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Matrix Mittag-Leffler function in fractional systems and its computation

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Abstract. Matrix Mittag-Leffler functions play a key role in numerous applications related to systems with fractional dynamics. That is why the methods for computing the matrix Mittag-Leffler function are so important. The matrix Mittag-Leffler function is a generalization of matrix exponential function. This implies that some of numerous existing methods for computing the matrix exponential can be adapted for matrix Mittag-Leffler functions as well. Unfortunately, the technique of scaling and squaring, widely used in computing of the matrix exponential, cannot be applied to matrix Mittag-Leffler functions, as the latter do not possess the semigroup property. Here we describe a method of computing the matrix Mittag-Leffler function based on the Jordan canonical form representation. This method is implemented with MATLAB code [1].

Key words: matrix Mittag-Leffler function, Jordan canonical form, fractional calculus, fractional differential equation.

1. Introduction

The theory of fractional differential equations is a new and important branch of differential equation theory, which has numerous applications and provides realistic models for many real-life processes and phenomena; see [2–5]. The Mittag-Leffler function $E_{\alpha,\beta}(z)$ plays the same role in fractional differential equations that the exponential function e^z plays in ordinary differential equations.

A natural extension of the exponential function to the case of matrix arguments proved to be extremely useful in studying the linear systems of ordinary differential equations arising in engineering, mechanics, control theory etc. Similarly, matrix Mittag-Leffler function is crucial in linear systems of fractional differential equations, allowing to represent explicitly their solutions. That is why the methods for computing the matrix Mittag-Leffler function are so important.

There is a wide range of methods for computing of the matrix exponential and many of them can be adapted for computation of the matrix Mittag-Leffler function. One of the methods is based on Jordan canonical form [6, 7]. Here we apply this approach and propose a technique for computation the matrix Mittag-Leffler function, which is implemented with MATLAB code [1].

The numerical methods based on Jordan canonical form have a major disadvantage, since the involved similarity transformation can be ill-conditioned. However, the numerical experiments on the benchmark example of the Bagley–Torvik equation [8] imply that the proposed approach allows to obtain satisfactory accuracy.

2. Fractional differential equations and matrix Mittag-Leffler functions

The matrix Mittag-Leffler function was probably first introduced in paper [9], where it was used in an explicit solution of a linear system of fractional order equation (FDEs)

$$D^{\alpha}z = Az + f, \quad 0 < \alpha \leq 1. \tag{1}$$

Here $D^{\alpha}z$ stands for the Riemann–Liouville fractional derivative of order α [10]. In general, if g is a function having absolutely continuous derivatives up to the order $m-1$, the Riemann–Liouville derivative of fractional order α, $m-1 < \alpha \leq m$, can be defined as follows:

$$D^{\alpha}g(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t \frac{g(\tau)}{(t-\tau)^{\alpha-m+1}}\, d\tau. \tag{2}$$

Hereafter A is a fixed real $n \times n$ matrix, and $z, f\colon [0, \infty) \to \mathbb{R}^n$ are measurable vector-functions taking values in $\mathbb{R}^n$.

If (1) is supplied with initial condition of the form

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{z(\tau)}{(t-\tau)^{\alpha}}\, d\tau \Big|_{t=0} = z^0, \tag{3}$$

then solution to the initial value problem (1), (3) can be written down in the form

$$z(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})z^0 + \\ + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-\tau)^{\alpha}\big)f(\tau)d\tau, \tag{4}$$

where

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$$E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \quad \beta \in \mathbb{C}, \tag{5}$$

denotes the matrix Mittag-Leffler function of A.

The expression (4) can be rewritten in a more compact form

$$z(t) = e_\alpha^{At} z^0 + \int_0^t e_\alpha^{A(t-\tau)} f(\tau) d\tau, \tag{6}$$

where $e_\alpha^{At} = t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha)$ is the matrix α-exponential function introduced in the monograph [11].

