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BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES
Vol. 55, No. 1, 2007

Stochastic controllability of linear systems with delay in control

J. KLAMKA*

Institute of Control Engineering, Silesian University of Technology, 16 Akademicka St., 44-100 Gliwice, Poland

Abstract. In the paper finite-dimensional stationary dynamical control systems described by linear stochastic ordinary differential state equations with single point delay in the control are considered. Using notations, theorems and methods taken directly from deterministic controllability problems, necessary and sufficient conditions for different kinds of stochastic relative controllability are formulated and proved. It will be proved that under suitable assumptions relative controllability of a deterministic linear associated dynamical system is equivalent to stochastic relative exact controllability and stochastic relative approximate controllability of the original linear stochastic dynamical system. Some remarks and comments on the existing results for stochastic controllability of linear dynamical systems with delays are also presented. Finally, minimum energy control problem for stochastic dynamical system is formulated and solved.

Key words: controllability, linear control systems, stochastic control systems, delayed controls, minimum energy control.

1. Introduction

Controllability is one of the fundamental concept in mathematical control theory and plays an important role both in deterministic and stochastic control theory [1-4]. Controllability is a qualitative property of dynamical control systems and is of particular importance in control theory. Systematic study of controllability was started at the beginning of sixties, when the theory of controllability based on the description in the form of state space for both time-invariant and time-varying linear control systems was worked out. Roughly speaking, controllability generally means, that it is possible to steer dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. In the literature there are many different definitions of controllability, both for linear [1–5] and nonlinear dynamical systems [6-9], which strongly depend on class of dynamical control systems and the set of admissible controls [10,11]. Therefore, for deterministic dynamical systems linear and nonlinear there exist many different necessary and sufficient conditions for global and local controllability [6,10–12].

In recent years various controllability problems for different types of linear dynamical systems have been considered in many publications and monographs. The extensive list of these publications can be found for example in the monograph [10] or in the survey papers [6,11,12]. However, it should be stressed, that the most literature in this direction has been mainly concerned with deterministic controllability problems for finite-dimensional linear dynamical systems with unconstrained controls and without delays.

For stochastic control systems both linear and nonlinear the situation is less satisfactory. In recent years the extensions of the deterministic controllability concepts to stochastic control systems have been recently discussed only in a rather few number of publications. In the papers [4,5,13–16] different kinds of stochastic controllability were discussed for linear finite dimensional stationary and nonstationary control systems. The papers [3,4,17,18] are devoted to a systematic study of approximate and exact stochastic controllability for linear infinite dimensional control systems defined in Hilbert spaces. Stochastic controllability for finite dimensional nonlinear stochastic systems has been discussed in the papers [19–23]. Using theory of nonlinear bounded operators and linear semigroups different types of stochastic controllability for nonlinear infinite dimensional control systems defined in Hilbert spaces have been considered in [8,9]. In the papers [5] and [24] Lyapunov techniques were used to formulate and prove sufficient conditions for stochastic controllability of nonlinear finite dimensional stochastic systems with point delays in the state variable. Moreover, it should be pointed out, that the functional analysis approach to stochastic controllability problems is also extensively discussed both for linear and nonlinear stochastic control systems in the papers [2–5,7–9,13–26].

In the present paper we shall study stochastic controllability problems for linear dynamical systems, which are natural generalizations of controllability concepts well known in the theory of infinite dimensional control systems [10,12]. More precisely, we shall consider stochastic relative exact and approximate controllability problems for finite-dimensional linear stationary dynamical systems with single constant point delay in the control described

*e-mail: jerzy.klamka @polsl.pl

by stochastic ordinary differential state equations. More precisely, using techniques similar to those presented in the papers [3,7,25] we shall formulate and prove necessary and sufficient conditions for stochastic relative exact controllability in a prescribed time interval for linear stationary stochastic dynamical systems with one constant point delay in the control.

