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Dynamical properties of Metzler systems

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Abstract. Spectral properties of nonnegative and Metzler matrices are considered. The conditions for existence of Metzler spectrum in dynamical systems have been established. An electric RL and GC ladder-network is presented as an example of dynamical Metzler system. The suitable conditions for parameters of these electrical networks are formulated. Numerical calculations were done in MATLAB.

Key words: Metzler matrices, ladder and ring electrical networks, nonnegative matrices, positive linear systems.

1. Introduction

Analysis of properties of positive matrices was considered by many authors [1–3]. These matrices are connected with positive discrete dynamical systems. Metzler matrices are connected with positive continuous-time dynamical systems. The essence of positive dynamical systems resides in spectrum of the state matrices. The form of state matrices depends on the basis of co-ordinate system. Depending on the basis we obtain suitable positive-cone systems [4].

The paper is organized as follows: in Section 2 and 3 spectral properties of nonnegative and Metzler matrices are considered. In Section 4 Metzler dynamical systems is investigated. The conditions for positivity of electrical ladder network of RL and GC type are established in Section 5. Concluding remarks are given in Section 6.

2. Spectral properties of nonnegative matrix

The real matrix $A = [a_{ij}] \in R^{n\times n}$ is nonnegative matrix if and only if $a_{ij} \geq 0$. If A is nonnegative matrix we will write $A \geq 0$. Vector x with nonnegative real components is called nonnegative vector. In this case we will write $x \geq 0$. If $a_{ij} > 0$, then matrix $A = [a_{ij}]$ is positive and we will write $A > 0$.

Let $\lambda(A)$ be the spectrum of the square matrix A. Let $\lambda_i(A) \in \lambda(A)$ be an eigenvalue of A. Denote by $\rho(A) = \max_i |\lambda_i(A)|$ the spectral radius of matrix A and denote by $\alpha(A) = \max_i \mathrm{Re}\lambda_i(A)$ the growth constant of A.

Remark 1. Let $\rho(A) > 0$. For any square real matrix $A \geq 0$ there exists a real number $\lambda_{\max}(A) \in \lambda(A)$ such that $\lambda_{\max} = \rho(A)$. The eigenvalue $\lambda_{\max} = \rho$ is a simple root of characteristic polynomial of matrix $A \geq 0$ with eigenvector $w > 0$. If matrix $A \geq 0$ has k eigenvalues $\lambda_{\max} = \rho$, $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}$ such that $|\lambda_i| = \rho$, then these eigenvalues are various roots of equation $\lambda^k - \rho^k = 0$. See Theorem of Frobenius [1].

Spectral properties of nonnegative square matrix are defined by roots of characteristic polynomial. For $A \geq 0$ (see Remark 1) we have polynomial in the following form:

$$W(\lambda, A) = \det[\lambda I - A] = (\lambda - \lambda_{\max})f(\lambda) \quad \text{and} \quad f(\lambda_{\max}) \neq 0, \tag{1}$$

where roots λ_i of polynomial $f(\lambda)$ satisfy the condition

$$\begin{cases} |\lambda_i| \leq \rho = \lambda_{\max} \\ |\lambda_k| = \rho \ \text{ are diffrent roots and } \ \lambda_k \neq \lambda_{\max} \end{cases} \tag{2}$$

For $A = [a_{ij}] \in R^{n\times n}$ degree of polynomial (1) is n. The conditions (1) and (2) define the spectrum $\lambda(A)$ of nonnegative square matrix A (see Fig. 1 for $n = 6$).

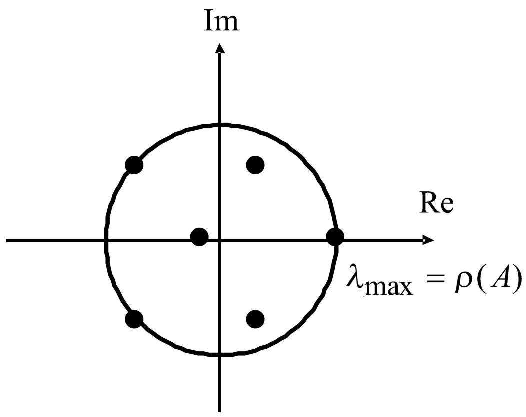

Fig. 1. Spectrum of nonnegative matrix

Now consider the following problem. Polynomial $\tilde{W}(\lambda)$ is given. We look for nonnegative matrix A such that $W(\lambda, A) = \tilde{W}(\lambda)$. Solution of this problem is shown in the following example.

