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Synchronization of fractional order Rabinovich-Fabrikant systems using sliding mode control techniques

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In this research article, we present the concepts of fractional-order dynamical systems and synchronization methodologies of fractional order chaotic dynamical systems using slide mode control techniques. We have analysed the different phase portraits and time-series graphs of fractional order Rabinovich-Fabrikant systems. We have obtained that the lowest dimension of Rabinovich-Fabrikant system is 2.85 through utilization of the fractional calculus and computational simulation. Bifurcation diagrams and Lyapunov exponents of fractional order Rabinovich-Fabrikant system to justify the chaos in the systems. Synchronization of two identical fractional-order chaotic Rabinovich-Fabrikant systems are achieved using sliding mode control methodology.

Key words: fractional-order chaotic system, chaos synchronization, Rabinovich-Fabrikant system, Lyapunov exponents

1. Introduction

Chaos, an inevitable phenomenon is the part of nonlinear systems. It is highly sensitive to the initial conditions. This sensitivity is popularly known as the butterfly effect [1]. Since Pecora and Carroll established the concept of chaos synchronization with different initial conditions, it (chaos synchronization) has been received much attention in the field of research. Synchronization of two or more than two chaotic dynamical systems is one of most important applications of chaos. Last several decades, chaos synchronization has been become the research subject in the field of nonlinear sciences due to its potential application in various disciplines such as – chemical reaction, power converters, aerospace, signal

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process, laser physics, secure communication, satellite systems and biological systems [1–5].

Fractional calculus having old topic belong to more than 300 years ago in history. In recent years, fractional-order dynamical systems have obtained much interest in physical modeling phenomena and in the synthesis of strengthening of controllers as well as secure communication and control processing. It has wide application in many area of physics and engineering. It (fractional-order dynamical systems) has attracted more attentions to scientists and researchers [6–8].

Sliding mode control (SMC) is a nonlinear control method. Its features has remarkable properties of accuracy, robustness, simple tuning and implementation. Sliding mode surface (SMS) systems are designed to drive the system states onto a particular surface in the state space which is called sliding surface. When the sliding surface is approached, sliding mode control methods applies the states on the close neighbourhood of the sliding surface. The sliding mode control methods (techniques) consist of the two part controller design. The first part includes the design of a sliding surface so that the sliding motion satisfies design specifications. The second involves the adaption of a sliding control methods that will design the switching surface fascinating to the system state [2, 9–11].

Rabinovich-Fabrikant physical model is nonlinear ordinary differential equations describing the stochasticity arising from the modulation instability in a non-equilibrium dissipative medium in 1979. This physical model have been introduced and analyzed by Rabinovich and Fabrikant. The system behavior depends sensibly on the parameters values [12, 13]. Khan and Tripathi have established the synchronization between a fractional order Coullet chaotic system and an integer order Rabinovich-Fabrikant chaotic system by using tracking control and stability theory of fractional order system [14]. Danca et al. have viewed more closely to the Rabinovich-Fabrikant system [13]. Srivastava et al have introduced their viewed on study of chaos of fractional order Rabinovich- Fabrikant system and chaos controlled and synchronization between chaotic fractional order Rabinovich-Fabrikant systems [15].

Motivated by the above discussions, we present the synchronization of fractional order Rabinovich-Fabrikant systems using sliding mode control. Having analyzed the different phase portraits and time-series graphs of fractional order Rabinovich-Fabrikant systems, we have obtained that the lowest dimension of Rabinovich-Fabrikant system is 2.85 through utilization of the fractional calculus and computational simulation. Bifurcation diagrams and Lyapunov exponents of fractional order Rabinovich-Fabrikant system is drawn to justify the chaos in the systems. Two identical fractional-order chaotic Rabinovich-Fabrikant systems are synchronized using sliding mode control techniques. These show the novelty of our research paper.

This paper is organized as follows: section 1 is introduction; section 2 describes the basic concepts of fractional derivatives and its approximation; systems description and sliding mode control methodology of fractional order systems are presented in section 3; in section 4, numerical simulations are used the synchronization between identical fractional order Rabinovich-Fabrikant systems using sliding mode control techniques; finally, conclusion is given in section 5.

2. Fractional derivatives and its approximation

Fractional calculus is a generalization concepts of integration and differentiation to a non-integer-order integro-differential operator ${}_aD_t^{\alpha_1}$. It is written as

$$ {}_aD_t^{\alpha_1} = \begin{cases} \dfrac{\mathrm{d}^{\alpha_1}}{\mathrm{d}t^{\alpha_1}}, & \text{if } \ R(\alpha_1) > 0, \\ 1, & \text{if } \ R(\alpha_1) = 0, \\ \displaystyle\int_a^t (\mathrm{d}\tau)^{-\alpha_1}, & \text{otherwise, i.e. } \ R(\alpha_1) < 0. \end{cases} \tag{1} $$

The generalized Riemann-Liouville denition [6] is defined as

$$ D^{\alpha_1} f(t) = \frac{\mathrm{d}^{\alpha_1}}{\mathrm{d}t^{\alpha_1}} J^{n-\alpha_1} f(t), \qquad \alpha_1 > 0, \tag{2} $$

where $n = [\alpha_1]$. n is taken as first integer which is not less than α_1, J^{β_1} is the β_1-order Riemann-Liouville integral operator. It is written as follows

$$ J^{\beta_1} f(t) = \frac{1}{\Gamma(\beta_1)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\beta_1}} \mathrm{d}\tau \tag{3} $$

for $0 < \beta_1 \leqslant 1$, where $\Gamma(.)$ is the gamma function.