Roughly speaking, it will be proved that under suitable assumptions relative controllability of a deterministic linear associated dynamical system is equivalent to stochastic relative exact controllability and stochastic relative approximate controllability of the original linear stochastic dynamical system. This is a generalization to control delayed case some previous results concerning stochastic controllability of linear dynamical systems without delays in the control [3,7,25]. It is well known, [10] that controllability concept for linear dynamical systems is strongly connected with so called minimum energy control problem. Therefore, finally, using quite general method presented in [10] and under the assumption that stochastic dynamical system is stochastically relatively exactly controllable minimum energy control problem is formulated and solved.

The paper is organized as follows: Section 2 contains mathematical model of linear, stationary stochastic dynamical system with single constant point delay in the control. Moreover, in this section basic definitions of stochastic relative exact and approximate controllability and some preliminary results are also included. In Section 3 using results and methods taken directly from deterministic controllability problems, necessary and sufficient conditions for exact and approximate stochastic relative controllability are formulated and proved. Section 4 is devoted to a study of minimum energy control problem. In this section we use some optimization methods to solve so called the minimum energy control problem and to show the analytic formula for minimum energy control. Finally, Section 5 contains concluding remarks and states some open controllability problems for more general stochastic dynamical systems.

2. System description

Throughout this paper, unless otherwise specified, we use the following standard notations. Let (Ω, F, P) be a complete probability space with probability measure P on Ω and a filtration $\{F_t | t \in [0,T]\}$ generated by n-dimensional Wiener process $\{w(s) : 0 \leq s \leq t\}$ defined on the probability space (Ω, F, P).

Let $L_2(\Omega, F_T, R^n)$ denotes the Hilbert space of all F_T-measurable square integrable random variables with values in R^n. Moreover, let $L_2^F([0,T], R^n)$ denotes the Hilbert space of all square integrable and F_t-measurable processes with values in R^n.

In the theory of linear, finite-dimensional, time-invariant stochastic dynamical control systems we use mathematical model given by the following stochastic ordinary differential state equation with single point delay in the control

$$dx(t) = (Ax(t) + B_0u(t) + B_1u(t-h))dt + \sigma dw(t) \quad \text{for} \quad t \in [0,T], \quad T > h \tag{1}$$

with initial conditions:

$$x(0) = x_0 \in L_2(\Omega, F_T, R^n) \text{ and } u(t) = 0 \text{ for } t \in [-h, 0) \tag{2}$$

where the state $x(t) \in R^n = X$ and the control $u(t) \in R^m = U$, A is $n \times n$ dimensional constant matrix, B_0 and B_1 are is $n \times m$ dimensional constant matrices, σ is $n \times n$ dimensional constant matrix, and $h > 0$ is a constant delay.

In the sequel for simplicity of considerations we generally assume that the set of admissible controls $U_{ad} = L_2^F([0,T], R^m)$.

It is well known (see e.g. [3,7,25] or [21]) that for a given initial conditions (2) and any admissible control $u \in U_{ad}$, for $t \in [0,T]$ there exist unique solution $x(t; x_0, u) \in L_2(\Omega, F_t, R^n)$ of the linear stochastic differential state equation (1) which can be represented in the following integral form

$$x(t;x_0,u) = \exp(At)x_0 + \int_0^t \exp(A(t-s))(B_0u(s) + B_1u(s-h))ds + \int_0^t \exp(A(t-s))\sigma dw(s).$$

Thus, taking into account zero initial control for $t \in [-h, 0]$, the solution for $t \in [0,h]$ has the following form [10]

$$x(t;x_0,u) = \exp(At)x_0 + \int_0^t \exp(A(t-s))B_0u(s)ds + \int_0^t \exp(A(t-s))\sigma dw(s)$$

Moreover, for $t > h$ we have

$$x(t;x_0,u) = \exp(At)x_0 + \int_0^t \exp(A(t-s))B_0u(s)ds + \int_0^{t-h} \exp(A(t-s-h))B_1u(s))ds + \int_0^t \exp(A(t-s))\sigma dw(s)$$

or equivalently

$$x(t;x_0,u) = \exp(At)x_0 + \int_0^{t-h} (\exp(A(t-s))B_0 + \exp(A(t-s-h))B_1)u(s)ds + \int_{t-h}^{t} \exp(A(t-s))B_0 u(s)ds + \int_0^t \exp(A(t-s))\sigma dw(s)$$

Now, for a given $T > h$, taking into account the form of the integral solution $x(t;x_0,u)$ let us introduce the following operators and sets [1].

