T x_t\right\}^{-1}\right), \tag{3.a}$$

and if $F(\cdot)$ is a cumulative distribution function of the logistic distribution ($F = \Lambda$), then:

$$\widehat{\beta}^{ML} \sim N\left(\beta, \left\{\sum_{t=1}^{T} \Lambda(x_t\beta)\,(1-\Lambda(x_t\beta))\, x_t^T x_t\right\}^{-1}\right), \tag{3.b}$$

where $\Lambda = \frac{\exp(x_t\beta)}{1+\exp(x_t\beta)}$. Results (2) and (3) are standard and well-known, likewise the consequences of asymptotic theory for the maximum likelihood estimator under satisfied regularity conditions. If a researcher wants to test restriction $\beta_k = b$ within a binary choice model with stationary regressors, they can use the critical values of the standardized normal distribution for large samples or the critical value of t-student distribution in smaller samples.

When regressors are nonstationary, asymptotic distributions (3.a) and (3.b) are invalid. The asymptotic distribution of the maximum likelihood estimator of a binary choice model with I(1) regressors was derived by Park and Phillips (2000). Before presenting the results obtained by Park and Phillips (2000), let us formulate the following assumptions:

Assumption 1. *A data generating process. Process x_t is generated in the following way:*

$$x_t = x_{t-1} + \nu_t, \quad t = 1, \dots, T$$

with $P(x_0 = 0) = 1$, where:

$$\nu_t = \Pi(L)\eta_t = \sum_{i=0}^{\infty} \Pi_i \eta_{t-i},$$

with $\Pi(1) \neq 0$ *and* $\sum_{i=0}^{\infty} i\|\Pi_i\| < \infty$. *The innovations* η_t *are i.i.d. with mean zero and* $E\|\eta_t\|^r < \infty$, *for some* $r > 8$, *have distributions that are absolutely continuous with respect to the Lebesgue measure and have the characteristic function* φ, *which satisfies* $\lim_{\|t\|\to\infty} \|t\|^p \varphi_i(t) = 0$ *for some* $p > 0$.

Assumption 2. *Denote by* $F(\cdot)$ *the cumulative distribution function of* ε_t. *Define the following functions:*

$$G = \frac{\dot{F}}{F(1-F)}; \quad K = G \cdot \dot{F} = G^2 \cdot F \cdot (1-F).$$

F *is three times differentiable so that* $\dot{F}$, $\ddot{F}$, $\dot{G}$, $\ddot{G}$ *all exist.*

As in Park and Phillips (2000), the regressor space is rotated using an orthogonal matrix $H = [h_1 H_2]$, where $h_1 = \frac{\beta}{\sqrt{\beta'\beta}}$. We then define the following processes:

$$V_1 = h_1' V,$$
$$V_2 = H_2' V,$$

where V is a Brownian motion in (Ω, F, P). Define chronological local time of the process V_1, see Park and Phillips (1999) and (2001), given by:

$$L_1(t,s) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{|V_1(r)-s|<\varepsilon\}}\, dr, \tag{4}$$

which measures the sojourn time in chronological units that the process spends in the vicinity of the spatial point s. For a more extended analysis of this process see Revuz and Yor (1994). After defining the foregoing quantities, we can formulate a theorem, which gives the asymptotic distribution of the maximum likelihood estimator in a binary choice model with I(1) regressors.

Theorem 1. *Consider the estimation of the parameters of a binary choice model (1) by maximum likelihood. Let assumptions 1 and 2 hold. Then the asymptotic distribution of the ML estimator is as follows:*

$$\widehat{\beta}^{ML} \sim N\left(\beta, T^{-\frac{1}{2}} P_\beta q_{11.2}^{-1}\right), \tag{5}$$

where: $P_\beta = \beta(\beta'\beta)^{-1}\beta'$, $q_{11.2} = q_{11} - q_{12} Q_{22}^{-1} q_{12}$ *and the elements of matrix* $Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & Q_{22} \end{bmatrix}$ *are:*

$$\begin{aligned}
q_{11} &= L_1(1,0) \textstyle\int_{-\infty}^{\infty} s^2 K(\alpha s)\, ds, \\
q_{12} &= \textstyle\int_0^1 dL_1(r,0) V_2(r)' \int_{-\infty}^{\infty} sK(\alpha s)\, ds, \\
q_{21} &= \textstyle\int_0^1 V_2(r) dL_1(r,0) \int_{-\infty}^{\infty} sK(\alpha s)\, ds, \\
Q_{22} &= \textstyle\int_0^1 V_2(r) V_2(r)' dL_1(r,0) \int_{-\infty}^{\infty} K(\alpha s)\, ds, \\
\alpha &= \sqrt{\beta'\beta}
\end{aligned}$$

This theorem was formulated and proved by Park and Phillips (2000).
It can be noticed that, if the binary choice model's regressors are integrated of order 1, then the asymptotic distribution of the ML estimator is normal too. Obviously, restriction testing in smaller samples can use the t-Student distribution. But with $b \neq 0$ the t-statistic for a model with stationary regressors is different from the t-statistic for a model with I(1) regressors. To derive the appropriate statistic, we present a corollary of Park and Phillips (2000) that results from Theorem 1.