The Caputo differential operator is defined as

$$ D^{\alpha_1} f(t) = J^{n-\alpha_1} f^n(t), \qquad \alpha_1 > 0, \tag{4} $$

where $n = [\alpha_1]$. The operator D^{α_1} is called the Caputo differential operator of order α_1. It has been used firstly for the solution of practical problems by Caputo [6, 7].

Some basic definitions and properties of fractional order derivatives and integrals

Definition 1 *[6–8]: A real function $f(t)$, $t > 0$ is said to be in Caputo space C_{α_1}, $\alpha_1 \in R$ if there exist a real number p $(> \alpha_1)$, such that $f(t) = t^p f_1(t)$; where $f_1(t) \in C[0,\infty)$.*

Definition 2 *[6–8]: A real function $f(t)$, $t > 0$ is said to be in Caputo space $C^m_{\alpha_1}$, $m \in \mathbb{N} \cup 0$ if $f^{(m)} \in C_{\alpha_1}$.*

Definition 3 *[6–8]: Let $f \in C_{\alpha_1}$ and $\alpha_1 \geqslant -1$, then the Riemann-Liouville integral of order α_1 $(\alpha_1 > 0)$ is given by*

$$I^{\alpha_1} f(t) = \frac{1}{\Gamma(\alpha_1)} \int_0^t (t - \alpha_1)^{\alpha_1 - 1} f(\tau)\,\mathrm{d}\tau, \qquad t > 0. \tag{5}$$

Definition 4 *[6–8]: The Caputo fractional order derivative of f, $f \in C_{-1}^m$, $m \in \mathbb{N} \cup 0$, is written as:*

$$\begin{gathered} D^{\alpha_1} f(t) = \frac{\mathrm{d}^m}{\mathrm{d}t^m} f(t), \qquad \alpha_1 = m = I^{m-\alpha_1} \frac{\mathrm{d}^m f(t)}{\mathrm{d}t^m}, \\ m - 1 < \alpha_1 < m, \qquad m \in \mathbb{N}. \end{gathered} \tag{6}$$

Note that for $m - 1 < \alpha_1 \leqslant m, \quad m \in \mathbb{N}$,

$$I^{\alpha_1} D^{\alpha_1} f(t) = f(t) - \sum_{k=0}^{m-1} \frac{\mathrm{d}^k f}{\mathrm{d}t^k}(0) \frac{t^k}{k!}, \qquad I^{\alpha_1} t^{\alpha_1} = \frac{\Gamma(v+1)}{\Gamma(\alpha_1 + v + 1)} t^{\alpha_1 + v}. \tag{7}$$

Consider the differential equations system as

$$\begin{aligned} \frac{\mathrm{d}^{\alpha_1} x}{\mathrm{d}t^{\alpha_1}} &= f(t, x), \qquad 0 \leqslant t \leqslant T \\ \text{and} \qquad x^k(0) &= x_0^{(k)}, \quad k = 0, 1, 2, \cdots, n-1. \end{aligned} \tag{8}$$

This differential equations system is similar to the Volterra integral equation [16, 17]

$$x(t) = \sum_{k=0}^{[\alpha_1]-1} x_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha_1)} \int_0^t \frac{f(\tau, x)}{(t-\tau)^{1-\alpha_1}} \mathrm{d}\tau. \tag{9}$$

Set $h = T/N, \quad t_n = nh (n = 0, 1, 2, \cdots, N)$. Then equation can be discretized as follows:

$$\begin{aligned} x_h(t_{n+1}) = &\sum_{k=0}^{[\alpha]-1} x_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{h^{\alpha_1}}{\Gamma(\alpha_1 + 2)} f(t_{n+1}, x_h^p(t_{n+1})) \\ &+ \frac{h^{\alpha_1}}{\Gamma(\alpha_1 + 2)} \sum a_{j,n+1} f(t_j, x_h(t_j)), \end{aligned} \tag{10}$$

where the predicted value $x_h^p(t_{n+1})$ is obtained by

$$x_h^p(t_{n+1}) = \sum_{k=0}^{[\alpha_1]-1} \frac{t_{n+1}^k}{k!} + \frac{1}{\Gamma(\alpha_1)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, x_h(t_j))$$

and

$$a_{j,n+1} = \begin{cases} n^{\alpha_1+1} - (n-\alpha_1)(n+1)^{\alpha_1+1}, & \text{if } j=0, \\ (n-j+2)^{\alpha_1+1} + (n-j^{\alpha_1+1}) - 2(n-j+1)^{\alpha_1+1}, & \text{if } 1 \leqslant j \leqslant n, \\ 1, & j = n+1; \end{cases}$$

$$b_{j,n+1} = \frac{h^{\alpha_1}}{\alpha_1}\left((n-j+1)^{\alpha_1} - (n-j)^{\alpha_1}\right).$$

The estimated errors of this approximation is written as $e = \mathrm{Max}|x(t_j) - x_h(t_j)| = O(h^p)(j = 0, 1, \cdots, N,)$ in which $p = \mathrm{Min}(2, 1+\alpha_1)$. Numerical solution of a fractional-order system is determined by implementing this method.