The linear bounded control operator

$$L_T \in L(L_2^F([0,T],R^m), L_2(\Omega,F_T,R^n))$$

defined by

$$L_T u = \int_0^{T-h} (\exp(A(T-s))B_0 + \exp(A(T-s-h))B_1)u(s)ds + \int_{T-h}^{T} \exp(A(T-s))B_0 u(s)ds$$

and its adjoint linear bounded operator

$$L_T^* \in L_2(\Omega,F_T,R^n) \to L_2^F([0,T],R^m)$$

$$L_T^* z = (B_0^* \exp(A^*(T-t)) + B_1^* \exp(A^*(T-t-h)))E\{z\,|F_t\} \quad \text{for} \quad t \in [0,T-h]$$

$$L_T^* z = B_0^* \exp(A^*(T-t))E\{z\,|F_t\} \quad \text{for} \quad t \in (T-h,T]$$

and the set of all states reachable from initial state $x(0) = x_0 \in L_2(\Omega,F_T,R^n)$ in time $T > 0$, using admissible controls

$$R_T(U_{ad}) = \{x(T;x_0,u) \in L_2(\Omega,F_T,R^n) : u \in U_{ad}\} = \exp(At)x_0 + ImL_T + \int_0^T \exp(A(T-s))\sigma dw(s)$$

Moreover, we introduce the concept of the linear controllability operator [6,7,10,25] $C_T \in L(L_2(\Omega,F_T,R^n), L_2(\Omega,F_T,R^n))$ which is strongly associated with the control operator LT and is defined by the following equality

$$C_T = L_T L_T^* = \int_0^{T-h} (\exp(A(T-t))B_0 B_0^* \exp(A^*(T-t)) + \exp(A(T-t-h)B_1 B_1^* \exp(A^*(T-t-h)))E\{\cdot|\,F_t\}dt + \int_{T-h}^{T} \exp(A(T-t))B_0 B_0^* \exp(A^*(T-t))E\{\cdot|\,F_t\}dt$$

Finally, let us recall the form of $n \times n$-dimensional deterministic controllability matrix [10]

$$G_T = \int_0^{T-h} (\exp(A(T-t))B_0 B_0^* \exp(A^*(T-t)) + \exp(A(T-t-h)B_1 B_1^* \exp(A^*(T-t-h)))dt + \int_{T-h}^{T} \exp(A(T-t))B_0 B_0^* \exp(A^*(T-t))dt$$

In the proofs of the main results we shall use also controllability operators $C_T(s)$ and controllability matrices $G_T(s)$ depending on time $s \in [0,T-h]$, and defining as follows,

$$C_T(s) = L_T(s)L_T^*(s) = \int_s^{T-h} (\exp(A(T-t))B_0 B_0^* \exp(A^*(T-t)) + \exp(A(T-t-h)B_1 B_1^* \exp(A^*(T-t-h)))E\{\cdot|\,F_t\}dt + \int_{T-h}^{T} \exp(A(T-t))B_0 B_0^* \exp(A^*(T-t))E\{\cdot|\,F_t\}dt$$

$$G_T(s) = \int_s^{T-h} (\exp(A(T-t))B_0 B_0^* \exp(A^*(T-t)) + \exp(A(T-t-h)B_1 B_1^* \exp(A^*(T-t-h)))dt + \int_{T-h}^{T} \exp(A(T-t))B_0 B_0^* \exp(A^*(T-t))dt$$