Since FDEs involving the Riemann–Liouville fractional derivative require initial conditions of the form (3) lacking clear physical interpretation, the regularized fractional derivative was introduced. The latter is often referred to as the Caputo derivative and defined as follows:

$$D^{(\alpha)} g(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{g^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, \quad m-1 > \alpha \leq m. \tag{7}$$

Initial value problem for FDEs involving the Caputo derivative

$$D^{(\alpha)} z = Az + f, \quad 0 < \alpha \leq 1, \tag{8}$$

requires standard initial conditions

$$z(0) = z^0, \tag{9}$$

and its solution can be explicitly written down in terms of matrix Mittag-Leffler functions as follows [12, 13]:

$$z(t) = E_{\alpha,1}(At^\alpha) z^0 + + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-\tau)^\alpha\big) f(\tau) d\tau. \tag{10}$$

As an example let us consider the well-known Bagley–Torvik equation [8] describing vibrations of a rigid plate immersed in Newtonian liquid:

$$ay''(t) + bD^{(3/2)} y(t) + cy(t) = f(t), \tag{11}$$

$$y(0) = y_0, \quad y'(0) = y'_0. \tag{12}$$

Its analytical solution obtained with the help of fractional Green's function in terms of scalar generalized Mittag-Leffler functions is cumbersome and involves evaluation of a convolution integral, containing a Green's function expressed as an infinite sum of derivatives of Mittag-Leffler functions, and for general functions f this cannot be evaluated conveniently. Namely [14]

$$y(t) = \int_0^t G_3(t-\tau) f(\tau) d\tau,$$

where the Green function G_3 is of the form

$$G_3(t) = \frac{1}{a} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{c}{a}\right)^k t^{2k+1} E^{(k)}_{\frac{1}{2}, 2+\frac{3k}{2}} \left(-\frac{b}{a}\sqrt{t}\right).$$

On the other hand, the equation of Bagley–Torvik is equivalent to the following system [15]

$$D^{(1/2)} z = Bz + Cf, \tag{13}$$

where $B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -c/a & 0 & 0 & -b/a \end{bmatrix}$, $C = (0, 0, 0, 1/a)^T$, $z = \big(y, D^{(1/2)}y, y', D^{(3/2)}y\big)^T$ under the initial conditions

$$z(0) = z^0 = (y_0, 0, y'_0, 0)^T.$$

Its solution in terms of matrix Mittag-Leffler functions is given by the following expression:

$$z(t) = E_{\frac{1}{2},1}(B\sqrt{t}) z^0 + \int_0^t E_{\frac{1}{2},\frac{1}{2}}(B\sqrt{t-\tau}) C \frac{f(\tau)d\tau}{\sqrt{t-\tau}}, \tag{14}$$

which can be easily evaluated.

The explicit expressions (4, 6), and (10) play a key role in numerous applications related to systems with fractional dynamics (5, 16, 17). That is why the methods for computing the matrix Mittag-Leffler function are so important.

Both the matrix Mittag-Leffler function and the matrix α-exponential functions are generalizations of matrix exponential function, since

$$E_{1,1}(At) = e_1^{At} = e^{At}.$$

This implies that some of numerous existing methods for computing the matrix exponential can be adapted for the matrix Mittag-Leffler functions as well. An overview and analysis of these methods can be found in the paper [7] and in the monograph [6]. Unfortunately, the technique of scaling and squaring, widely used in computing of the matrix exponential, cannot be applied for the matrix Mittag-Leffler and α-exponential functions, as the latter do not possess the semigroup property.

Here we describe a method of computing the matrix Mittag-Leffler function based on the Jordan canonical form representation. This method is implemented with MATLAB code [1].

3. Matrix functions

There is a number of equivalent definitions of a matrix function. The following classic definition in terms of interpolation polynomials is applied according to [18]. Let

$$\psi(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_1)^{m_2} \ldots (\lambda - \lambda_s)^{m_s}$$

be the minimal polynomial of A, where $\lambda_1, \lambda_2, \ldots, \lambda_s$ are all the distinct eigenvalues of A. The degree of this polynomial is $m = \sum_{k=1}^{s} m_k$.