Now, the fractional-order system is defined as

$$\begin{aligned} \frac{\mathrm{d}^{\alpha_1} x}{\mathrm{d}t^{\alpha_1}} &= f_1(x,y,z), \\ \frac{\mathrm{d}^{\alpha_1} y}{\mathrm{d}t^{\alpha_1}} &= f_2(x,y,z), \\ \frac{\mathrm{d}^{\alpha_1} z}{\mathrm{d}t^{\alpha_1}} &= f_3(x,y,z), \end{aligned} \tag{11}$$

for $0 < \alpha_1 \leqslant 1$ with the initial condition x_0, y_0, z_0. System (8) can be written as:

$$\begin{aligned} x_{n+1} &= x_0 + \frac{h^{\alpha_1}}{\Gamma(\alpha_1+2)}\left[f_1(x^p_{n+1}, y^p_{n+1}, z^p_{n+1}) + \sum_{j=0}^{n} \gamma_{1,j,n+1} f_1(x_j, y_j, z_j)\right], \\ y_{n+1} &= y_0 + \frac{h^{\alpha_1}}{\Gamma(\alpha_1+2)}\left[f_2(x^p_{n+1}, y^p_{n+1}, z^p_{n+1}) + \sum_{j=0}^{n} \gamma_{2,j,n+1} f_2(x_j, y_j, z_j)\right], \\ z_{n+1} &= z_0 + \frac{h^{\alpha_1}}{\Gamma(\alpha_1+2)}\left[f_3(x^p_{n+1}, y^p_{n+1}, z^p_{n+1}) + \sum_{j=0}^{n} \gamma_{3,j,n+1} f_3(x_j, y_j, z_j)\right], \end{aligned}$$

where

$$\begin{aligned} x^p_{n+1} &= x_0 + \frac{1}{\Gamma(\alpha_1)} \sum_{j=0}^{n} \omega_{1,j,n+1} f_1(x_j, y_j, z_j), \\ y^p_{n+1} &= y_0 + \frac{1}{\Gamma(\alpha_1)} \sum_{j=0}^{n} \omega_{2,j,n+1} f_2(x_j, y_j, z_j), \\ z^p_{n+1} &= z_0 + \frac{1}{\Gamma(\alpha_1)} \sum_{j=0}^{n} \omega_{3,j,n+1} f_3(x_j, y_j, z_j), \end{aligned}$$

$$\gamma_{i,j,n+1} = \begin{cases} n^{\alpha_1+1} - (n-\alpha_1)(n+1)^{\alpha_1+1}, & \text{if } \; j = 0, \\ (n-j+2)^{\alpha_1+1} + (n - j^{\alpha_1+1}) - 2(n-j+1)^{\alpha_1+1}, & \text{if } \; 1 \leqslant j \leqslant n, \\ 1, & \text{if } \; j = n+1; \end{cases}$$

$$\omega_{i,j,n+1} = \frac{h^{\alpha_1}}{\alpha_1}\left((n-j+1)^{\alpha_1} - (n-j)^{\alpha_1}\right) \quad 0 \leqslant j \leqslant n, \quad i = 1, 2, 3.$$

3. Systems description and methodology by sliding mode control

Consider the chaotic system as

$$\frac{\mathrm{d}^{\alpha_1} x(t)}{\mathrm{d}t^{\alpha_1}} = \mathbf{A}x(t) + f(x(t)), \tag{12}$$

where $x \in R^n$ represents the state vector of the system. $\mathbf{A}$ represents the $n \times n$ matrix of the system of parameters and $f : R^n \to R^n$ denotes the nonlinear part of the system. System (12) is considered as the drive (master) system.

Corresponding response (slave) chaotic system is defined as:

$$\frac{\mathrm{d}^{\alpha_1} y(t)}{\mathrm{d}t^{\alpha_1}} = \mathbf{A}y(t) + f(y(t)) + u(t), \tag{13}$$

where $y \in R^n$ represents the state vector of the slave system and $u(t) \in R^m$ is the controller to be designed. For that,the synchronization error system is written as

$$e(t) = y(t) - x(t).$$

The error dynamics is written as

$$\frac{\mathrm{d}^{\alpha_1} e(t)}{\mathrm{d}t^{\alpha_1}} = \mathbf{A}e(t) + \theta(x, y) + u(t). \tag{14}$$

Here, $\theta(x, y)$ represents the nonlinear parts of the fractional order chaotic system. The controller $u(t)$ for chaos synchronization problem is designed such that

$$\lim_{t\to\infty} \|e(t)\| = 0 \qquad \text{for} \quad e(0) \in R^n. \tag{15}$$

For this, the controller $u(t)$ is defined such that

$$u(t) = -\theta(x, y) + \mathbf{B}\upsilon(t), \tag{16}$$

where $\mathbf{B}$ is a constant gain vector selected such that $(\mathbf{A}, \mathbf{B})$ is controllable. $\upsilon(t) \in R^n$ represents the control input.

From (14) and (16), the error dynamics is rewritten as

$$\frac{\mathrm{d}^{\alpha_1} e(t)}{\mathrm{d}t^{\alpha_1}} = \mathbf{A}e(t) + \mathbf{B}\upsilon(t). \tag{17}$$

The system (17) is the linear time-invariant controller techniques system with single input $\upsilon(t)$. The synchronization of fractional order systems is determined by an equivalent systm of stabilizing the zero solution $e(t) \to 0$ of the system (17) through a suitable approaching of the sliding mode control techniques.