In the theory of dynamical systems with delays in control or in the state variables, it is necessary to distinguish between two fundamental concepts of controllability, namely: relative controllability and absolute controllability (see e.g. [6,10], or [12] for more details). In this paper we shall concentrate on the weaker concept relative controllability. On the other hand, since for the stochastic dynamical system (1) the state space $L_2(\Omega,F_t,R^n)$ is in fact infinite-dimensional space, we distinguish exact or strong controllability and approximate or weak controllability. Using the notations given above for the stochastic dynamical system (1) we define the following stochastic relative exact and approximate controllability concepts.

Definition 1. The stochastic dynamical system (1) is said to be stochastically relatively exactly controllable on [0,T] if $R_T(U_{ad}) = L_2(\Omega,F_T,R^n)$ that is, if all the points in $L_2(\Omega,F_T,R^n)$ can be exactly reached from arbitrary initial condition $x_0 \in L_2(\Omega,F_T,R^n)$ at time T.

Definition 2. The stochastic dynamical system (1) is said to be stochastically relatively approximately controllable on $[0,T]$ if $\overline{R_T(U_{ad})} = L_2(\Omega,F_T,R^n)$ that is, if all the points in $L_2(\Omega,F_T,R^n)$ can be approximately reached from arbitrary initial condition $x_0 \in L_2(\Omega, F_T, R^n)$ at time T.

Remark 1. From the definitions 1 and 2 directly follows, that stochastic relative exact controllability is generally a stronger concept than stochastic relative approximate controllability. However, there are many cases when these two concepts coincide.

Remark 2. Since the stochastic dynamical system (1) is linear, then without loss of generality in the above two definitions it is enough to take zero initial condition $x_0 = 0 \in L_2(\Omega, F_T, R^n)$.

Remark 3. It should be pointed out, that in the case of delayed controls the above controllability concepts essentially depend on the length of the time interval $[0, T]$.

Remark 4. Since for $T \leq h$ stochastic dynamical system (1) is in fact a system without delay therefore, in the sequel we generally assumed that the final time $T > h$.

Remark 5. Since the dynamical system (3) is stationary, therefore in fact controllability matrix $G_T(s)$ has the same rank at least for all $s \in [0, T-h]$, [10].

Remark 6. From the form of the controllability operator C_T immediately follows, that this operator is self-adjoint.

In the sequel we study the relationship between the controllability concepts for the stochastic dynamical system (1) and controllability of the associated deterministic dynamical system of the following form

$$y'(t) = Ay(t) + B_0 v(t) + B_1 v(t-h) \quad t \in [0, T] \tag{3}$$

where the admissible controls $v \in L_2([0, T], R^m)$.

Therefore, let us recall the following lemma concerning relative controllability of deterministic system (3).

Lemma 1. [10]. The following conditions are equivalent:
(i) deterministic system (3) is relatively controllable on $[0, T]$,
(ii) controllability matrix G_T is nonsingular,
(iii) $\text{rank}[B_0, B_1, AB_0, AB_1, A^2B_0, A^2B_1, ..., A^{n-1}B_0, A^{n-1}B_1] = n$,

Now, let us formulate auxiliary well known lemma taken directly from the theory of stochastic processes, which will be used in the sequel in the proofs of the main results.

Lemma 2. [7,21,25]. For every $z \in L_2(\Omega, F_T, R^n)$, there exists a process $q \in L_2^F([0, T], R^{n\times n})$ such that

$$C_T z = G_T E z + \int_0^T G_T(s) q(s) dw(s)$$

Taking into account the above notations, definitions and lemmas in the next section we shall formulate and prove conditions for stochastic relative exact and stochastic relative approximate controllability for stochastic dynamical system (1).

3. Stochastic relative controllability

In this section, using lemmas given in Section 2 we shall formulate and prove the main result of the paper, which says that stochastic relative exact and in consequence also approximate controllability of stochastic system (1) is in fact equivalent to relative controllability of associated linear deterministic system (3).