Example 1. Following roots of polynomial $\tilde{W}(\lambda) = \lambda^3 - 0.3333\lambda^2 - 0.1953\lambda - 0.4714$ are given

$$\lambda_{\max} = 1, \quad \lambda_{1,2} = -0.3333 \pm 0.6002\,\mathrm{j}, \quad \mathrm{j}^2 = -1, \quad |\lambda_{1,2}| = 0.4714. \tag{3}$$

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It is clear, that $W(\lambda, F) = \tilde{W}(\lambda)$, where

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.4714 & 0.1953 & 0.3333 \end{bmatrix} \geq 0 \qquad (4)$$

is nonnegative Frobenius matrix. For eigenvalue $\lambda_{\max} = 1 = \rho(F)$ we have eigenvector $w = [0.5774 \;\; 0.5774 \;\; 0.5774]^T > 0$.

Example 2. Consider a discrete-time linear system described by the equations

$$\begin{array}{ll} x(k+1) = Ax(k), \quad x(0) \in R^n \;\Leftrightarrow & x(i) = A^i x(0) \\ k = 0,\, 1,\, 2,\, \ldots & i = 0,\, 1,\, 2,\, \ldots \end{array} \qquad (5)$$

If A is nonnegative matrix, then equations (5) generate nonnegative dynamical system. If $A \geq 0$ and $x(0) \geq 0$ then $x(i) \geq 0$ for $i = 0,\ 1,\ 2,\ \ldots$. System (5) is asymptotically stable if and only if $\rho(A) \in [0,\ 1)$.

Remark 2. Let $\eta \in R$ and $A = [a_{ij}] \geq 0$. Real number η is greater than eigenvalue $\lambda_{\max} = \rho(A)$ of nonnegative matrix A ($\lambda_{\max}(A) < \eta$) if and only if all principal minors of the matrix $\eta I - A$ are greater than zero, i.e. $M_i[\eta I - A] > 0$, $i = 1,\ 2,\ \ldots\ n$, where $M_1[\eta I - A] = \eta - a_{11}$, $M_2[\eta I - A] = \det \begin{bmatrix} \eta - a_{11} & -a_{12} \\ -a_{21} & \eta - a_{22} \end{bmatrix}$, $\ldots$, $M_n[\eta I - A] = \det[\eta I - A]$ (see [1, p. 349]).

3. Spectral properties of Metzler matrix

The real square matrix $M = [m_{ij}] \in R^{n \times n}$ is called the Metzler matrix if its all off-diagonal entries are nonnegative, i.e. $m_{ij} \geq 0, \quad i \neq j$.

Remark 3. Every nonnegative matrix is the Metzler matrix. Let $\gamma = \min_i m_{ii}$. For every Metzler matrix M there exists a real number $\eta \geq \gamma$ such that matrix $\eta I + M = A$ is nonnegative matrix. Let $s \in \lambda(A)$. Then $s - \eta = \lambda \in \lambda(M)$. Thus spectrum $\lambda(A)$ is copy of spectrum $\lambda(M)$ and is shifted (see Fig. 2) [5].

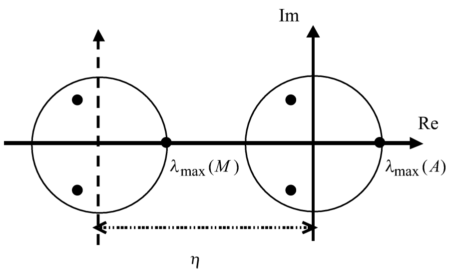

Fig. 2. Spectrum of matrices A and M

We consider real square $n \times n$ Frobenius matrix $F \geq 0$. The characteristic polynomial of $F \geq 0$ is given by the following equation:

$$W(s, F) = \det[sI - F] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \quad (6)$$

where $a_i \leq 0$ for $i = 0,\ 1,\ 2,\ \ldots\ ,\ n-1$.

Let $s_{\max} = \rho(F)$. Let $s = \lambda + \eta$, where $\eta > s_{\max}$. Thus from polynomial (6) we obtain new polynomial of variable λ

$$\tilde{W}(\lambda) = \lambda^n + b_{n-1}\lambda^{n-1} + \ldots + b_1\lambda + b_0. \qquad (7)$$

It is obvious that $M = F - \eta I$ is the Metzler matrix such that $W(\lambda, M) = \det[\lambda I - M] = \tilde{W}(\lambda)$ and additionally M is asymptotically stable, i.e. $\mathrm{Re}\lambda_i(M) < 0$, $i = 1,\ 2,\ 3,\ \ldots,\ n$.

