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Invariant properties of positive linear electrical circuits

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Abstract: The invariant properties of the stability, reachability, observability and transfer matrices of positive linear electrical circuits with integer and fractional orders are investigated. It is shown that the stability, reachability, observability and transfer matrix of positive linear systems are invariant under their integer and fractional orders.

Key words: invariance, positive linear system, reachability, stability, transfer matrix

1. Introduction

An electrical circuit system is called fractional if it is described by a fractional order differential equation. The fundamentals of fractional calculus and fractional systems have been given in [23, 26–32]. The stability of fractional linear systems have been analyzed in [3−5].

In positive electrical circuits the inputs, state variables and outputs take only nonnegative values. Examples of positive systems are electrical circuits, industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc. An overview of state of the art in theory of positive systems is given in the monographs [2, 6, 13].

The determination of the matrices A, B, C, D of the state equations of linear systems for given transfer matrices is called the realization problem. The realization problem has been investigated in [12, 24, 25, 27]. A tutorial on the positive realization problem has been given in the paper [1] and in the books [6, 13]. The positive realization problem for linear systems with delays has been analyzed in [7, 8, 14, 20, 21, 27], for cone systems in [10] and positive stable realizations in [9, 15–17]. The existence and determination of the set of Metzler matrices for given stable polynomials have been considered in [11]. The realization problem for positive 2D hybrid systems has been addressed in [19]. For fractional linear systems the realization problem has been considered in [18, 26].

In this paper the invariant properties of the stability, reachability, observability and transfer matrices of positive linear systems and electrical circuits with integer and fractional orders will be investigated.

The paper is organized as follows. In section 2 the invariance of stability of the positive linear electrical circuits with integer and fractional orders is investigared. The invariance of the reachability of the positive linear systems is analyzed in section 3 and of the observability in section 4. The invariance of transfer matrices of positive linear systems is considered in section 5. The realization problem for positive asymptotically stable electrical circuits is addressed in section 6. Concluding remarks are given in section 7.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n \times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n \times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n – the set of $n \times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n \times n$ identity matrix.

2. Stability invariance of positive linear electrical circuits

Consider the autonomous linear electrical circuit described by the differential equation:

$$\dot{x}(t) = Ax(t), \tag{2.1}$$

where $x(t) \in \Re^n$ is the state vector and $A \in \Re^{n\times n}$.

As the state variables (components of the state vector) the voltages on the capacitors and the currents in the coils are usually chosen.

The electrical circuit described by (2.1) is called (internally) positive if $x(t) \in \Re^n$, $t \geq 0$ for any initial conditions $x(0) \in \Re_+^n$.

A matrix $A = [a_{ij}] \in \Re^{n\times n}$ is called the Metzler matrix if $a_{ij} \geq 0$ for $i \neq j$.

Theorem 2.1. [2, 6, 13] The electrical circuit (2.1) is positive if and only if A is a Metzler matrix.

The positive electrical circuit (2.1) is called asymptotically stable (the matrix A Hurwitz) if

$$\lim_{t\to\infty} x(t) = 0 \quad \text{for all} \quad x(0) \in \Re_+^n. \tag{2.2}$$

The positive electrical circuit (2.1) is asymptotically stable if and only if all real parts of eigenvalues s_k of the matrix A are negative, i.e. $\Re s_k < 0$ for $k = 1, \ldots, n$.

Theorem 2.2. [13] For the positive electrical circuit (2.1) the following conditions are equivalent:

1. The positive electrical circuit (2.1) is asymptotically stable (the Metzler matrix A is Hurwitz).
2. All coefficients of the characteristic polynomial

$$\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{2.3}$$

are positive, i.e. $a_i > 0$ for $i = 0, 1, \ldots, n-1$.

3. All principal minors M_i, $i = 1, \ldots, n$ of the matrix $-A$ are positive, i.e.

$$M_1 = |-a_{11}| > 0, \quad M_2 = \begin{vmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{vmatrix} > 0, \ldots, \quad M_n = \det[-A] > 0. \tag{2.4}$$

4. There exists strictly positive vector $\lambda = [\lambda_1 \ \cdots \ \lambda_n]$, $\lambda_k > 0$, $k = 1, \ldots, n$ such that

$$A\lambda < 0. \tag{2.5}$$

Remark 2.1. From (2.5) it follows that the positive electrical circuit (2.1) is asymptotically stable only if all diagonal entries of A are negative.

