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Ideal observability for bilinear discrete-time systems with and without delays in observation

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An ideal observability subspace expression is stated for bilinear abstract system with bounded operator in Hilbert spaces. The case of finite dimentional space is also treated. However, it's noticed that the state ideal observability can never be fulfilled within an infinite dimensional phase space in the case of scalar output. The case of bilinear discrete-time system with delays in observation is also described. To illustrate this work some examples are presented.

Key words: bilinear systems, Hilbert spaces, ideal observability, observability subspace, delayed observation

1. Introduction

In control theory, observability is how well the states of a system can be deduct from knowledge of its outputs. The concept of observability was introduced by Kalman, R. E. in [10] for linear dynamic systems.

The ideal terminology was initially introduced by [15] who defined the finite dimensional precess as an ideally observable system if and only if its initial state $x(0)$ can be determined only from the output $y(.)$. This ideal concept was revisited afterward by [12] and [9], in [12] the notion of relative ideal observability is concerned with the determination of the whole trajectory $x(.)$, the criterion giving the rebuilding of $x(0)$ of an autonomous system is enough to largely recover all the state trajectory $x(.)$ of such a system.

The extension to Hilbert spaces of the finite dimensional approach is developed in [4]. In [1] the characterization of the Lack ideal observability is given and the ideal observability subspace is described as the intersection of a family of Kalman observability subspaces. A criteria of ideal observability and conditional

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ideal observability of linear stationary regular differential-algebraic systems are proved in [14].

Bilinear systems was introduced into control theory in the 1960s. This type of system is simpler and better understood than most other nonlinear systems. Among the examples of bilinear systems found in the control of industrial processes we cite a switched circuits, mechanical brakes, controlled suspension systems and in biology such as population growth, immunological systems, enzymatic kinetics, etc.

The observability of bilinear systems was tackled in many works among which we can cite [19] and [8]. The necessary and sufficient conditions to achieve the property of observability for bilinear systems have been established using essentially geometric or algebraic tools. Many studies deal with the observability of bilinear systems using generally either geometric tools or linear time-varying system theory [6, 7] and [17].

Discrete systems is one of the most important fields in the theory of systems. However, it seems that the study of ideal observability for such systems was neglected and hence their applicability is severely limited. We suggest in this paper to develop the ideal observability concept for discrete-time bilinear system described by

$$\begin{cases} x_{i+1} = Ax_i + B_0 f_i + \sum_{j=1}^{p} e_i^j B_j x_i, & 0 \leqslant i \leqslant N-1, \\ x_0 \end{cases} \tag{1}$$

the corresponding output is

$$y_i = Cx_i, \qquad 0 \leqslant i \leqslant N, \tag{2}$$

where $x_i \in X$ is the state of system (1) and $f_i \in U$, $e_i = (e_i^j)_{1 \leqslant j \leqslant p} \in \mathbb{R}^p$ are unknown perturbations which affect the system because of it's connection with his environment, $y_i \in Y$ is the output variable (X, U and Y are Hilbert spaces). Moreover we suppose that $A, B_j \in \mathscr{L}(X)$, $B_0 \in \mathscr{L}(U, X)$, and $C \in \mathscr{L}(X, Y)$ where $\mathscr{L}(E, F)$ is the bounded linear operator spaces defined from E to F and $\mathscr{L}(E) = \mathscr{L}(E, E)$.

The notation (S) will design the observed system (1), (2) and we write (S_0) instead of (S) in the particular case where $X = \mathbb{R}^n$, $U = \mathbb{R}^m$, and $Y = \mathbb{R}^k$.

Ideal observability problem consists in collect the maximum of information on the system trajectory in the state space X where the parameters A, B_0, $B_1, \ldots$, B_p, C and the output y_i, $i \in \{0, \ldots, N\}$ are knows while the perturbations f_i, e_i and the initial state x_0 are unknowns.

Then we devote this paper to the construction of the maximal subspaces E_0, E_1, $\ldots$, E_N of X such that we can identify the projection of the state x_i of system (S) on E_i when $i \in \{0, \ldots, N\}$.

The determination of the ideal observability spaces $E_0,\ E_1,\ \ldots,\ E_N$ is processed in the section 3. In the section 4 we give the necessary and sufficient conditions under which the systems S and S_0 are ideally observable, the notion of G-ideal observability is also treated. To illustrate this work some examples are presented in section 5. In section 6 we described the case of bilinear discrete-time system with delays in observation. Finally a conclusion is summarized in section 7.

2. Preliminary

In this paragraph we define the ideal observability subspaces, the ideal observability and the G-ideal observability.