Corollary 1. *Let assume that 1 and 2 hold. If $\int_{-\infty}^{\infty} sK(s)\,ds = 0$ and $\beta \neq 0$ then*

$$T^{\frac{1}{4}}\left(\widehat{\beta}^{ML} - \beta\right) \xrightarrow{d} \frac{\beta}{\|\beta\|}\left(L_1(1,0)\int_{-\infty}^{\infty} s^2 K\left(s\|\beta\|\right)\,ds\right)^{-\frac{1}{2}} W(1), \tag{6}$$

where $\|\beta\| = \sqrt{\beta^T\beta}$ and W are a univariate standard Brownian motion independent of V.

This corollary was formulated and proved by Park and Phillips (2000).

3 Restriction testing based on a traditionally calculated t-Student statistic

Consider a binary choice model and the following hypothesis:

$$\begin{array}{ll} H_0: & \beta_k = b, \\ H_1: & \beta_k \neq b \end{array}, \tag{7}$$

where $b \neq 0$. According to asymptotic distributions (3.a), (3.b) of the maximum likelihood estimator of a binary choice model with stationary regressors, the testing of hypothesis (7) can utilize t-statistic given by the formula:

$$t_{\widehat{\beta}_k} = \frac{\widehat{\beta}_k^{ML} - b}{d_{kk}}, \tag{8}$$

where d_{kk} is the k-th diagonal element of matrix $\left\{\sum_{t=1}^{T} \frac{\varphi(x_t\widehat{\beta}^{ML})^2}{\left(1-\Phi(x_t\widehat{\beta}^{ML})\right)\Phi(x_t\widehat{\beta}^{ML})} x_t^T x_t\right\}^{-1}$ for a probit model and the k-th diagonal element of matrix $\left\{\sum_{t=1}^{T} \Lambda(x_t\widehat{\beta}^{ML})\left(1-\Lambda\left(x_t\widehat{\beta}^{ML}\right)\right) x_t^T x_t\right\}^{-1}$ in the case of a logit model. For regressors integrated of order 1 and $b = 0$, testing is also based on statistic (8).
However, the latter is inappropriate if the binary choice model (1) has nonstationary regressors. To make it clear, let us run a Monte Carlo experiment for the logit model. In our experiment, explanatory variables are generated according to the following formula:

$$x^i_{jt} = x^i_{jt-1} + \varepsilon^i_{jt},\ \ t = 1,\dots,T,\ \ x^i_{j0} = 0,\ \ \varepsilon^i_{jt} \sim N(0,1),\ \ j = 1,2 \tag{9}$$

where $i = 1, 2, \dots, I$ denotes the number of replication. An unobservable dependent variable is generated by formula:

$$y^{i*}_t = 1 + 0.001x^i_{1t} + 0.001x^i_{2t} + \xi^i_t,\ \ t = 1,\dots,T,\ \ \xi^i_t \sim \Lambda. \tag{10}$$

The observable values of variable y_t are obtained as follows:

$$y^i_t = \mathbf{1}_{\{y^{i*}_t > 0\}},\ \ t = 1,\dots,T. \tag{11}$$

In the next step of our experiment, parameters β_0, β_1, β_2 of the binary choice model:

$$y^{i*}_t = \beta_0 + \beta_1 x^i_{1t} + \beta_2 x^i_{2t} + \xi^i_t,\ \ t = 1,\dots,T,\ \ \xi^i_t \sim \Lambda \tag{12}$$

are estimated for each replication. Consider now the following statistic:

$$t_{\widehat{\beta}^i_1} = \frac{\widehat{\beta}^i_1 - 0.001}{\widehat{\sigma}_{\widehat{\beta}^i_1}}, \tag{13}$$

where $\widehat{\beta}^i_1$ is the maximum likelihood estimate of parameter β_1 for the i-th iteration. $\widehat{\sigma}_{\widehat{\beta}^i_1}$ is the estimate of the standard deviation of the maximum likelihood estimator for the i-th iteration. This estimate is a square root of the second diagonal element of the following matrix:

$$\widehat{D}^2\left(\widehat{\beta}^i\right) = \left(\sum_{t=1}^{T} \Lambda(x^i_t\widehat{\beta}^i)\left(1 - \Lambda\left(x^i_t\widehat{\beta}^i\right)\right)\left(x^i_t\right)^T x^i_t\right)^{-1}. \tag{14}$$

It is clear that if the size of the restriction test based on statistic (13) is equal to the significance level, then:

$$\lim_{I\to\infty} \frac{1}{I}\sum_{i=1}^{I} \mathbf{1}_{\left\{|t_{\widehat{\beta}^i}| > t_{1-\frac{\lambda}{2};T-1}\right\}} = \lambda, \tag{15}$$

where $t_{1-\frac{\lambda}{2};T-1}$ denotes an appropriate percentile of t-student distribution with $T-1$ degrees of freedom. We run 100 000 replications for each case and calculate - for the different numbers of observations and the different significance levels - the following quantity:

$$S(T,\lambda) = \frac{1}{100\ 000}\sum_{i=1}^{100\ 000} \mathbf{1}_{\left\{|t_{\widehat{\beta}^i}| > t_{1-\frac{\lambda}{2};T-1}\right\}}. \tag{16}$$