In the sliding mode control techniques, we take the variable [2] as,

$$s(e) = Me(t) = m_1 e_1 + m_2 e_2 + \cdots + m_n e_n, \tag{18}$$

where $M = (m_1, m_2, \cdots, m_n)$ is the determined constant row vectors. The motion of the system (17) to the sliding manifold is designed as

$$\mathbf{S} = (x \in R^n \qquad \text{such that} \quad s(e(t)) = 0). \tag{19}$$

The invariant flow of the error dynamics (17) satisfies the conditions $s(e(t)) = 0$ is written in the sliding manifold as

$$\Rightarrow \quad \frac{\mathrm{d}^{\alpha_1} s(e(t))}{\mathrm{d}t^{\alpha_1}} = 0. \tag{20}$$

This represents the necessary condition for the state trajectory $e(t)$ of (17) to be on the sliding manifold $\mathbf{S}$. From (17) and (18), the equation (20) is rewritten as,

$$\frac{\mathrm{d}^{\alpha_1} s(e(t))}{\mathrm{d}t^{\alpha_1}} = M(\mathbf{A}e(t) + \mathbf{B}\upsilon(t)) = 0. \tag{21}$$

Solving (21) for υ, the equivalent control law is written as

$$\upsilon_{eq}(t) = -(M\mathbf{B})^{-1} M\mathbf{A}e(t), \tag{22}$$

where M is taken such that $M\mathbf{B} \neq 0$. Substituting the equation (22) in the error system (17), we have

$$\frac{\mathrm{d}^{\alpha_1} e(t)}{\mathrm{d}t^{\alpha_1}} = \left(I - B(M\mathbf{B})^{-1} M\right) \mathbf{A}e(t). \tag{23}$$

Here, I denotes the identity matrix. The row vector, M is taken such that the system matrix of the controlled dynamics $\left(I - \mathbf{B}(M\mathbf{B})^{-1} M\right) \mathbf{A}$ satisfies Hurwitz stability criterion. That is, all eigenvalues of the above system has negative real parts. This means that the error dynamical system (17) is asymptotically stable.

The sliding mode controller techniques for the error dynamics system (17) and the constant plus proportional rate reaching law is designed as

$$\frac{\mathrm{d}^{\alpha_1} s(e(t))}{\mathrm{d}t^{\alpha_1}} = -\rho\, sign(s) - \sigma s, \tag{24}$$

where $sign(.)$ is the sign function and gain $\rho > 0$, $\sigma > 0$ are determined such that sliding condition is satisfied.

From (21) and (24), the controlled invariant inputs $\upsilon(t)$ is obtained as

$$\upsilon(t) = -(M\mathbf{B})^{-1}\left(M(\sigma I + \mathbf{A})e(t) + \rho\, sign(s)\right), \tag{25}$$

which gives

$$\upsilon(t) = \begin{cases} -(M\mathbf{B})^{-1}(M(\sigma I + \mathbf{A})e(t) + \rho s(e(t))), & \text{if} \quad s(e(t)) > 0, \\ -(M\mathbf{B})^{-1}(M(\sigma I + \mathbf{A})e(t) - \rho s(e(t))), & \text{if} \quad s(e(t)) < 0. \end{cases} \tag{26}$$

Lemma 4 *[2, 11]: If the motion of sliding mode is asymptotically stable then the following condition hold:*

$$s^T(e(t))\frac{\mathrm{d}^{\alpha_1} s(e(t))}{\mathrm{d}t^{\alpha_1}} < 0. \tag{27}$$

Theorem 1 *[2]: The fractional order master system* (12) *and the fractional order slave system* (13) *are asymptotically synchronized with the initial conditions* $x(0), y(0) \in R^n$ *through the feedback control law,*

$$u(t) = -\theta(x, y) + \mathbf{B}\upsilon(t), \tag{28}$$

where $\upsilon(t)$ *is defined by* (26) *and* $\mathbf{B}$ *is a column vector such that* $(\mathbf{A}, \mathbf{B})$ *is controllable. As well as the slide mode gain* κ *and* ρ *are taken as positive.*

Proof. Substituting the (27) and (25) into the error dynamics (23), the closed-loop error dynamics is written as

$$\frac{\mathrm{d}^{\alpha_1} e(t)}{\mathrm{d}t^{\alpha_1}} = \mathbf{A}e(t) - \mathbf{B}(M\mathbf{B})^{-1}(M(\sigma I + \mathbf{A})e(t) + \rho\, sign(s)). \tag{29}$$

To prove the error dynamics (23) is asymptotically stable, we consider the Lyapunov function,

$$\mathbf{V}(e(t)) = \frac{1}{2}s^2(e(t)). \tag{30}$$

It is noted that

$$\mathbf{V}(e(t)) \geqslant 0, \quad \text{for } e(t) \in R^n \qquad \text{and} \qquad \mathbf{V}(e(t)) = 0 \;\Leftrightarrow\; e(t) = 0.$$

Thus, it implies that V is a positive definite function on R^n. Differentiated V along the trajectories of (17) of the equivalent dynamics (24), we have

$$\frac{\mathrm{d}^{\alpha_1}\mathbf{V}(e(t))}{\mathrm{d}t^{\alpha_1}} = s^T(e(t))\dot{s}(e(t)) = -\sigma s^2 - \rho\, sign(s)s, \tag{31}$$

which is the negative definite function of R^n. These determined that V is the positive definite Lyapunov function for the error dynamics (17) and $\dfrac{\mathrm{d}^{\alpha_1}\mathbf{V}(e(t))}{\mathrm{d}t^{\alpha_1}}$ is the negative definite function.