Definition 1 *The ideal observability subspaces associated to system (S) and denoted by $E_0,\ E_1,\ \ldots,\ E_N$ are the maximal closed subspaces, in the senses of the inclusion, such that the orthogonal projection of the state x_i on E_i can be recognized for each $i \in \{0,\ldots,N\}$ with the help of the observation $(y_i)_{0\leqslant i\leqslant N}$ in presence of the perturbation $(f_i,e_i)_{1\leqslant i\leqslant N}$.*

Definition 2 *The system (S) will be told ideally observable if we can determine the state x_i, $i \in \{0,\ldots,N\}$ from the observation $(y_i)_i$. In the other term (S) is ideally observable if $E_0 = E_1 = \ldots = E_N = X$ where $E_0,\ E_1,\ \ldots,\ E_N$ are the ideal observability subspaces.*

Definition 3 *If $G \in \mathscr{L}(X,Z)$ where Z is a Hilbert space, the system (S) is said to be G-ideally observable if we can determine the vector $Gx_i \in Z$ for each $i \in \{0,\ldots,N\}$ from the output $(y_i)_i$ and in the presence of the perturbation $(f_i,e_i)_{1\leqslant i\leqslant N}$.*

Remark 1

i) *If (S) is ideally observable, $B_0 = 0$ and $B_1 = \ldots = B_p = 0$ then it is obvious that (S) is observable in the sense of Kalman;*

ii) *It is clear that if $G = id_X$, we have the equivalence*

$$(S)\ \text{ is G-ideally observable } \Longleftrightarrow\ (S)\ \text{ is ideally observable.}$$

3. Ideal observability subspace

Now we present a series of lemmas which will be used in the sequel

Lemma 1 *Let V and W Hilbert spaces, v an element of V and $D \in \mathscr{L}(V,W)$. The two following propositions are equivalent* *a) we know the vector Dv;*

b) we know the orthogonal projection of the vector v on the subspace $\overline{ran\,D^*}$, where $\overline{ran\,D^*}$ is the closure of range of D^* and D^* is the adjoint of D.

Proof. see [2]. □

Lemma 2 *If we denote $Z_i \in \mathscr{L}(X, Y^{2^i})$, for $i \in \{0, \dots, N\}$ the family of operators given by*

$$\begin{cases} Z_0 = C, \\ Z_{i+1} = \begin{bmatrix} Z_i \\ Q_{i+1} Z_i A \end{bmatrix}, \quad i \in \{0, \dots, N-1\}, \end{cases}$$

where Q_i is the orthogonal projection operator on the subspace $(ran(Z_{i-1}B_0) \cup ran(Z_{i-1}B_1) \cup \dots \cup ran(Z_{i-1}B_p))^{\perp}$ we have

$$z_j(i) = Z_j x_i, \qquad i \in \{0, 1, \dots, N-j\}, \tag{3}$$

where z_j is the map defined from $\{0, 1, \dots, N-j\}$ to Y^{2^i} by

$$\begin{cases} z_0(i) = y_i, & \forall i \in \{0, 1, \dots, N\}, \\ z_{j+1}(i) = \begin{bmatrix} z_j(i) \\ Q_{j+1} z_j(i+1) \end{bmatrix}, & j \in \{0, \dots, N-1\}. \end{cases}$$

Proof. It is clear that property (3), cited in lemma (2), is verified for $j = 0$. Let's suppose that

$$z_j(i) = Z_j x_i, \qquad \forall i \in \{0, \dots, N-j\} \text{ for every } j \in \{0, \dots, N-1\} \tag{4}$$

and prove that $z_{j+1}(i) = Z_{j+1} x_i$, $i \in \{0, 1, \dots, N-j-1\}$ where

$$z_j(i) = \begin{bmatrix} z_{j-1}(i) \\ Q_j z_{j-1}(i+1) \end{bmatrix}, \quad \forall i \in \{0, 1, \dots, N-j\} \text{ and } Z_j = \begin{bmatrix} Z_{j-1} \\ Q_j Z_{j-1} A \end{bmatrix}.$$

Using equations (4) and (1) we have

$$z_j(i+1) = Z_j x_{i+1} = Z_j A x_i + Z_j B_0 f_i + \sum_{j=1}^{p} e_i^j Z_j B_j x_i, \qquad \forall i \in \{0, \dots, N-j-1\}$$

and by projecting on the subspace $(ran(Z_j B_0) \cup ran(Z_j B_1) \cup \dots \cup ran(Z_j B_p))^{\perp}$ we obtain

$$Q_{j+1} z_j(i+1) = Q_{j+1} Z_j A x_i, \qquad i \in \{0, \dots, N-j-1\} \tag{5}$$

which implies that

$$z_{j+1}(i) = \begin{bmatrix} z_j(i) \\ Q_{j+1}z_j(i+1) \end{bmatrix} = \begin{bmatrix} Z_jx_i \\ Q_{j+1}Z_jAx_i \end{bmatrix} = Z_{j+1}x_i.$$