Example 3. Consider the characteristic polynomial of positive Frobenius matrix F given in Eq. (4),

$$W(s, F) = s^3 - 0.3333s^2 - 0.1953s - 0.4714. \qquad (8)$$

Let $s = \lambda + \eta$, where $\eta = 1.1 > \rho(F) = 1$. Thus we have

$$\tilde{W}(\lambda) = \lambda^3 + 2.9667\,\lambda^2 + 2.7014\,\lambda + 0.2415. \qquad (9)$$

The matrix $M = F - \eta I$,

$$M = \begin{bmatrix} -1.1 & 1 & 0 \\ 0 & -1.1 & 1 \\ 0.4714 & 0.1953 & -0.7667 \end{bmatrix} \qquad (10)$$

is the Metzler matrix such that $W(\lambda, M) = \det[\lambda I - M] = \tilde{W}(\lambda)$ and $\lambda_{\max} = -0.1$, $\lambda_{1,2} = -1.4334 \pm 0.6002\,j$. For the eigenvalue $\lambda_{\max} = -0.1$ we have the eigenvector $w = [0.5774 \;\; 0.5774 \;\; 0.5774]^T > 0$.

4. Metzler dynamical systems

Let us consider a continuous-time linear system described by the equations

$$\begin{array}{rl} \dot{x}(t) = A\,x(t), & x(0) \in R^n \\ t \geq 0 \quad \Leftrightarrow & x(t) = e^{A\,t}x(0). \end{array} \qquad (11)$$

If A is the Metzler matrix, then equations (11) generate the Metzler dynamical system. It has been shown [2, 6] that $e^{At} \geq 0$ if and only if $A \in R^{n \times n}$ is the Metzler matrix. If A is the Metzler matrix and $x(0) \geq 0$ then $x(t) \geq 0$ for $t \geq 0$ (see [6]).

Denote by $\alpha(A) = \max_i \mathrm{Re}\lambda_i(A)$ the growth constant of A. For any Metzler matrix M there exists a real number $\lambda_{\max} \in \lambda(M)$ such that $\lambda_{\max} = \alpha(M)$, where $\alpha(M) = \max_i \mathrm{Re}\lambda_i(M)$, $i = 1, 2, \ldots, n$, is the growth constant of M (see [9]). The number $\lambda_{\max}(M)$ is a single eigenvalue of the Metzler matrix M (see Fig. 2) and $\alpha(M) = \lambda_{\max}(M)$.

The Metzler matrix M is asymptotically stable if and only if $\alpha(M) < 0$. Let

$$\begin{aligned} W(\lambda, M) &= \det[\lambda I - M] \\ &= \lambda^n + b_{n-1}\lambda^{n-1} + \ldots + b_1\lambda + b_0. \end{aligned} \qquad (12)$$

In this case $\alpha(M) < 0$ if and only if $b_i > 0$ for $i = 0,\ 1,\ 2,\ \ldots,\ n-1$ [7]; see also [5].

It is well know that the system (11) is asymptotically stable if and only if $\mathrm{Re}\lambda_i(A) < 0$, $i = 1, \ldots, n$. When A is the Metzler matrix then $\alpha(A) = \lambda_{\max}(A)$ and $\mathrm{Re}\lambda_i(A) < 0$ if and only if $M_i[-A] > 0$, $i = 1,\ 2,\ \ldots,\ n$ (see Remark 2 with $\eta = 0$). We can notice that $M_i[-A] > 0$, $i = 1, 2, \ldots, n$ if and only if $M_1[A] < 0$, $M_2[A] > 0$, $\ldots$, $(-1)^n M_n[A] > 0$ [1]; see also [8].

Consider linear transformation of variables $x(t) \in R^n$ given by following equations with matrix P such that $\det P \neq 0$

$$x(t) = Pz(t), \qquad z(t) = P^{-1}x(t). \tag{13}$$

Thus from (11) we obtain

$$\dot{z}(t) = P^{-1}APz(t). \tag{14}$$

It is clear that the spectrum $\lambda(P^{-1}AP) = \lambda(A)$. If A is the Metzler matrix then the system (11) or (14) has spectrum of the Metzler matrix. Structure of the matrix $P^{-1}AP$ depends on chosen base of space R^n (columns of matrix P).