Thus, by the Lyapunov stability theory [18], the error dynamics (17) is asymptotically stable with the initial condition $e(0) \in R^n$. We have

$$\lim_{t\to\infty} e(t) = 0.$$

Therefore, the fractional order master system (12) and the fractional order slave system (13) are asymptotically synchronized and stable with the initial conditions $x(0),\ y(0) \in R^n$.

4. Numerical results

Synchronization of identical fractional order Robinovich-Fabricant systems by sliding mode control

Consider the chaotic Robinovich-Fabricant system [13]

$$\begin{cases} \dot{x} = y(z - 1 + xx) + ax, \\ \dot{y} = x(3z + 1 - xx) + ay, \\ \dot{z} = -2z(b + xy). \end{cases} \tag{32}$$

Based on the above descriptions, we consider the new version fractional Robinovich-Fabricant system as follows:

$$\begin{cases} \dfrac{\mathrm{d}^{\alpha_1}x}{\mathrm{d}t^{\alpha_1}} = y(z - 1 + xx) + ax, \\ \dfrac{\mathrm{d}^{\alpha_1}y}{\mathrm{d}t^{\alpha_1}} = x(3z + 1 - xx) + ay, \\ \dfrac{\mathrm{d}^{\alpha_1}z}{\mathrm{d}t^{\alpha_1}} = -2z(b + xy), \end{cases} \tag{33}$$

where α_1 is the fractional order satisfying $0 < \alpha_1 \leqslant 1$.

Two identical fractional order master (drive) and slave (response) systems for Robinovich-Fabricant system (33) are rewritten with the subscripts of x and y respectively as:

Master system is:

$$\begin{cases} \dfrac{d^{\alpha_1} x_1}{dt^{\alpha_1}} = x_2(t)(x_3(t) - 1 + x_1(t)x_1(t)) + ax_1(t), \\ \dfrac{d^{\alpha_1} x_2}{dt^{\alpha_1}} = x_1(t)(3x_3(t) + 1 - x_1(t)x_1(t)) + ax_2(t), \\ \dfrac{d^{\alpha_1} x_3}{dt^{\alpha_1}} = -2x_3(t)(b + x_1(t)x_2(t)). \end{cases} \tag{34}$$

Its fractional-order slave Robinovich-Fabricant system is:

$$\begin{cases} \dfrac{d^{\alpha_1} y_1}{dt^{\alpha_1}} = y_2(t)(y_3(t) - 1 + y_1(t)y_1(t)) + ay_1(t) + u_1(t), \\ \dfrac{d^{\alpha_1} y_2}{dt^{\alpha_1}} = y_1(t)(3y_3(t) + 1 - y_1(t)y_1(t)) + ay_2(t) + u_2(t), \\ \dfrac{d^{\alpha_1} y_3}{dt^{\alpha_1}} = -2y_3(t)(b + y_1(t)y_2(t)) + u_3(t), \end{cases} \tag{35}$$

where, $u(t) = (u_1, u_2, u_3)$ is the controller. Our aim is to determine the controller $u(t)$ such that the fractional-order master system synchronize the fractional-order slave system. For this, we define the error signal for (34) and (35). The error dynamics is written as:

$$\begin{aligned} &e(t) = [e_1(t), e_2(t), e_3(t)]^T \\ &e_1(t) = y_1(t) - x_1(t), \quad e_2(t) = y_2(t) - x_2(t), \quad e_3(t) = y_3(t) - x_3(t). \end{aligned} \tag{36}$$

The error dynamics is written as

$$\begin{cases} \dfrac{d^{\alpha_1} e_1(t)}{dt^{\alpha_1}} = \dfrac{d^{\alpha_1} y_1(t)}{dt^{\alpha_1}} - \dfrac{d^{\alpha} x_1(t)}{dt^{\alpha}} = ae_1(t) - e_2(t) \\ \qquad + (y_2(t)y_3(t) - y_2(t)y_3(t) + y_2(t)y_1^2(t) - x_2(t)x_1^2(t)) + u_1(t), \\ \dfrac{d^{\alpha_1} e_2(t)}{dt^{\alpha_1}} = \dfrac{d^{\alpha_1} y_2(t)}{dt^{\alpha_1}} - \dfrac{d^{\alpha} x_2(t)}{dt^{\alpha}} = e_1(t) + ae_2(t) \\ \qquad + (3y_1(t)y_3(t) - y_1^3(t) - 3x_1(t)x_3(t) + x_1^3(t)) + ay_2 + u_2(t), \\ \dfrac{d^{\alpha_1} e_3(t)}{dt^{\alpha_1}} = \dfrac{d^{\alpha_1} y_3(t)}{dt^{\alpha_1}} - \dfrac{d^{\alpha} x_3(t)}{dt^{\alpha}} \\ \qquad = -2be_3(t) - 2(y_1(t)y_2(t)y_3(t) - x_1(t)x_2(t)x_3(t)) + u_3(t). \end{cases} \tag{37}$$

The error dynamical system in the matrix notation is written as

$$\frac{d^{\alpha_1} e(t)}{dt^{\alpha_1}} = \mathbf{A}e(t) + \theta(x, y) + u(t), \tag{38}$$

where,

$$\mathbf{A} = \begin{bmatrix} a & -1 & 0 \\ 1 & a & 0 \\ 0 & 0 & -2b \end{bmatrix}; \qquad \theta(x, y) = \begin{bmatrix} (y_2 y_3 - x_2 x_3 + y_2 y_1^2 - x_2 x_1^2) \\ (3y_1 y_3 y_1^3 - 3x_1 x_3 + x_3^3) \\ (-2y_1 y_2 y_3 + 2x_1 x_2 x_3) \end{bmatrix};$$

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}.$$

Setting the the sliding mode controller, $u(t)$ as,

$$u(t) = -\theta(x, y) + \mathbf{B}\upsilon(t), \tag{39}$$

where B is taken such that $(\mathbf{A}, \mathbf{B})$ is controllable. We take $\mathbf{B}$ as $\mathbf{B} = (1, 1, 1)^T$.