Then we deduce that

$$z_{j+1} = Z_{j+1}x_i, \qquad \forall i \in \{0,\dots,N-j-1\},$$

consequently we have $z_j = Z_jx_i$, $i \in \{0,\dots,N-j\}$. □

It is easy to establish that the adjoint operator Z_j^* of Z_j, defined from Y^{2^j} to X, are given by

$$\begin{cases} Z_0^* &= C^*, \\ Z_{j+1}^* &= [Z_j^*,\, A^*Z_j^*Q_{j+1}^*] : \quad Y^{2^{j+1}} \longrightarrow X,\ \forall j \in \{0,\dots,N-1\}, \\ & \qquad\qquad\qquad\qquad\quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \longrightarrow Z_j^*y_1 + A^*Z_j^*Q_{j+1}^*y_2\,. \end{cases}$$

The next result, gives a characterization of $(ran Z_j^*)_{0\leqslant j\leqslant N}$.

Lemma 3 *Under the above assumption, we have*

$$ran Z_{j+1}^* = ran Z_0^* + A^*(ran Z_j^* \cap Ker B_0^* \cap Ker B_1^* \cap \ldots \cap Ker B_p^*).$$

Proof. Since the operator Q_{j+1} is self-adjoint, we have

$$ran\, Q_{j+1} = ran\, Q_{j+1}^* = (ran(Z_jB_0) \cup ran(Z_jB_1) \cup \ldots \cup ran(Z_jB_p))^{\perp}$$

so

$$\begin{aligned} ran Z_j^*Q_{j+1}^* &= Z_j^*[ran\, Q_{j+1}^*] \\ &= Z_j^*[(ran(Z_jB_0) \cup ran(Z_jB_1) \cup \ldots \cup ran(Z_jB_p))^{\perp}] \\ &= ran Z_j^* \cap Ker B_0^* \cap Ker B_1^* \cap \ldots \cap Ker B_p^* \end{aligned}$$

then

$$\begin{aligned} ran Z_{j+1}^* &= ran Z_j^* + ran A^*Z_j^*Q_{j+1}^* \\ &= ran Z_j^* + A^*(ran Z_j^* \cap Ker B_0^* \cap Ker B_1^* \cap \ldots \cap Ker B_p^*). \end{aligned}$$

Consequently, it is easy to see that

$$ran Z_{j+1}^* = ran Z_0^* + \sum_{i=0}^{j} A^*(ran Z_i^* \cap Ker B_0^* \cap Ker B_1^* \cap \ldots \cap Ker B_p^*). \qquad (6)$$

On the other hand the equality $ranZ_{j+1}^* = ranZ_j^* + ranA^*Z_j^*Q_{j+1}^*$ implies that $ranZ_j^* \subset ranZ_{j+1}^*$. Consequently, for all $i \in \{0,\ldots,N\}$, we have the following inclusion

$$A^*(ranZ_{i-1}^* \cap KerB_0^* \cap KerB_1^* \cap \ldots \cap KerB_p^*) \subset A^*(ranZ_i^* \cap KerB_0^* \cap KerB_1^* \cap \ldots \cap KerB_p^*) \qquad (7)$$

combining (6) and (7), we deduce that we have

$$ranZ_{j+1}^* = ranZ_0^* + A^*(ranZ_j^* \cap KerB_0^* \cap KerB_1^* \cap \ldots \cap KerB_p^*). \qquad \square$$

The results developed in lemma 2 and lemma 3 allow us to state the following theorem.

Theorem 1 *The ideal observability subspaces, $E_0, E_1, \ldots, E_N$, are given by*

$$E_i = \overline{X}_{N-i}, \qquad 0 \leqslant i \leqslant N,$$

where

$$X_0 = ranC^*, \qquad X_i = X_0 + A^*(X_{i-1} \cap KerB_0^* \cap KerB_1^* \cap \ldots \cap KerB_p^*), \quad 0 \leqslant i \leqslant N.$$

Proof. Since $X_0 = ranC^* = ranZ_0^*$, we deduce from the equalities

$$\begin{cases} ranZ_{j+1}^* = ranZ_0^* + A^*(ranZ_j^* \cap KerB_0^* \cap KerB_1^* \cap \ldots \cap KerB_p^*) \\ \text{and} \\ X_i = X_0 + A^*(X_{i-1} \cap KerB_0^* \cap KerB_1^* \cap \ldots \cap KerB_p^*), \quad 0 \leqslant i \leqslant N \end{cases}$$

that

$$X_j = ranZ_j^*, \qquad \forall j \in \{0, 1, \ldots, N\}. \qquad (8)$$

On the other hand, according to lemma 1 and lemma 2 the equation (2) allows to determine the projection $P_j x_i$ of x_i on the subspace $\overline{ranZ_j^*}$ when $i \in \{0,\ldots,N-j\}$ and $j \in \{0,\ldots,N\}$.