Let us consider a sufficiently smooth function $f(\lambda)$ of scalar argument and call the m numbers

$$f(\lambda_k), f'(\lambda_k), \ldots, f^{(m_k-1)}(\lambda_k) \qquad (k = 1, \ldots, s) \tag{15}$$

the values of the function f on the spectrum of the matrix A and the set of all these values will be denoted symbolically by $f(\Lambda_A)$. If for some function f the values (15) exist, then we will say that the function f is defined on the spectrum of the matrix A.

Definition 1. (matrix function via interpolation polynomial [18]). Let $f(\lambda)$ be a function defined on the spectrum of a matrix A and $r(\lambda)$ the corresponding interpolation polynomial such that $f(\Lambda_A) = r(\Lambda_A)$. Then

$$f(A) = r(A).$$

Let us recall the following.

Theorem 1. Any constant $n \times n$ matrix A is similar to a matrix J in Jordan canonical form. That is, there exists an invertible matrix P such that the $n \times n$ matrix $J = Z^{-1}AZ$ is in the canonical form

$$J = \operatorname{diag}\{J_1, J_2, \ldots, J_s\} \tag{16}$$

where each Jordan block matrix J_k, $k = 1, \ldots, s$, is a square matrix of the form

$$J_k = \begin{pmatrix} \lambda_k & 1 & 0 & \ldots & 0 \\ 0 & \lambda_k & 1 & \ldots & 0 \\ 0 & 0 & \lambda_k & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & \lambda_k \end{pmatrix}.$$

It is shown (see e.g. [18]) that Definition 1 is equivalent to the following definition based on the Jordan canonical form. We will use the latter for computing the matrix Mittag-Leffler function.

Definition 2. (matrix function via Jordan canonical form). Let the function f be defined on the spectrum of A and let $A = ZJZ^{-1}$, where J is the Jordan canonical form (16). Then

$$\begin{aligned} f(A) &= Zf(J)Z^{-1} = \\ &= Z\operatorname{diag}\{f(J_1), f(J_2), \ldots, f(J_s)\}Z^{-1}, \end{aligned} \tag{17}$$

where

$$f(J_k) = \begin{pmatrix} f(\lambda_k) & f'(\lambda_k) & \frac{f''(\lambda_k)}{2} & \ldots & \frac{f^{(m_k-1)}(\lambda_k)}{(m_k-1)!} \\ 0 & f(\lambda_k) & f'(\lambda_k) & \ldots & \frac{f^{(m_k-2)}(\lambda_k)}{(m_k-2)!} \\ 0 & 0 & f(\lambda_k) & \ldots & \frac{f^{(m_k-3)}(\lambda_k)}{(m_k-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & f'(\lambda_k) \\ 0 & 0 & 0 & \ldots & f(\lambda_k) \end{pmatrix}. \tag{18}$$

3.1. Generalized Mittag-Leffler functions. The generalized (scalar) Mittag-Leffler function also known as Prabhakar function is defined for complex $z, \alpha, \beta, \rho \in \mathbb{C}$, and $\Re(\alpha) > 0$ by

$$E^{\rho}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_k}{\Gamma(\alpha k + \beta)} \frac{z_k}{k!}, \tag{19}$$

where $(\rho)_k = \rho(\rho+1)\ldots(\rho+k-1)$ is the Pochhammer symbol.

In particular, when $\rho = 1$, it coincides with the Mittag-Leffler function (5):

$$E^{1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z).$$

Since the expression (18) involves derivatives, the following equation [11] is important for the purpose of computing the matrix Mittag-Leffler function:

$$\left(\frac{d}{dt}\right)^m E_{\alpha,\beta}(t) = m!E^{m+1}_{\alpha,\beta+\alpha m}(t), \qquad m \in \mathbb{N}. \tag{20}$$