Theorem 1. The following conditions are equivalent:
(i) Deterministic system (3) is relatively controllable on $[0, T]$,
(ii) Stochastic system (1) is stochastically relatively exactly controllable on $[0, T]$
(iii) Stochastic system (1) is stochastically relatively approximately controllable on $[0, T]$.

Proof. (i) $\Rightarrow$ (ii) Let us assume that the deterministic system (3) is relatively controllable on $[0, T]$. Then, it is well known (see e.g. [5,6,10]) that the relative deterministic controllability matrix $G_T(s)$ is invertible and strictly positive definite at least for all $s \in [0, T-h]$ [10]. Hence, for some $\gamma > 0$ we have

$$\langle G_T(s)x, x\rangle \geqslant \gamma \|x\|^2$$

for all $s \in [0, T-h]$ and for all $x \in R^n$. In order to prove stochastic relative exact controllability on $[0, T]$ for the stochastic system (1) we use the relationship between controllability operator C_T and controllability matrix G_T given in Lemma 2, to express $E\langle C_T z, z\rangle$ in terms of $\langle G_T Ez, Ez\rangle$. First of all we obtain

$$\begin{aligned}
&E\langle C_T z, z\rangle \\
&= E\left\langle G_T Ez + \int_0^T G_T(s) q(s) dw(s), Ez + \int_0^T q(s) dw(s)\right\rangle \\
&= \langle G_T Ez, Ez\rangle + E\int_0^T \langle G_T(s) q(s), q(s)\rangle\, ds \geqslant \gamma \\
&\times \left(\|Ez\|^2 + E\int_0^T \|q(s)\|^2\, ds\right) = \gamma E\|z\|^2
\end{aligned}$$

Hence, in the operator sense we have the following inequality $C_T \geqslant \gamma I$, which means that the operator C_T is strictly positive definite and thus, that the inverse linear operator C_T^{-1} is bounded. Therefore, stochastic relative exact controllability on $[0, T]$ of the stochastic dynamical system (1) directly follows from the results given in [10]. Moreover, in the next section using fact that the operator C_T^{-1} is bounded we shall construct the control $u^0(t)$, $t \in [0, T]$ which steers stochastic dynamical system (1) from given initial state x_0 to the desired final state x_T at time T.
(ii) $\Rightarrow$ (iii) This implication is obvious (see e.g. [10] and [3,4,8]).
(iii) $\Rightarrow$ (i) Assume that the stochastic dynamical system (1) is stochastically relatively approximately controllable

on $[0,T]$, and hence its controllability operator is positive definite, i.e. $C_T > 0$ [10]. Then, using the resolvent operator $R(\lambda, C_T)$ and following directly the functional analysis method given in [7,21,25] and for stochastic dynamical systems without delays we obtain that deterministic system (3) is approximately relatively controllable on [0.T]. However, taking into account that the state space for deterministic dynamical system (3) is finite dimensional, i.e. exact and approximate controllability coincide [10], we conclude that deterministic dynamical system (3) is relatively controllable on $[0,T]$.

Remark 7. Let us observe, that for a special case when the final time $T \leqslant h$, stochastic relative exact or approximate controllability problems in $[0,T]$ for stochastic dynamical system with delay in the control (1) are reduced to the standard stochastic exact or stochastic approximate controllability problems for the stochastic dynamical system without delays in the control [10].

COROLLARY 1. [7,25]. Suppose that $T \leq h$. Then the stochastic dynamical control system (1) is stochastically relatively exactly controllable in [0,T] if and only if

$$\operatorname{rank}[B_0, AB_0, A^2B_0, ..., A^{n-1}B_0] = n,$$

COROLLARY 2. [25]. Stochastic dynamical system without delay ($B_1 = 0$) is stochastically exactly controllable in any time interval if and only if associated deterministic dynamical system without delay is controllable.