Then the ideal observability subspace E_i on which the orthogonal projection of x_i is completely determined, is given by

$$E_i = \bigcup_{0 \leqslant j \leqslant N-i} \overline{ranZ_j^*}$$

i.e., $E_i = \bigcup_{0 \leqslant j \leqslant N-i} \overline{ranZ_j^*}$. As it is obvious to see that $X_0 \subset X_1 \subset \ldots \subset X_N$ we deduce that

$$E_i = \overline{X_{N-i}} \qquad \text{for} \quad i \in \{0,\ldots,N\}.$$

On the other hand, the subspaces E_i are maximal in the sense of definition 1. Indeed, if we suppose that exist $i \in \mathbb{N}$ such that E_i is not maximal knowing that the solution x_i can be under the forme $x_i = x_1(i) + x_2(i)$, where $x_1(i) \in E_i$ and $x_2(i) \in E_i^{\perp}$, the fact that E_i is not maximal implies exists a subspace G that contains E_i, such that $x_2(i) = y_1(i) + y_2(i)$, $y_2(i) \neq 0$ where $x_1(i) + y_1(i)$ is the orthogonal projection of x_i on the subspace G and which is recognized with the help of the observation $(y_i)_{0 \leqslant i \leqslant N}$ and in presence of the perturbation $(f_i, e_i)_{1 \leqslant i \leqslant N}$. Thus $x_1(i) + y_1(i)$ is known with the help of equation (3) and we deduce that it is known with the help of system

$$z_{N-i}(i) = Z_{N-i} x_i.$$

Then, by lemma 1 we have

$$x_1(i) + y_1(i) = \overline{Z}_{N-i}^{-1} z_{N-i},$$

where $\overline{Z}_{N-i}$ is the restriction of Z_{N-i} on $Ker Z_{N-i}$ which is invertible on the left. On the other hand, we have

$$z_{N-i}(i) = Z_{N-i} x_i = Z_{N-i} x_1(i)$$

because $x_2(i) \in E_i^{\perp}$, then

$$x_1(i) + y_1(i) = \overline{Z}_{N-i}^{-1} z_{N-i} = \overline{Z}_{N-i}^{-1} Z_{N-i} x_1(i) = x_1(i)$$

which implies that $y_1(i) = 0$, that contradicted the hypothesis. □

Remark 2 *For the case of linear systems with $B_0 = B$ and $B_i = 0$ for all $i \in \{1, \ldots, p\}$, the ideal observability spaces are given by $E_i = \overline{X}_{N-i}$, $i \in \{0, \ldots, N\}$ where $X_0 = ran\, C^*$ and $X_i = X_0 + A^*(X_{i-1} \cap Ker\, B^*)$, $i \in \{0, \ldots, N\}$.*

4. Ideal observability criteria

We obtain in this paragraph a criterion of ideal observability, and a result concerning finite dimension spaces.

Corollary 1 *The system (S) is ideally observable if and only if $X = \overline{ran\, C^*}$.*

Proof. The system (S) is ideally observable if and only if $E_0 = E_1 = \ldots = E_N = X$, as $X_0 \subset X_1 \subset \ldots \subset X_N$ and $E_i = X_{N-i}$ for $i \in \{0, \ldots, N\}$ then $E_N \subset E_{N-1} \subset \ldots \subset E_1$. Consequently the system (S) is ideally observable if and only if $X = E_N = \overline{X_0} = \overline{ran\, C^*}$. □

Remark 3 *We see that the ideal observability depend only of the injection of the operator C.*

We can give a necessary and sufficient conditions to establish the G-ideally observable

Corollary 2 *Let $G \in \mathcal{L}(X,Z)$, where Z is a Hilbert space. The system (S) is G-ideally observable if and only if $ran\, G^* \subset \overline{ran\, C^*}$.*

Proof. If the system is G-ideally observable, the output $(y_i)_i$ allows to identify Gx_i, then by virtue of lemma 1, we know the projection of x_i on $\overline{ran\, G^*}$, for all $i \in \{0,\dots,N\}$. Since E_i are the maximal subspaces such that a projection of x_i on E_i can be identified with the help of output $(y_i)_i$, we has therefore $ran\, G^* \subset E_i$ for all $i \in \{0,\dots,N\}$ what implies that $ran\, G^* \subset \bigcap_{i=0}^{N} E_i = E_N$. Consequently $\overline{ran\, G^*} \subset \overline{ran\, C}$.