In view of (20), the formulas (17, 18) take on the form

$$\begin{aligned} &E_{\alpha,\beta}(A) = \\ &= Z\operatorname{diag}\{E_{\alpha,\beta}(J_1),\ E_{\alpha,\beta}(J_2), \ldots, E_{\alpha,\beta}(J_s)\}Z^{-1}, \end{aligned} \tag{21}$$

$$f(J_k) = \\ = \begin{pmatrix} E_{\alpha,\beta}(\lambda_k) & E^{2}_{\alpha,\beta+\alpha}(\lambda_k) & \ldots & E^{m_k}_{\alpha,\beta+(m_k-1)\alpha}(\lambda_k) \\ 0 & E_{\alpha,\beta}(\lambda_k) & \ldots & E^{m_k-1}_{\alpha,\beta+(m_k-2)\alpha}(\lambda_k) \\ 0 & 0 & \ldots & E^{m_k-2}_{\alpha,\beta+(m_k-3)\alpha}(\lambda_k) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & E^{2}_{\alpha,\beta+\alpha}(\lambda_k) \\ 0 & 0 & \ldots & E_{\alpha,\beta}(\lambda_k) \end{pmatrix}. \tag{22}$$

4. Software implementation

The formulas (21, 22) can be used for computing the matrix Mittag-Leffler function and were implemented in the form of MATLAB routine `mlfm.m` [1]. For computing generalized Mittag-Leffler functions of the form $E^{m}_{\alpha,\beta+(m-1)\alpha}(\lambda_k)$, the MATLAB routine by R. Garrappa is used, which implements the optimal parabolic contour (OPC) algorithm described in [19] and based on the inversion of the Laplace transform on a parabolic contour suitably chosen in one of the regions of analyticity of the Laplace transform.

To verify the accuracy of the `mlfm.m` routine, one can consider the Bagley–Torvik equation (11, 12).

Figure 1 shows analytical solution to (11, 12) obtained by direct evaluation of the expression (14) using the `mlfm.m` routine. For comparison, Fig. 2 represents numerical solution derived using discretization on the basis of triangular strip matrices [20].