The sliding mode variable is written as

$$s = Me = [-3, 2, 3]e = -3e_1 + 2e_2 + 3e_3, \tag{40}$$

we take $M = [4\ 4\ 1]$ such that the all eigenvalues of the matrix $E = [I - \mathbf{B}(M\mathbf{B})^{-1}M]\mathbf{A}$ are negative real part or zero. That is,

$$eig(E) = (-4.9365,\ -0.5135,\ 0)$$

which determines that the sliding mode state equation system is asymptotically stable. Taking the sliding mode gain as $\kappa = 4$, $\rho = 0.1$ and I is the identity matrix of order 3. It is found that the larger value of κ can cause chattering (noise) and an appropriate value of ρ is taken to increase the time for the sliding manifold as well as to reduce the system chattering (noise) [2].

From (24), time-invariant controlled signal with single input, $\upsilon(t)$ is obtained as

$$\begin{aligned} \upsilon(t) &= -(M\mathbf{B})^{-1}(M(\sigma I + \mathbf{A})e(t) + \rho\, sign(s)) \\ &= -\frac{1}{2}\left(\begin{bmatrix} -3 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4.86 & 1 & 0 \\ 1 & 4.86 & 0 \\ 0 & 0 & -2.4 \end{bmatrix} e(t) + \rho\, sign(s)\right); \end{aligned} \tag{41}$$

$$\upsilon(t) = \begin{bmatrix} 3.35 & -4.4 & -6.6 \end{bmatrix} e(t) - 0.05 sign(s).$$

Thus, the required sliding mode controller is obtained as

$$u(t) = -\theta(x, y) + \mathbf{B}\upsilon(t).$$

Thus, the following results is obtained.

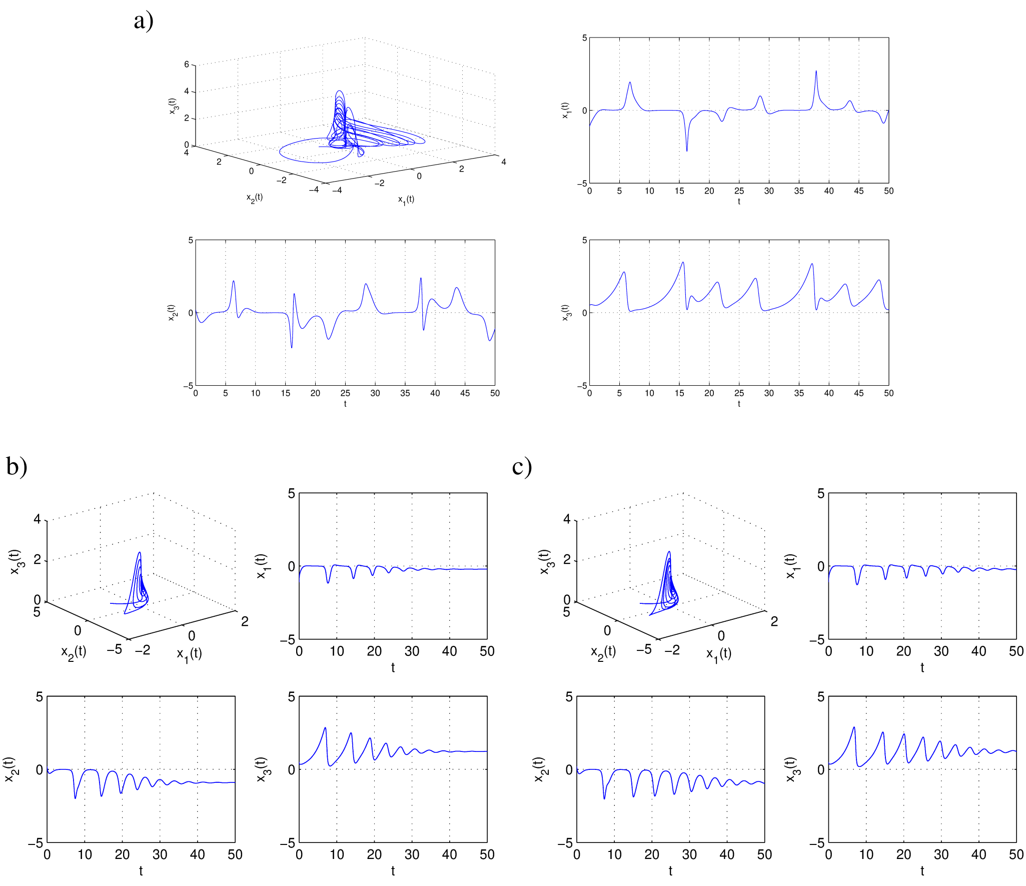
Remark 1 *The identical fractional order chaotic Robinovich-Fabricant systems* (34) *and* (35) *are asymptotically synchronized and stable with initial conditions by the sliding mode controller $u(t)$ which is defined in the equation* (39).