In the next section we shall formulate and solve minimum energy control problem for stochastic relative exact controllable stochastic dynamical system (1).

4. Minimum energy control

Minimum energy control problem is strongly connected with controllability concept (see e.g. [1,2] and [10] for more details). First of all, let us observe, that for exactly controllable on $[0,T]$ linear control system there exists generally many different admissible controls $u(t)$, defined for $t \in [0,T]$ and transferring given initial state x_0 to the desired final state x_T at time T. Therefore, we may ask which of these possible admissible controls are optimal one according to given a priori criterion. In the sequel we shall consider minimum energy control problem for stochastic dynamical system (1) with the optimality criterion representing the energy of control. In this case optimality criterion has the following simple form

$$J(u) = E\int_0^T \|u(t)\|^2\,dt$$

THEOREM 2. Assume that the stochastic dynamical system (1) is relatively exactly controllable on [0,T]. Then, for arbitrary $x_T \in L_2(\Omega, F_T, R^n)$ and arbitrary σ, the control

$$u^0(t) = B_0^* \exp(A^*(T-t))E$$
$$\times \left\{ C_T^{-1}\left(x_T - \exp(AT)x_0 - \int_0^{T-h} \exp(A(T-s))\sigma dw(s)\right) |F_t \right\}$$

for $t \in [0,h]$

$$u^0(t) = (B_0^* \exp(A^*(T-t)) + B_1^* \exp(A^*(T-h-t)))$$
$$\times\ E\left\{ C_T^{-1}\left(x_T - \exp(AT)x_0 - \int_{T-h}^{T} \exp(A(T-h-s))\sigma dw(s)\right) |F_t \right\}$$

for $t \in (h,T]$ transfers the system (1) from initial state x_0 to the final state x_T at time $T > h$.

Moreover, among all admissible controls $u^a(t)$ transferring initial state x_0 to the final state x_T at time $T > h$, the control $u^0(t)$ minimizes the integral performance index

$$J(u) = E\int_0^T \|u(t)\|^2\,dt$$

Proof. First of all let us observe, that since the stochastic dynamical system (1) is stochastically relatively exactly controllable on $[0,T]$, then the controllability operator C_T is invertible and its inverse C_T^{-1} is a linear and bounded operator, i.e. $C_T^{-1} \in L(L_2(\Omega, F_T, R^n), L_2(\Omega, F_T, R^n))$. Substituting the control $u^0(t)$ into the solution formula of the differential state equation, one can easily obtain

$$x(t; x(0), u^0(t)) = \exp(At)x_0$$
$$\int_0^t \exp(A(t-s))B_0 B_0^* \exp(A^*(t-s))$$
$$\times E\left\{ C_T^{-1}\left(x_T - \exp(AT)x(0) - \int_0^{T-h} \exp(A(T-s))\sigma dw(s)\right)\right\} |F_s\ ds$$
$$+ \int_0^t \exp(A(t-s))\sigma dw(s)$$

for $t \in [0,h]$

$$x(t; x(0), u^0(t)) = \exp(At)x(0)$$
$$+ \int_0^{t-h} (\exp(A(t-s))B_0 B_0^* \exp(A^*(T-s)$$
$$+ \exp(A(T-h-s))B_1 B_1^* \exp(A^*(T-h-s)))$$
$$\times E\left\{ C_T^{-1}\left(x_T - \exp(AT)x(0) - \int_{t-h}^{t} \exp(A(T-s)\sigma dw(s)\right)\right\} |F_s ds$$
$$+ \int_{t-h}^{t} \exp(A(t-s))B_0 B_0^* \exp(A^*(T-s))$$
$$\times E\left\{ C_T^{-1}\left(x_T - \exp(AT)x(0) - \int_0^{T-h} \exp(A(T-s)\sigma dw(s)\right)\right\} |F_s\ ds$$
$$+ \int_0^t \exp(A(t-s))\sigma dw(s)$$