Conversely, if $ran\, G^* \subset \overline{ran\, C^*}$, then the projection of x_i on $\overline{ran\, G^*}$ is known for all $i \in \{0,\dots,N\}$ which implies that Gx_i is known. □

Remark 4 *If the space X is infinite dimensional, separable and $Y = \mathbb{R}$, the system (S) will not be ideally observable because otherwise we will have $X = ran\, C^* = lin(h)$, (where $lin(h)$ is the linear envelope of h and $h \in X$ is the vectorial representation of the function $C : X \longrightarrow \mathbb{R}$ deriving of the Riesz representation theorem), and this contradicted the made hypothesis assumption. Thus, if the output is scalar and the system is submitted to a non null perturbation, there can be ideally observable.*

In the finite case, we determine the ideal observability subspaces in the following corollary

Corollary 3 *The ideal observability subspaces $E_0, E_1, \dots, E_N$ associate to the system (S_0) are given by*

i) If $N \leqslant n-1$ then, $E_i = X_{N-i}$, $\forall i \in \{0,\dots,N\}$,
where $X_0 = ran\, C^*$
and $X_i = X_0 + A^*(X_{i-1} \cap Ker\, B_0^* \cap Ker\, B_1^* \cap \dots \cap Ker\, B_p^*)$, $i \in \{0,\dots,N\}$;

ii) If $N > n-1$ then, $E_i = \begin{cases} X_{N-i} & \forall i \in \{N-n+2,\dots,N\}, \\ X_{n-1} & \forall i \in \{0,\dots,N-n+1\}. \end{cases}$

Proof. The space X being finished dimension, we has $E_i = X_{N-i}$, $0 \leqslant i \leqslant N$, on the other hand, we see that if $X_i = X_{i+1}$ for some i, then $X_{i+j} = X_i$ for all integer j.

Then seen that $dimX = n$ and $rangC \geqslant 1$, we has necessarily $X_{n-1+j} = X_{n-1}$, $j \in \mathbb{N}$, otherwise $X_j = X_{n-1}$, $j \geqslant n-1$. Consequently we deduces that

If $N \leqslant n-1$ we have, $E_i = X_{N-i}, \quad \forall i \in \{0,\dots,N\}$.

If $N > n-1$ we have, $E_i = \begin{cases} X_{N-i} & \forall i \in \{N-n+2,\dots,N\}, \\ X_{n-1} & \forall i \in \{0,\dots,N-n+1\}. \end{cases}$ □

In the separable phase spaces case, the ideal propriety allows the reconstruction of the state x_i in the form of Fourier series decomposition according to an orthonormal base, as shown by the following example.

Examples

1) We consider the system

$$\begin{cases} x_{i+1} = Ax_i + B_0 f_i + e_i B_1 x_i, & 0 \leqslant i \leqslant N-1 \\ x_0 \end{cases} \tag{9}$$

the corresponding output is

$$y_i = Cx_i, \qquad 0 \leqslant i \leqslant N \tag{10}$$

with the following parameter: $X = l_2 = \left\{ x = (x_i)_{i=1}^{\infty},\ x_i \in \mathbb{R},\ \sum_{i=1}^{\infty} x_i^2 < \infty \right\}$; the operators A and B_1 are defined by: $A : x \to Ax = (x_2, x_3, \dots)$; $B_1 : x \to B_1 x = (x_1, 0, 0, \dots)$. C and B_0 are given under the matrix form

$$C = \begin{bmatrix} 1 & 0 & 0 & \dots & \dots \\ 0 & 1 & 0 & \dots & \dots \end{bmatrix}, \qquad B_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \vdots \end{bmatrix}.$$

Let $(e_i)_{i=1}^{\infty}$ the canonical basis of l_2, we have then $Ae_1 = 0$ and $Ae_i = e_{i-1}$, $i = 2, 3, \dots$ a simple calculation allows to obtain from analogous expression for A^* : $A^* e_i = e_{i+1}$, for $i = 1, 2, \dots$, and $B_1^* = B_1$. Otherwise, $ran B_0 = lin(e_1)$ and $Ker\ B_1^* = Ker\ B_0^* = \overline{lin}(e_2, e_3, \dots)$. The calculate of the H^* allows to obtain $ran\, C^* = lin(e_1, e_2)$.