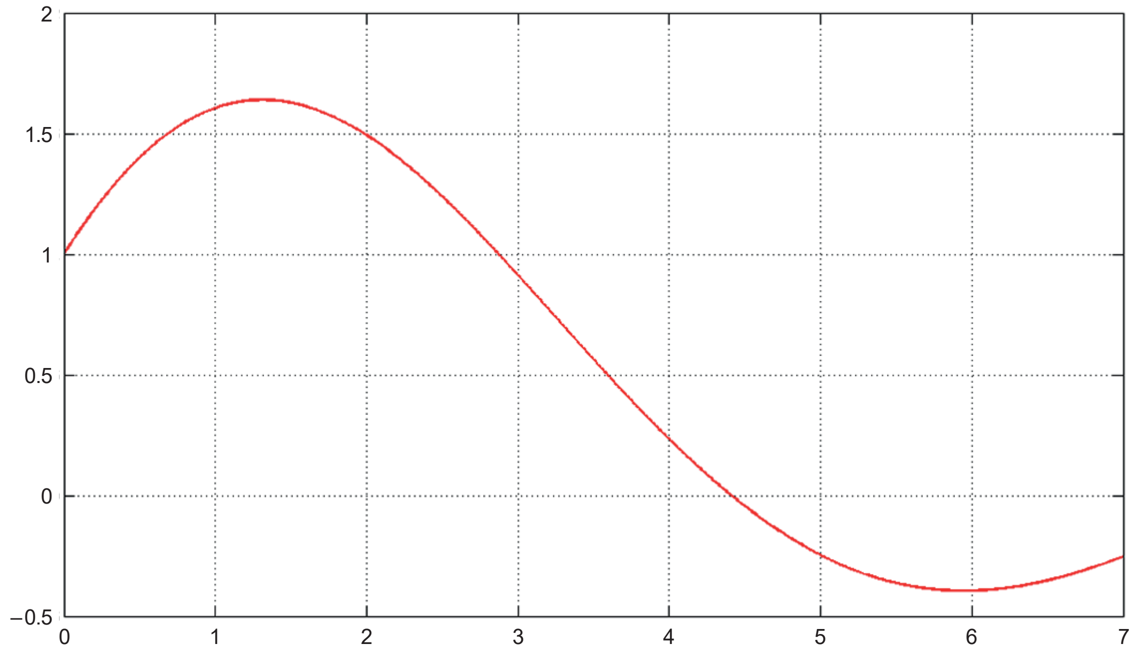

Fig. 1. Analytical solution using the `mlfm.m` routine

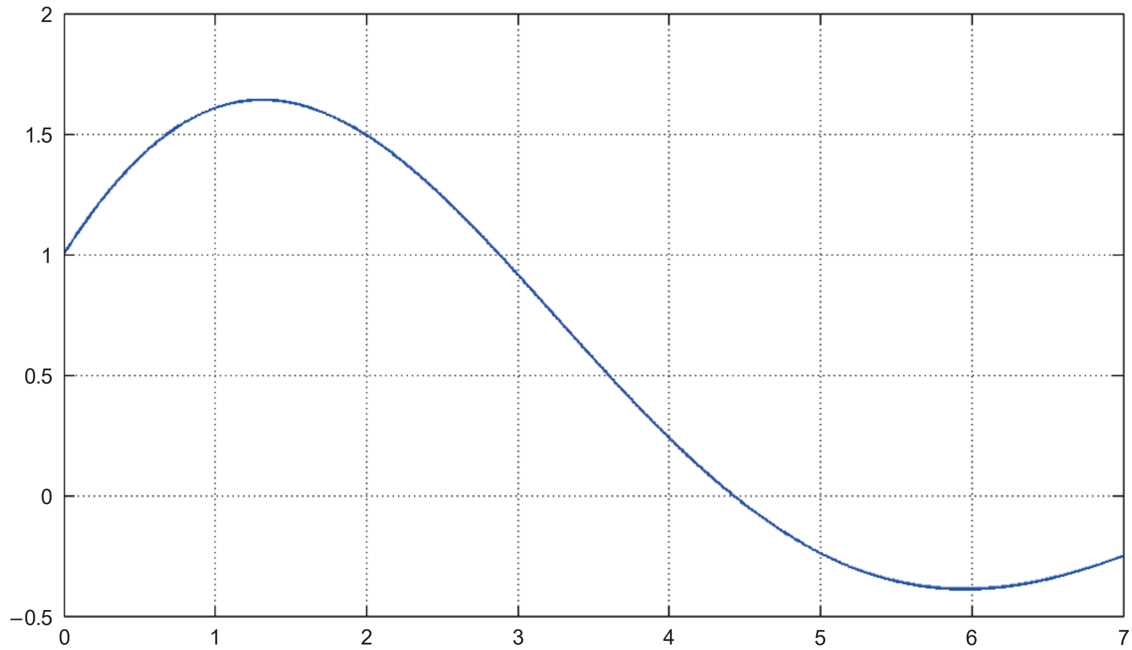

Fig. 2. Numerical solution

If $c = 0$, the matrix Mittag-Leffler functions appearing in (14) can be found analytically.

Indeed, if $c = 0$ the matrix B in (13) takes on the form

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & p \end{bmatrix}, \tag{23}$$

where $p = -b/a$.

Hence,

$$B^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & p \\ 0 & 0 & 0 & p^2 \end{pmatrix}, \quad B^k = \begin{pmatrix} 0 & 0 & 0 & p^{k-3} \\ 0 & 0 & 0 & p^{k-2} \\ 0 & 0 & 0 & p^{k-1} \\ 0 & 0 & 0 & p^k \end{pmatrix}, \quad k = 3, 4, \ldots$$