for $t \in (h,T]$

Hence, for $t = T$ we have

$$x(T; x(0), u^0(t)) = \exp(AT)x_0$$
$$+ \int_0^{T-h} (\exp(A(T-s))B_0B_0^* \exp(A^*(T-s))$$
$$+ \exp(A(T-h-s))B_1B_1^* \exp(A^*(T-h-s)))$$
$$\times E\left\{ C_T^{-1} \left(x_T - \exp(AT)x_0 - \int_{T-h}^{T} \exp(A(T-s))\sigma dw(s) \right) \right\} |F_s ds$$
$$+ \int_{T-h}^{T} \exp(A(T-s))B_0B_0^* \exp(A^*(T-s))$$
$$\times E\left\{ C_T^{-1} \left(x_T - \exp(AT)x_0 - \int_0^{T-h} \exp(A(T-s))\sigma dw(s) \right) \right\} |F_s\ ds$$
$$+ \int_0^T \exp(A(T-s))\sigma dw(s)$$

Thus, taking into account the form of the operator C_T we have

$$x(T; x(0), u^0(t)) = \exp(AT)x_0 + C_T C_T^{-1}$$
$$\times \left(x_T - \exp(AT)x(0) - \int_0^T \exp(A(T-s))\sigma dw(s) \right)$$
$$+ \int_0^T \exp(A(T-s))\sigma dw(s)$$
$$= \exp(AT)x_0 + x_T - \exp(AT)x(0)$$
$$- \int_0^T \exp(A(T-s))\sigma dw(s) + \int_0^T \exp(A(T-s))\sigma dw(s) = x_T$$

Therefore, for $t = T$ we see that the control $u^0(t)$ transfers dynamical system (1) from given initial state $x_0 \in L_2(\Omega, F_T, R^n)$ to the desired final state $x_T \in L_2(\Omega, F_T, R^n)$ at time $T > h$.

In the second part of the proof using the method presented in [10] we shall show that the control $u^0(t)$, $t \in [0, T]$ is optimal according to performance index J. In order to do that, let us suppose that $u'(t)$, $t \in [0, T]$ is any other admissible control which also steers the initial state x_0 to the final state x_T at time T. Hence using controllability operator defined in section 2 we have

$$L_T(u^0(\cdot)) = L_T(u'(\cdot))$$

Subtracting from both sides and using the properties of scalar product in the space R^n and the form of controllability operator L_T we obtain the following equality

$$E \int_0^T \langle (u'(t) - u^0(t)), u^0(t) \rangle\, dt = 0$$

Moreover, using once again properties of the scalar product in R^n we have

$$E \int_0^T \|u'(t)\|^2 dt$$
$$= E \int_0^T \|u'(t) - u^0(t)\|^2 dt + E \int_0^T \|u^0(t)\|^2 dt$$

Since $E \int_0^T \|u'(t) - u^0(t)\|^2 dt \geqslant 0$, we conclude that for any admissible control $u'(t)$, $t \in [0, T]$ the following inequality holds

$$E \int_0^T \|u^0(t)\|^2 dt \leqslant E \int_0^T \|u'(t)\|^2 dt$$

Hence, the control $u^0(t)$, $t \in [0, T]$ is optimal control according to the performance index J, and thus it is minimum energy control.

5. Concluding remarks

In the paper sufficient conditions for stochastic relative exact controllability for linear stationary finite-dimensional stochastic control systems with single constant point delay in the control have been formulated and proved. These conditions extend to the case of one constant point delay in control, known stochastic exact controllability conditions for dynamical control systems without delays recently published in the papers [7,8] and [25]. Finally, it should be pointed out, that using the standard techniques presented in the monograph [10] it is possible to extend the results presented in this paper for more general nonstationary linear stochastic control systems with many time variable delays in the control. Moreover, the extension for stochastic absolute exact controllability and stochastic absolute approximate controllability in a given time interval is also possible.

Acknowledgements. This paper has been supported by the Ministry of Science and Higher Education under grant no 3T 11A 002 27.

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