We deduce

$$\begin{aligned} X_0 &= ran\, C^* = lin(e_1, e_2) \\ X_1 &= X_0 + A^*(X_0 \cap Ker\ B_0^* \cap Ker\, B_1^*) = lin(e_1, e_2, e_3); \end{aligned}$$

and the same manner we have

$$X_i = lin(e_j, \;\; j = 1,2,\ldots,i+2), \qquad i = 0,\ldots,N.$$

Consequently it is clear that

$$\begin{aligned} E_N &= lin(e_1,e_2), \\ E_i &= lin(e_j, \; j = 1,\ldots,N-i+2), \qquad i = 0,\ldots,N-1. \end{aligned}$$

The system considered is not ideally observable, in this case we can give the expression of the $N-i+2$ first component of x_i in l_2 for $i \in \{0,\ldots,N\}$.
Let $y_1(i)$ and $y_2(i)$ the component of the vector y_i, we have then

$$\langle e_1,x_i\rangle = y_1(i)$$

and

$$\langle e_j,x_i\rangle = y_2(i+j-2) \quad \text{where } i \in \{0,\ldots,N-j+2\} \quad \text{and} \quad j \in \{2,\ldots,N\}.$$

2) If we consider now the operator C described by

$$\begin{aligned} C \; : \; l_2 &\longrightarrow \qquad l_2 \\ x &\longrightarrow Cx = ((Cx)_i)_{i\geqslant 1} \end{aligned}$$

where $(Cx)_1 = x_1$ and $(Cx)_i = x_i + x_{i-1}$.
In this case we deduce that $ranC^* = l_2$ and consequently the system (S) is ideally observable. We can in this case give the explicit expression of x_i according to y_i

$$\langle e_k,x_i\rangle = \sum_{j=1}^{k}(-1)^{k-j} < e_j,y_i >, \quad 0 \leqslant i \leqslant N, \quad k \geqslant 1$$

consequently the vector x_i allows in the form

$$x_i = \sum_{k=1}^{\infty}\sum_{j=1}^{k}(-1)^{k-j} < e_j,y_i > e_k, \quad 0 \leqslant i \leqslant N.$$

5. Discrete system with delays in observation

We suggest to develop the concept of ideal observability for a system defined by equation

$$\begin{cases} x_{i+1} = Ax_i + B_0 f_i + \sum_{j=1}^{q} e_i^j B_j x_i, \qquad 0 \leqslant i \leqslant N-1, \\ x_0, \end{cases} \tag{11}$$

where A, $B_j \in \mathscr{L}(X)$ for $j \in \{1,\dots,q\}$, $B_0 \in \mathscr{L}(U,X)$, $e_i = (e_i^j)_{1\leqslant j\leqslant q} \in \mathbb{R}^q$ and $f_i \in U$ are the unknown perturbations, X and U are the Hilbert spaces. The output is defined by

$$\begin{cases} y_i = \sum_{j=0}^{p} C_j x_{i-j}, & 0 \leqslant i \leqslant N, \\ x_r = e_r = 0, & \text{pour } \; r < 0. \end{cases} \tag{12}$$

$y_i \in Y$, $C_j \in \mathscr{L}(X,Y)$, $\forall j \in \{0,\dots,p\}$ and Y is a Hilbert space. We consider operators $(Q_{i,j})_{0\leqslant i\leqslant N}$ for $0 \leqslant j \leqslant p$ defined by

for $i = 0$,

$$\begin{cases} Q_{0,0} & \text{is the identity function on } \; Y, \\ Q_{0,j} & \text{is the projection on } \; (ran\, C_0 \cup \ldots \cup ran\, C_{j-1})^{\perp}, \quad 1 \leqslant j \leqslant p; \end{cases}$$

for $1 \leqslant i \leqslant N-p$,

$$\begin{cases} Q_{i,0} & \text{the orthogonal projection operator on } (ran\, C_1 \cup \ldots \cup ran\, C_i)^{\perp}, \\ Q_{i,k} & \text{projection on } (ran\, C_0 \cup \ldots \cup ran\, C_{k-1} \cup ran\, C_{k+1} \cup \ldots \cup ran\, C_{i+k})^{\perp}, \\ & \hfill \text{for } 1 \leqslant k \leqslant p-i, \\ Q_{i,k} & \text{projection on } (ran\, C_0 \cup \ldots \cup ran\, C_{k-1} \cup ran\, C_{k+1} \cup \ldots \cup ran\, C_p)^{\perp}, \\ & \hfill \text{for } p-i+1 \leqslant k \leqslant p-1, \\ Q_{i,p} & \text{the orthogonal projection operator on } (ran\, C_0 \cup \ldots \cup ran\, C_{p-1})^{\perp}; \end{cases}$$

for $N-p+1 \leqslant i \leqslant N-1$,

$$\begin{cases} Q_{i,0} & \text{the orthogonal projection operator on } (ran\, C_1 \cup \ldots \cup ran\, C_p)^{\perp}, \\ Q_{i,k} & \text{projection on } (ran\, C_0 \cup \ldots \cup ran\, C_{k-1} \cup ran\, C_{k+1} \cup \ldots \cup ran\, C_p)^{\perp}, \\ & \hfill \text{for } 1 \leqslant k \leqslant N-i; \end{cases}$$

for $i = N$,

$$Q_{N,0} \text{ is the orthogonal projection operator on } (ran\, C_1 \cup \ldots \cup ran\, C_p)^{\perp}.$$