Quantity (16) is interpreted as the true size of the significance test (with statistic (8)), when the regressors in a binary choice model are integrated of order 1. Table

11 presents the values taken by this quantity. As most economic time series have numbers of observations varying between 20 and 300, the samples range from 20 to 300. The values are reported for the most frequently used statistical inference significance levels, namely ($\alpha = 0.1; 0.05; 0.02; 0.01$).

Table 1: Simulated quantity (16) for logit model regressors integrated of order 1 and a traditional t-statistic

		λ			
		0.1	0.05	0.02	0.01
T	20	0.071	0.019	0.000	0.000
	50	0.086	0.037	0.006	0.003
	100	0.081	0.031	0.009	0.003
	200	0.084	0.030	0.008	0.003
	300	0.086	0.032	0.010	0.004

The results show that the differences between the true size and the significance level are large, so traditional t-Student (8) shall not be used for inferring about significance in the binary choice model with I(1) regressors.

4 A new statistic for restriction testing involving binary choice model with I(1) regressors

The results of the Monte Carlo experiment have shown that the traditional t-student statistic shall not be used for verifying restriction in the binary choice models with I(1) regressors. This section of the paper proposes an alternative solution, which is based on corollary 1. According to the corollary, if we set hypothesis (7), we shall use the following statistic:

$$t_{\widehat{\beta}_k} = \frac{\widehat{\beta}_k - b}{\sqrt{\left(\sum_{t=1}^{T} K\left(x_t \widehat{\beta}^{ML}\right) x_{kt}^2\right)^{-1}}} \frac{\sqrt{\widehat{L_1}(1,0)}}{\widehat{w}_k}, \tag{17}$$

where:

$$\widehat{w}_k = \frac{\widehat{\beta}_k}{\sqrt{\sum_{i=1}^{K} \widehat{\beta}_i^2}}, \tag{18}$$

$$\widehat{L_1}(1,0) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^1 \mathbf{1}_{\{|\widehat{V_1}(r)| < \varepsilon\}}\, dr \tag{19}$$

and

$$\forall_{r\in[0,1]}\ V_1(r) = \frac{\sum_{i=1}^{K}\widehat{\beta}_i}{\sqrt{\sum_{i=1}^{K}\widehat{\beta}_i^2}}V(r). \tag{20}$$

$K(s) = \frac{\varphi(s)^2}{\Phi(s)(1-\Phi(s))}$ in the case of a probit model and $K(s) = \frac{1}{\Lambda(s)(1-\Lambda(s))}$ in the logit case. Because statistic (17) differs from statistic (8), hence with a non-zero restriction imposed on parameter β_k we shall not use statistic (8), because the obtained results would be inappropriate then.

In order to demonstrate the superiority of statistic (17) when restriction (7) is tested in a binary choice model with I(1) regressors, we shall conduct a Monte Carlo experiment to compare the true size with its significance level, when restriction (7) is tested and statistic (17) is used. Following the setup of the first Monte Carlo investigation, the samples range from 20 to 300, the data are generated according to formulas (9)-(12) and quantity (16) is computed.

Table 2: Simulated quantity (16) values for logit model regressors integrated of order 1 and statistic (17)

		λ			
		0.1	0.05	0.02	0.01
T	20	0.102	0.050	0.020	0.011
	50	0.099	0.050	0.020	0.010
	100	0.100	0.050	0.020	0.010
	200	0.100	0.050	0.020	0.010
	300	0.100	0.050	0.020	0.010

The second Monte Carlo experiment shows that testing restriction (7) in a binary choice model with I(1) regressors and using $\beta \neq 0$ and statistic (17) we obtain more credible results than for the traditional statistic. The true size and the significance level are nearly equal in most cases, proving the superiority of statistic (17).

As mentioned above, statistic (8) is appropriate when restriction $\beta_k = 0$ is verified. The same observation was made by Guerre and Moon (2002) and Grabowski (2009).

5 Conclusions

Because of the problem of non-stationary regressors appearing sometimes in the binary choice models, restriction $\beta_k = b$ has to be dealt with by means of an appropriate statistic. For $b \neq 0$, the ML estimator of the parameters of a binary choice model with I(1) regressors has a mixed normal limit distribution. Limit distribution (6) shows, however, that in a binary choice model with I(1) regressors the ML estimator converges to the parameter more slowly than in a model with stationary regressors. Therefore, a larger sample is needed.

The results of the Monte Carlo investigation show that the true size of a restriction test differs from the significance level when a traditional t-statistic is used in a binary choice model with I(1) regressors. Accordingly, this paper proposes an alternative statistic whose superiority has been proven by the second Monte Carlo investigation.

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