Consider the autonomous fractional linear electrical circuit described by the equation:

$$\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = Ax(t), \quad 0 < \alpha < 1, \tag{2.6}$$

where $x(t) \in \Re^n$ is the state vector and $A \in \Re^{n\times n}$, and

$$\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \frac{\dot{x}(\tau)}{(t-\tau)^\alpha} \mathrm{d}\tau, \quad \Gamma(x) = \int\limits_0^\infty e^{-t} t^{x-1} \mathrm{d}t, \tag{2.7}$$

is the Caputo definition of α order of $x(t)$.

The electrical circuit (2.6) is called positive if $x(t) \in \Re_+^n$, $t \geq 0$ for any initial conditions $x(0) \in \Re_+^n$.

Theorem 2.3. [23] The fractional electrical circuit is positive if and only if A is a Metzler matrix.

The positive electrical circuit (2.6) is called asymptotically stable (the matrix A Hurwitz) if

$$\lim_{t\to\infty} x(t) = 0 \quad \text{for all} \quad x(0) \in \Re_+^n. \tag{2.8}$$

The positive fractional electrical circuit (2.6) is asymptotically stable if and only if the real parts of all eigenvalues s_k of the matrix A are negative, i.e. $\Re\, s_k < 0$ for $k = 1, \ldots, n$ [23].

Theorem 2.4. For the positive fractional electrical circuit (2.6) the following conditions are equivalent:

1. The positive electrical circuit (2.6) is asymptotically stable (the Metzler matrix A is Hurwitz).
2. All coefficients of the characteristic polynomial

$$\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{2.9}$$

are positive, i.e. $a_i > 0$ for $i = 0, 1, \ldots, n-1$.

3. All principal minors M_i, $i = 1, \ldots, n$ of the matrix $-A$ are positive, i.e.

$$M_1 = |-a_{11}| > 0, \quad M_2 = \begin{vmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{vmatrix} > 0, \ldots, \quad M_n = \det[-A] > 0. \tag{2.10}$$

4. There exists strictly positive vector $\lambda = [\lambda_1 \ \cdots \ \lambda_n]$, $\lambda_k > 0$, $k = 1, \ldots, n$ such that

$$A\lambda < 0. \tag{2.11}$$

Remark 2.2. From (2.11) it follows that the positive fractional electrical circuit (2.6) is asymptotically stable only if all diagonal entries of A are negative.

From comparison of Theorems 2.1 and 2.2 with Theorems 2.3 and 2.4 we have the following important collorary respectively.

Collorary 2.1. The stability of positive linear electrical circuits is invariant under their (integer and fractional) orders.

These considerations can be extended to positive linear electrical circuits with delays in state vectors.

3. Reachability invariance of the positive linear electrical circuits

Consider the standard linear electrical circuit described by the equation:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{3.1}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$ are the state and input vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$.

Definition 3.1. [13, 23] The linear electrical circuit (3.1) is called (internally) positive if $x(t) \in \Re_+^n$ and all $u(t) \in \Re_+^m$, $t \geq 0$.

Theorem 3.1. [13, 23] The linear electrical circuit (3.1) is positive if and only if

$$A \in M_n, \quad B \in \Re_+^{n\times m}. \tag{3.2}$$

Definition 3.2. [13, 23] The positive electrical circuit (3.1) is called reachable in the time $[0, t_f]$, $t_f > 0$, if there exists the input $u(t) \in \Re_+^m$ for $t \in [0, t_f]$ which steers the state of electrical circuit from $x(0) = 0$ to the given final state $x_f \in \Re_+^n$, i.e. $x(t_f) = x_f$.

Theorem 3.2. [13, 23] The linear positive electrical circuit (3.1) is reachable in the time $[0, t_f]$ if and only if the reachability matrix

$$R(t_f) = \int_0^{t_f} e^{A\tau} BB^T e^{A^T\tau}(\tau)\,\mathrm{d}\tau \in \Re_+^{n\times n} \tag{3.3}$$

is a monomial matrix.

The input $u(t) \in \Re_+^m$, $t \in [0, t_f]$ which steers the state of system from $x(0) = 0$ to the given final state $x_f \in \Re_+^n$, is given by

$$u(\tau) = B^T e^{A^T(t_f-\tau)} R^{-1}(t_f)\, x_f \in \Re_+^m, \quad \tau \in [0, t_f]. \tag{3.4}$$

Consider the fractional continous-time linear system:

$$\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = Ax(t) + Bu(t), \quad 0 < \alpha < 1, \tag{3.5}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$ are the state and input vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$ and the Caputo derivative of $x(t)$ is defined by (2.7).