Simulation:

We take initial condition for master and slave fractional order chaotic Robinovich-Fabricant systems as

$$x(0) = (x_1(0), x_2(0), x_3(0)) = (-1.1, 0.2, 0.5)^T \text{ and}$$
$$y(0) = (y_1(0), y_2(0), y_3(0)) = (-1.5, 0.25, 0.75)^T.$$

Take the parameters values $a = -1.1$, $b = -0.2$ and the lowest fractional order $\alpha_1 = 0.95$. In Figure 1 (a–g) is shown the 3-D phase portrait with time series figures chaotic Robinovich-Fabricant systems at different fractional order $\alpha_1 = 1$, $\alpha_1 = 0.89$, $\alpha_1 = 0.90$, $\alpha_1 = 0.91$, $\alpha_1 = 0.92$, $\alpha_1 = 0.94$ and $\alpha_1 = 0.95$ respectively. Figure 2 (a–c) shown as bifurcation diagrams with respect to the parameters a along the x_1, a along the x_2 and b along the x_3 respectively. We have computed the Lyapunov exponent of chaotic Robinovich-Fabricant system at



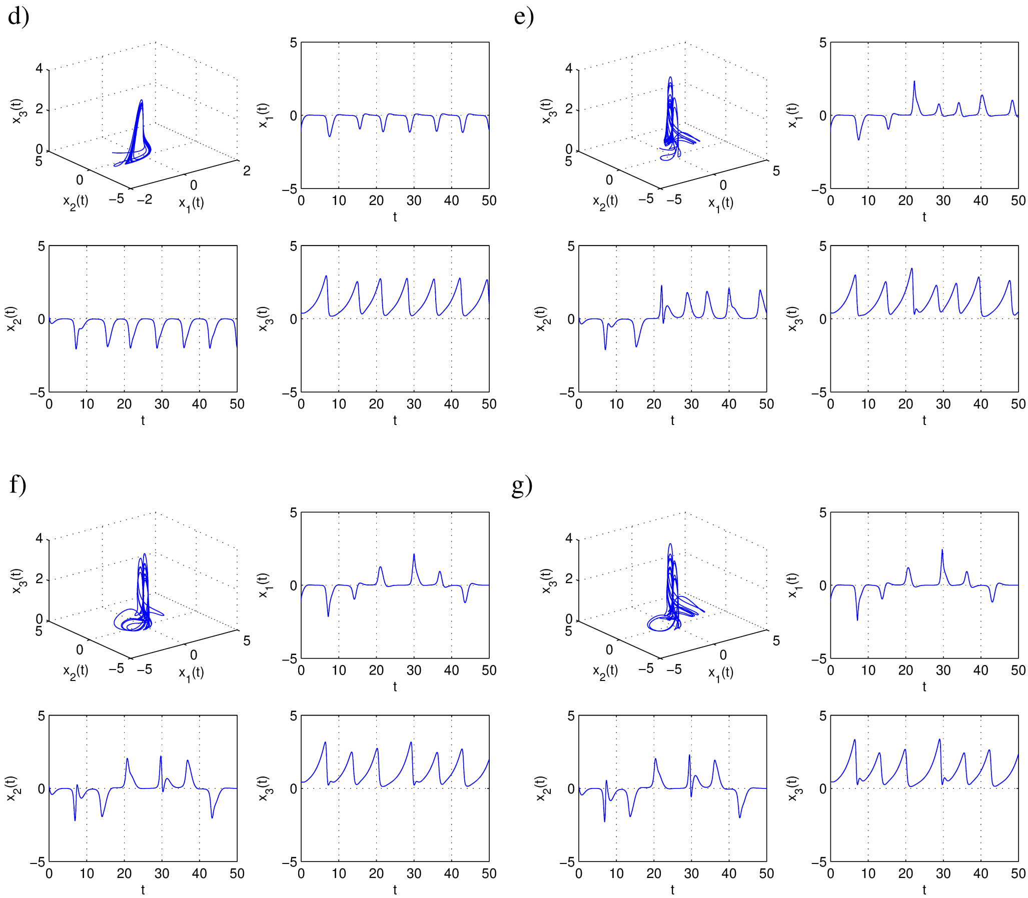


Figure 1: 3 Dimensional phase portrait with time series graphs of chaotic Robinovich-Fabricant systems (without controller).

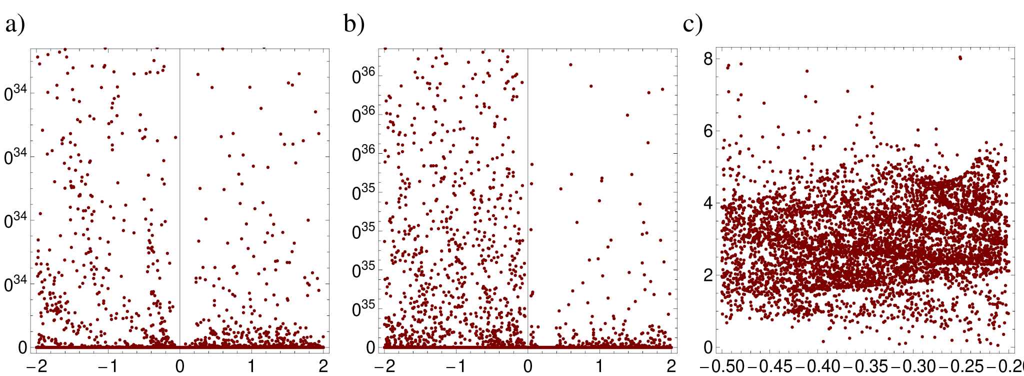


Figure 2: Bifurcation diagrams with with the parameter a along x_1 axis, a along x_2 axis and b along x_3 respectively