We applies for $i \in \{0,\dots,N-p\}$ the operator $Q_{i,j}$ on two member of the $(i+j)$-equation of the equation (12) for $j \in \{0,\dots,p\}$, and similarly we applies for $i \in \{N-p+1,\dots,N\}$ the operator $Q_{i,j}$ on the two member of the $(i+j)$-equation (12) for $j \in \{0,\dots,N-i\}$, we obtains then the following equation

$$z_{i,0} = K_{i,0} x_i\,, \qquad 0 \leqslant i \leqslant N, \tag{13}$$

where $(z_{i,0})_i$ and $(K_{i,0})_i$ are described by

$$z_{i,0} = \begin{bmatrix} Q_{i,0} y_i \\ Q_{i,1} y_{i+1} \\ \vdots \\ Q_{i,p} y_{i+p} \end{bmatrix}, \qquad K_{i,0} = \begin{bmatrix} Q_{i,0} C_0 \\ Q_{i,1} C_1 \\ \vdots \\ Q_{i,p} C_p \end{bmatrix} \quad \text{for } \; i \in \{0,\dots,N-p\}, \tag{14}$$

and

$$z_{i,0} = \begin{bmatrix} Q_{i,0}y_i \\ \vdots \\ Q_{i,N-i}y_N \end{bmatrix}, \qquad K_{i,0} = \begin{bmatrix} Q_{i,0}C_0 \\ \vdots \\ Q_{i,N-i}C_{N-i} \end{bmatrix} \quad \text{for } i \in \{N-p+1,\dots,N\}.$$

In the same way as lemmas (2) and (3), were proved, we obtain two following lemmas

Lemma 4 *If we consider the operator $z_{i,0}$ and $K_{i,0}$ defined by equation (14) for $i \in \{0,\dots,N\}$, we define the family of operators*

$$K_{i,j} = \begin{bmatrix} K_{i,j-1} \\ W_{i,j}K_{i+1,j-1}A \end{bmatrix}, \quad i \in \{1,\dots,N\},$$

where $W_{i,j+1}$ is the orthogonal projection operator on the subspace $(ran(K_{i+1,j}B_0) \cup ran(K_{i+1,j}B_1) \cup \dots \cup ran(K_{i+1,j}B_q))^{\perp}$ we have

$$z_{i,j} = K_{i,j}x_i, \qquad i \in \{0,\dots,N-j\} \tag{15}$$

where z_j is the signal given by

$$z_{i,j} = \begin{bmatrix} z_{i,j-1} \\ W_{i,j}z_{i+1,j-1} \end{bmatrix}, \quad j \in \{1,\dots,N\}.$$

Lemma 5 *Under the above assumption, we have*

$$ran\,K_{i,j}^* = ran\,K_{i,0}^* + A^*(ran\,K_{i+1,j-1}^* \cap Ker\,B_0^* \cap Ker\,B_1^* \cap \dots \cap Ker\,B_q^*).$$

The determination of ideal observability subspace $E_0, E_1, \dots, E_N$ what allows identify the trajectory x_i of system (S) will be given in the following theorem.

Theorem 2 *The observability subspaces associate to (S) are given by the following expression*

$$E_i = \overline{X}_{i,N-i}, \qquad 0 \leqslant i \leqslant N$$

where for $j \in \{1,\dots,N\}$ and $i \in \{0,\dots,N-j\}$ we have

$$X_{i,j} = X_{i,0} + A^*(X_{i+1,j-1} \cap Ker\,B_0^* \cap Ker\,B_1^* \cap \dots \cap Ker\,B_q^*),$$

and

$$X_{i,0} = ran\,K_{i,0}^*, \qquad \forall i \in \{0,\dots,N\}.$$

Proof. According to the lemma 4, the equation (13) allows to determine the projection $P_j x_i$ of x_i on the subspace $\overline{X_{i,j}} = \overline{ran\, K^*_{i,j}}$ when $i \in \{0,\ldots,N-j\}$ and $j \in \{0,\ldots,N\}$.