Definition 3.3. [13, 23] The fractional positive electrical circuit (3.5) is called reachable in the time $[0, t_f]$, $t_f > 0$, if there exists an input $u(t) \in \Re^m$ for $t \in [0, t_f]$ which steers the state of electrical circuit from $x(0) = 0$ to the given final state $x_f \in \Re_+^n$, i.e. $x(t_f) = x_f$.

Theorem 3.3. The fractional positive electrical circuit (3.5) is reachable in the time $[0, t_f]$ if and only if the reachability matrix

$$\overline{R}(t_f) = \int_0^{t_f} \Phi(\tau) B B^T \Phi^T(\tau)\,\mathrm{d}\tau \in \Re_+^{n\times n} \tag{3.6}$$

is a monomial matrix, where

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}\,.$$

The input $u(t)$ which steers the state of the system from $x(0) = 0$ to $x_f = x(t_f) \in \Re^n$ is given by

$$u(\tau) = B^T \Phi^T(t_f - \tau)\overline{R}^{-1}(t_f) x_f \in \Re_+^m, \qquad \tau \in [0, t_f]. \tag{3.7}$$

Proof. It is well-known [23] that $\overline{R}^{-1}(t_f) \in \Re_+^{n\times n}$ if and only if the matrix (3.6) is monomial. Substituting (3.7) into

$$x(t_f) = \int_0^{t_f} \Phi(t_f - \tau) B u(\tau)\,\mathrm{d}\tau \tag{3.8}$$

we obtain

$$x(t_f) = \int_0^{t_f} \Phi(t_f-\tau) B B^T \Phi^T(t_f-\tau)\overline{R}^{-1}(t_f) x_f\,\mathrm{d}\tau = \int_0^{t_f} \Phi(\tau) B B^T \Phi^T(\tau)\,\mathrm{d}\tau\,\overline{R}^{-1}(t_f) x_f = x_f\,. \tag{3.9}$$

Therefore, the input (3.7) steers the state of the electrical circuit from $x(0) = 0$ to $x(t_f) = x_f$.

Theorem 3.4. The fractional positive linear electrical circuit is reachable in the time $[0, t_f]$ if and only if the positive linear electrical circuit (3.1) is reachable in the same interval $[0, t_f]$.

Proof. Note that the reachability matrices (3.3) and (3.6) of the positive electrical circuit (3.1) and of fractional positive electrical circuit (3.5) differ only by the transition matrices e^{At} for the electrical circuit and $\Phi(t)$ (defined by (3.6)) for the fractional electrical circuit. Using the well-known Cayley–Hamilton theorem or the Lagrange–Sylvester formula [12, 23] it is possible to write the transition matrices in the forms:

$$e^{At} = \sum_{k=0}^{n-1} c_k(t) A^k \tag{3.10}$$

and

$$\Phi(t) = \sum_{k=0}^{n-1} \overline{c}_k(t) A^k, \tag{3.11}$$

where $c_k(t)$ and $\overline{c}_k(t)$ for $k = 0, 1, \ldots, n-1$ are nonzero linearly independent functions of time t [22].

Therefore, the reachability matrix (3.6) is monomial if and only if the reachability matrix (3.3) is monomial. By Theorems 3.2 and 3.3 the fractional positive electrical circuit (3.5) is reachable in the time $[0, t_f]$ if and only if the positive electrical circuit (3.1) is reachable in the interval $[0, t_f]$.

Therefore, from Theorem 3.4 we have the following important conclusion.

Conclusion 3.1. The reachability of positive linear electrical circuits is invariant under their (integer and fractional) orders.

4. Observability invariance of the positive electrical circuits

Consider the linear electrical circuit described by the equations:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{4.1a}$$

$$y(t) = Cx(t), \tag{4.1b}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$.

Definition 4.1. [13] The electrical circuit (4.1) is called (internally) positive if and only if $x \in \Re_+^n$, $y(t) \in \Re_+^p$, $t \geq 0$ for any $u(t) \in \Re_+^m$, $t \geq 0$ and all initial conditions $x(0) \in \Re_+^n$.