$t = 200$. We have $\lambda_1 = 0.19941$, $\lambda_2 = -0.038729$ and $\lambda_3 = -1.9607$. We observe that out of these three Lyapunov exponent values, one is positive, one is negative and one of these tends to zero which is required condition for chaotic systems. It represents that fractional order Robinovich-Fabricant system is chaotic. It is shown in Figure 3. Figures 4 (a–c) show the tracking the trajectories of slave to master fractional order Robinovich-Fabricant systems in x_1y_1, x_2y_2 and x_3y_3 with sliding mode controllers. Trajectories of master and slave identical fractional order Robinovich-Fabricant systems are synchronized with sliding mode control techniques in these figure 4 (a–c). Figure 5 is shown the synchronization of error dynamic of identical fractional order Robinovich-Fabricant systems in the form of $e_1e_2e_3$ with sliding mode control with respect to time t at the initial condition $e(0) = (e_1(0), e_2(0), e_3(0)) = (-0.4, 0.05, 0.25)^T$. That is,

$$\lim_{t\to\infty} e(t) = 0.$$

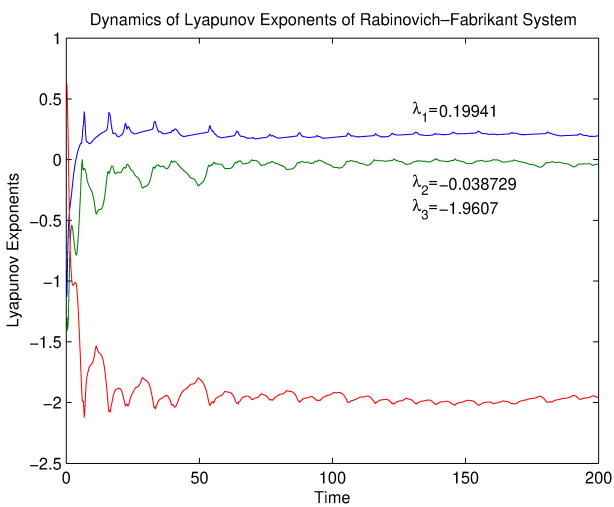


Figure 3: Lyapunov exponent of chaotic Robinovich-Fabricant systems

a) b) c)

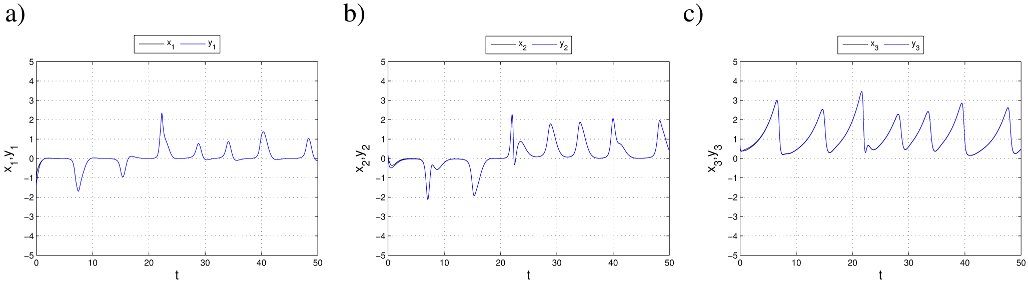


Figure 4: Tracking the trajectory of the master to the salve chaotic Robinovich-Fabricant systems with order $\alpha_1 = 0.95$ (with controller)

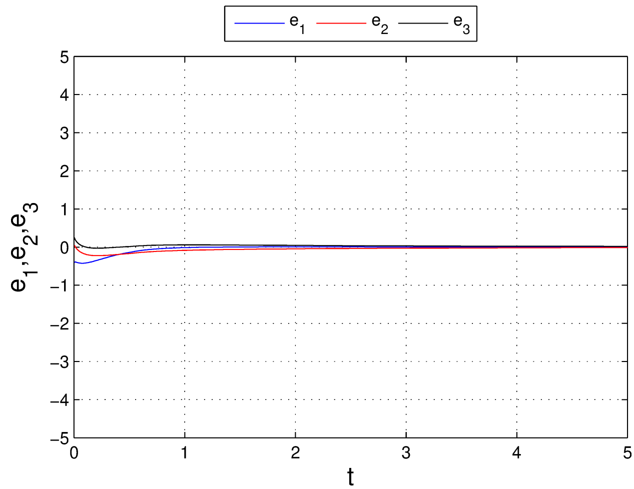


Figure 5: Synchronization of error dynamics of identical Robinovich-Fabricant systems

5. Conclusions

In this research article, we have addressed fractional order chaotic systems using sliding mode control techniques and the behaviour of fractional order Robinovich-Fabricant systems. We have obtained the lowest fractional order of Robinovich-Fabricant system through utilization of fractional calculus and computational simulations. It is 2.85. We have shown bifurcation diagrams and Lyapunov exponents fractional order chaotic Robinovich-Fabricant system. We have established the synchronization of two identical fractional order Robinovich-Fabricant systems using sliding mode control method.

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