Example 4. Consider the electrical network shown in Fig. 3. Parameters of the network $R > 0$, L and $C > 0$ are known. The system shown in Fig. 3 is described by equation

$$LC\,\ddot{x}_1(t) + RC\,\dot{x}_1(t) + x_1(t) = 0. \tag{15}$$

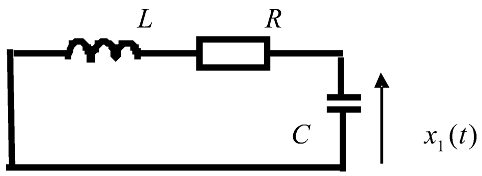

Fig. 3. Electrical network of RLC type

If $R > 2\sqrt{L/C}$ then the system (15) has a spectrum of the Metzler matrix. Let $x(t) = [x_1(t)\ x_2(t)]^T$. From (15) we obtain equation (11) with

$$A = \begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix}. \tag{16}$$

Let $L = 1$, $C = 1/6$, $R = 5$ and

$$T = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix},$$
$$Q = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}, \quad \det Q = 1, \quad Q^{-1} = Q^T. \tag{17}$$

Thus $T^{-1}AT = diag\,(-2, -3)$ and

$$M = Q^{-1}T^{-1}ATQ = \begin{bmatrix} \cos^2\varphi - 3 & -0.5\sin 2\varphi \\ -0.5\sin 2\varphi & -\cos^2\varphi - 2 \end{bmatrix}. \tag{18}$$

If $\sin 2\varphi \leq 0$, then M given in (18) is the Metzler matrix. Consequently M is the Metzler matrix for $\varphi \in \left[-\frac{\pi}{2} + \pi\right] + k\pi$, $k = 0, \pm 1, \pm 2, \ldots$ (see [9]).

5. Electrical ladder network of RL and GC type

We consider the electrical ladder network of the RL and GC type shown in Fig. 4. The parameters of the network $L > 0$, $G > 0$, $R > 0$ and $C > 0$ are known [10, 11], see also [12].

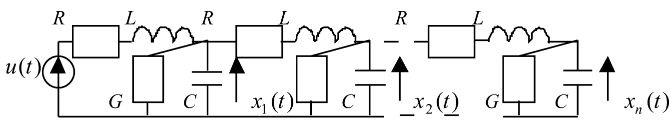

Fig. 4. Electric ladder network of RL and GC type

Let $u(t) = 0$. The eigenvalues of ladder network we can obtain from following equation (see [11] and [10])

$$LC\lambda^2 + (LG + RC)\lambda + RG + 2(1 - \cos\varphi_k) = 0, \tag{19}$$

where

$$\varphi_k = \frac{2k-1}{2n+1}\pi, \quad k = 1, 2, \ldots, n. \tag{20}$$

If $|LG - RC| > 2\sqrt{2\,LC}(1 - \cos\varphi_n)$, then ladder network has a spectrum of the Metzler matrix.

Let $u(t) = x_n(t)$. In this case the electrical network shown in Fig. 4 is the ring network of RL and GC type [11]. The eigenvalues of the ring network we obtain from (19) for

$$\varphi_k = \frac{2k}{n}\pi, \quad k = 1, 2, \ldots, n. \tag{21}$$

If $|LG - RC| > 2\sqrt{2\,LC}(1 - \cos\varphi_{[n/2]})$, then the ring network has a spectrum of the Metzler matrix.

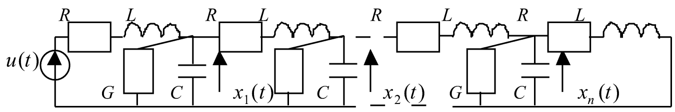

Fig. 5. Ladder network of RL and GC type

The eigenvalues of the ladder network shown in Fig. 5 with $u(t) = 0$ we can obtain from equation (19), where

$$\varphi_k = \frac{k}{n+1}\pi, \quad k = 1, 2, \ldots, n. \tag{22}$$

If $|LG - RC| > 2\sqrt{2\,LC}(1 - \cos\varphi_n)$, then the ladder network shown in Fig. 5 has a spectrum of the Metzler matrix.

6. Concluding remarks

In this paper a spectral properties (see (2) and Remark 3) of finite dimensional continuous-time Metzler systems are considered. The considerations have been illustrated by examples of electrical ladder networks of RL and GC type. The conditions for positivity of the electrical ladder network of RL and GC type are given in Section 5 (see also Example 4 for electrical network of RLC type given in Fig. 3).

The positive ladder networks can be applied in approximation of some positive distributed parameters systems [13].

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