Theorem 4.1. [13] The electrical circuit (4.1) is positive if and only if

$$A \in M_n, \quad B \in \Re_+^{n\times m}, \quad C \in \Re_+^{p\times n}. \tag{4.2}$$

Definition 4.2. The positive electrical circuit (4.1) is called (strongly) observable in the interval of $[0, t_f]$ if by knowing the input $u(t)$ and output $y(t)$ for $[0, t_f]$ it is possible to find the unique $x(0) \in \Re_+^n$ of the electrical circuit.

Theorem 4.2. The positive electrical circuit (4.1) is observable in the interval $[0, t_f]$ if and only if the matrix

$$W_f = \int_0^{t_f} e^{A^T t} C^T C e^{At}\,\mathrm{d}t \tag{4.3}$$

is monomial.

Proof. Assuming $B = 0$ and premultiplying the equation

$$y(t) = Ce^{At}x(0) \tag{4.4}$$

by $e^{At}C^T$ we obtain

$$e^{A^T t} C^T C e^{At} x(0) = e^{A^T t} C^T y(t). \tag{4.5}$$

Integrating (4.5) on the interval $[0, t_f]$ we obtain

$$\int_0^{t_f} e^{A^T t} C^T C e^{At}\,\mathrm{d}t\, x(0) = \int_0^{t_f} e^{A^T t} C^T y(t)\,\mathrm{d}t \tag{4.6}$$

and

$$x(0) = W_f^{-1} \int_0^{t_f} e^{A^T t} C^T y(t)\,\mathrm{d}t \in \Re_+^n, \tag{4.7}$$

if and only if the matrix (4.3) is monomial.

Consider the fractional linear electrical circuit

$$\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = Ax(t) + Bu(t), \quad 0 < \alpha < 1, \tag{4.8a}$$

$$y(t) = Cx(t), \tag{4.8b}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$ and the Caputo derivative of $x(t)$ is defined by (2.7).

Definition 4.3. [13, 23] The fractional electrical circuit (4.8) is called (internally) positive if $x \in \Re_+^n$, $y(t) \in \Re_+^p$, $t \geq 0$ for any $u(t) \in \Re_+^m$, $t \geq 0$ and all initial conditions $x(0) \in \Re_+^n$.

Theorem 4.3. The fractional electrical circuit (4.8) is positive if and only if

$$A \in M_n, \quad B \in \Re_+^{n\times m}, \quad C \in \Re_+^{p\times n}. \tag{4.9}$$

Definition 4.4. The fractional positive electrical circuit (4.8) is called observable in the interval $[0, t_f]$ if by knowing the input $u(t)$ and output $y(t)$ for $[0, t_f]$ it is possible to find the unique $x(0) \in \Re_+^n$ of the electrical circuit.

Theorem 4.4. The solution of Equation (4.8a) has the form:

$$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)\,\mathrm{d}\tau, \tag{4.10a}$$

where

$$\Phi_0(t) = \sum_{i=0}^{\infty} \frac{A^i t^{i\alpha}}{\Gamma(i\alpha + 1)}, \quad \Phi(t) = \sum_{i=0}^{\infty} \frac{A^i t^{(i+1)\alpha - 1}}{\Gamma[(i+1)\alpha]}. \tag{4.10b}$$

Theorem 4.5. The positive fractional electrical circuit (4.8) is observable in the interval $[0, t_f]$ if and only if the matrix

$$W_\alpha = \int_0^{t_f} \Phi_0^T(t) C^T C \Phi_0(t)\,\mathrm{d}t \tag{4.11}$$

is monomial.

Proof. Using (4.10a) for $B = 0$ and (4.8) and premultiplying the equation

$$y(t) = C\Phi_0 x(0), \tag{4.12}$$

by $\Phi_0^T(t)C^T$ we obtain

$$\Phi_0^T(t) C^T C \Phi_0(t) x(0) = \Phi_0^T(t) C^T y(t). \tag{4.13}$$

Integrating (4.13) on the interval $[0, t_f]$ we obtain

$$\int_0^{t_f} \Phi_0^T(t) C^T C \Phi_0(t)\,\mathrm{d}t\, x(0) = \int_0^{t_f} \Phi_0^T(t) C^T y(t)\,\mathrm{d}t \tag{4.14}$$

and

$$x(0) = W_\alpha^{-1} \int_0^{t_f} \Phi_0^T(t) C^T y(t)\,\mathrm{d}t, \tag{4.15}$$

if and only if the matrix (4.11) is monomial.