Thus by lemma 5, we can easily verified that the subspace $X_{i,j}$ verifies for $1 \leqslant j \leqslant N,\ 0 \leqslant i \leqslant N-j$

$$X_{i,j} = X_{i,0} + A^*(X_{i+1,j-1} \cap Ker\, B^*_0 \cap Ker\, B^*_1 \cap \ldots \cap Ker\, B^*_q),$$

$$X_{i,0} = ran\, K^*_{i,0}.$$

Then we deduce that the subspace E_i on which we can determine complitely the projection of x_i is defined by

$$E_i = \bigcup_{0 \leqslant j \leqslant N-i} \overline{X_{i,j}}$$

and as the $(X_{i,j})_j$ are fit together then

$$E_i = \overline{X_{i,N-i}} \qquad \text{pour} \quad i \in \{0,\ldots,N\}.$$

On the other hand, the subspaces E_i are maximal in the sense of definition 1. Indeed, suppose that exist $i \in \{0,\ldots,N\}$ such that E_i is not maximal knowing that the solution x_i can be in the form $x_i = x_1(i) + x_2(i)$ where $x_1(i) \in E_i$ and $x_2(i) \in E_i^\perp$. The supposition that we have made implies that the componant $x_2(i)$ affect the output $(y_i)_{i \leqslant j \leqslant i+j}$ is $i \in \{0,\ldots,N-p\}$, and $(y_i)_{i \leqslant j \leqslant N}$ if $j \in \{N-p+1,\ldots,N\}$, then affect $z_{i,0}$, and that it can be identified on a part of $E_i^\perp$. As we can easily to see that $E_i^\perp$ is contained in $Ker\ K^*_{i,0}$ thus $z_{i,0} = K_{i,0}x_i = K_{i,0}x_1(i)$. □

Example We consider the system (9) with the output described by

$$y_i = Cx_{i-1}, \qquad 0 \leqslant i \leqslant N.$$

The orthogonal projection operators are defined by

$i = 0$

$$\begin{cases} Q_{0,0} & \text{is the identity function on } Y \\ Q_{0,1} & \text{is the identity function on } Y \end{cases}$$

$1 \leqslant i \leqslant N-1$

$$\begin{cases} Q_{i,0} & \text{the ortrhogonal projection operator on } (ran\, C)^\perp = Ker\, C^* \\ Q_{i,1} & \text{is the identity function on } Y \end{cases}$$

$i = N$

$$\{\, Q_{N,0} \ \ \text{the orthogonal projection operator on } (ran\, C)^\perp = Ker\, C^*$$

then we deduce that

$$\begin{cases} K_{i,0} &= \begin{bmatrix} Q_{i,0}C \\ Q_{i,1}C \end{bmatrix} = \begin{bmatrix} 0 \\ Q_{i,1}C \end{bmatrix} \qquad \forall i \in \{0,\dots,N-1\}, \\ K_{N,0} &= Q_{N,0}C, \end{cases}$$

then

$$ran\, K^*_{i,0} = ran\, C^* = lin(e_1, e_2), \quad i \in \{0,\dots,N-1\},$$
$$ran\, K^*_{N,0} = \{0\},$$

then we obtain

$$X_{i,0} = \begin{cases} K_{i,0} = lin(e_1,e_2), \\ K_{N,0} = \{0\}; \end{cases}$$

and

$$\begin{aligned} X_{i,1} &= X_{i,0} + A^*(X_{i+1,0} \cap Ker\, B^*_0 \cap Ker\, B^*_1), \quad i \in \{0,\dots,N-1\}, \\ &= \begin{cases} lin(e_1,e_2) + A^*(lin(e_1,e_2) \cap lin(e_2,e_3,\dots)), & i \in \{0,\dots,N-2\}, \\ lin(e_1,e_2), \end{cases} \\ &= \begin{cases} lin(e_1,e_2,e_3), & i \in \{0,\dots,N-2\}, \\ lin(e_1,e_2) \end{cases} \end{aligned}$$

and the same way we deduce that for $i \in \{1,\dots,N\}$ we have

$$X_{i,j} = \begin{cases} lin(e_1,e_2,\dots,e_{j+2}), & i \in \{0,\dots,N-j-1\}, \\ lin(e_1,e_2,\dots,e_{j+1}), & i \in \{0,\dots,N-j\} \end{cases}$$

it is easily to see that

$$\begin{cases} E_N &= \{0\}, \\ E_i &= \overline{lin}(e_j,\ j = 1,1,\dots,N-i+2), \qquad i = 0,\dots,N-1. \end{cases}$$

We can give the expression of $N-i+2$ first component of x_i in l_2 for $i \in \{0,\dots,N\}$.

Indeed let $y_1(i)$ the vector components of y_i, then we have

$$\langle e_1, x_i \rangle = y_1(i)$$

and

$$\langle e_j, x_i \rangle = y_2(i+j-2) \qquad \text{where } \ i \in \{0,\dots,N-j+2\} \quad \text{and} \quad j \in \{2,\dots,N-1\}.$$

6. Conclusion

In this work, we give an explicit expression of the ideal observability subspaces. A criterion of ideal observability is established and a result concerning finite dimension spaces are also given. In the separate spaces case, the property of ideal observability allows the rebuilding of the state in the form of a decomposition in Fourier series along an orthonormal basis. The case of bilinear discrete-time with delays in observation is also described. Finally some examples are presented to illustrate this work.

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