Therefore,

$$E_{\frac{1}{2},1}(B) = \sum_{k=0}^{\infty} \frac{B^k}{\Gamma(k/2+1)} =$$

$$= \begin{pmatrix} 1 & \frac{1}{\Gamma(3/2)} & 1 & p^{-3}E_{\frac{1}{2},1}(p) - p^{-3} - \frac{1}{p^2\Gamma(3/2)} - \frac{1}{p} \\ 0 & 0 & \frac{1}{\Gamma(3/2)} & p^{-2}E_{\frac{1}{2},1}(p) - p^{-2} - \frac{1}{p\Gamma(3/2)} \\ 0 & 0 & 1 & p^{-1}E_{\frac{1}{2},1}(p) - p^{-1} \\ 0 & 0 & 0 & E_{\frac{1}{2},1}(p) \end{pmatrix},$$

$$E_{\frac{1}{2},\frac{1}{2}}(B) = \sum_{k=0}^{\infty} \frac{B^k}{\Gamma((k+1)/2)} =$$

$$= \begin{pmatrix} \frac{1}{\Gamma(1/2)} & 1 & \frac{1}{\Gamma(3/2)} & p^{-3}E_{\frac{1}{2},\frac{1}{2}}(p) - \sum_{i=1}^{3} \frac{1}{p^i\Gamma\left(\frac{4-i}{2}\right)} \\ 0 & \frac{1}{\Gamma(1/2)} & 1 & p^{-2}E_{\frac{1}{2},\frac{1}{2}}(p) - \frac{1}{p^2\Gamma(1/2)} - \frac{1}{p} \\ 0 & 0 & \frac{1}{\Gamma(1/2)} & p^{-1}E_{\frac{1}{2},\frac{1}{2}}(p) - \frac{1}{p\Gamma(1/2)} \\ 0 & 0 & 0 & E_{\frac{1}{2},\frac{1}{2}}(p) \end{pmatrix}.$$

And taking into account the following properties of gamma-function and scalar Mittag-Leffler function

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2},$$

$$E_{\frac{1}{2},1}(z) = e^{z^2}\operatorname{erfc}(-z),$$

$$E_{\frac{1}{2},\frac{1}{2}}(z) = zE_{\frac{1}{2},1}(z) + \frac{1}{\Gamma(1/2)} = ze^{z^2}\operatorname{erfc}(-z) + \frac{1}{\sqrt{\pi}},$$

we arrive at the following explicit expressions

$$E_{\frac{1}{2},1}(B) =$$

$$= \begin{pmatrix} 1 & 2\sqrt{\frac{1}{\pi}} & 1 & \frac{e^{p^2}}{p^3}\operatorname{erfc}(-p) - \frac{1}{p^3} - \frac{2}{p^2}\sqrt{\frac{1}{\pi}} - \frac{1}{p} \\ 0 & 1 & 2\sqrt{\frac{1}{\pi}} & \frac{e^{p^2}}{p^2}\operatorname{erfc}(-p) - \frac{1}{p^2} - \frac{2}{p}\sqrt{\frac{1}{\pi}} \\ 0 & 0 & 1 & \frac{e^{p^2}}{p}\operatorname{erfc}(-p) - \frac{1}{p} \\ 0 & 0 & 0 & e^{p^2}\operatorname{erfc}(-p) \end{pmatrix}. \tag{24}$$

$$E_{\frac{1}{2},\frac{1}{2}}(B) = \begin{pmatrix} \frac{1}{\sqrt{\pi}} & 1 & \frac{2}{\sqrt{\pi}} & \frac{e^{p^2}}{p^2}\operatorname{erfc}(-p) - \frac{1}{p^2} - \frac{2}{p\sqrt{\pi}} \\ 0 & \frac{1}{\sqrt{\pi}} & 1 & \frac{e^{p^2}}{p}\operatorname{erfc}(-p) - \frac{1}{p} \\ 0 & 0 & \frac{1}{\sqrt{\pi}} & e^{p^2}\operatorname{erfc}(-p) \\ 0 & 0 & 0 & pe^{p^2}\operatorname{erfc}(-p) + \frac{1}{\sqrt{\pi}} \end{pmatrix}. \tag{25}$$

Here erfc stands for the complementary error function, an entire function defined by

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2} d\tau.$$

Applying the `mlfm.m` routine to the matrix (23) and then comparing the result to the reference matrices (24, 25) implies that the proposed approach ensures absolute error which does not exceed 10^{-15} for this particular case.

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