Theorem 4.6. The positive fractional electrical circuit (4.8) is observable in the interval $[0, t_f]$ if and only if the positive electrical circuit (4.1) is observable in the same interval $[0, t_f]$.

Proof. Using (4.10b) we obtain

$$\Phi_0(t) = \sum_{i=0}^{\infty} \frac{A^i t^{i\alpha}}{\Gamma(i\alpha + 1)} \in \Re_+^{n\times n} \quad \text{for} \quad t \geq 0 \quad \text{and} \quad 0 < \alpha < 1 \tag{4.16}$$

if and only if

$$e^{At} = \sum_{i=0}^{\infty} \frac{A^i t^i}{i!} \in \Re_+^{n\times n}, \quad t \geq 0. \tag{4.17}$$

From (4.16) it follows that $\Phi_0(t) \in \Re_+^{n\times n}$, $t \geq 0$ if and only if $e^{At} \in \Re_+^{n\times n}$, $t \geq 0$ and this implies that the matrix (4.11) is monomial if and only if the matrix (4.2) is monomial. Therefore, the positive electrical circuit (4.8) is observable if and only if the positive electrical circuit (4.1) is observable.

Therefore, from Theorem 4.6 we have the following important conclusion.

Conclusion 4.1. The observability of positive linear electrical circuits is invariant under their (integer and fractional) orders.

5. Transfer matrix invariance of the positive linear electrical circuits

The transfer matrix of the electrical circuit (4.1) is given by

$$T(s) = C[I_n s - A]^{-1} B. \tag{5.1}$$

The matrices (4.2) called the positive realization of the transfer matrix $T(s)$ if they satisfy (5.1) and it is called asymptotically stable realization if the matrix A is an asymptotically stable Metzler matrix (Hurwitz Metzler matrix).

Theorem 5.1. [6, 13, 23] The positive realization (4.2) is asymptotically stable if and only if all coefficients of the polynomial

$$p_A(s) = \det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{5.2}$$

are positive, i.e. $a_i > 0$ for $i = 0, 1, \ldots, n-1$.

The positive realization problem can be stated as follows. Given the proper transfer matrix $T(s)$ find its positive realization (4.2).

Theorem 5.2. [27] If (4.2) is a positive realization of (5.1) then the matrices

$$\bar{A} = PAP^{-1}, \quad \bar{B} = PB, \quad \bar{C} = CP^{-1} \tag{5.3}$$

are also a positive realization of (5.1) if the matrix $P \in \Re_+^{n\times n}$ is a monomial matrix.

Now let us consider the positive fractional electrical circuit (4.3)

The transfer matrix of the electrical circuit (4.3) is given by

$$T(\lambda) = C[I_n\lambda - A]^{-1}B, \quad \lambda = s^\alpha. \tag{5.4}$$

The positive realization problem for the fractional electrical circuit (4.3) can be stated in a similar way as for the positive electrical circuit (5.1) substituting $\lambda = s^\alpha$.

Theorem 5.3. If the matrix $A \in M_n$ is Hurwitz and $B \in \Re_+^{n\times m}$, $C \in \Re_+^{p\times n}$ then all coefficients of the transfer matrix (5.4) are positive.

Proof is given in [27].

Example 5.1. Consider the positive linear electrical circuit shown in Fig. 1 with known resistances R_1, R_2, R_3 inductances L_1, L_2 and source voltages $e_1 = e_1(t)$, $e_2 = e_2(t)$. The currents $i_1 = i_1(t)$, $i_2 = i_2(t)$ in the inductances are chosen as the state variables.

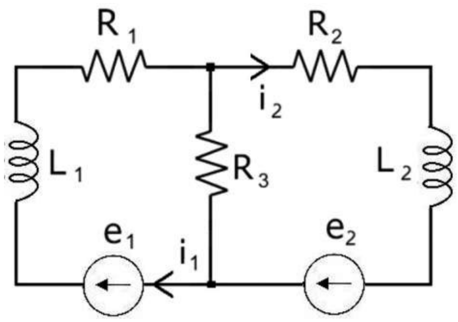

Fig. 1. Positive electrical circuit

Using Kirchhoff's laws we may write the equations:

$$\begin{aligned} e_1 &= R_1 i_1 + L_1\frac{\mathrm{d}i_1}{\mathrm{d}t} + R_3(i_1 - i_2),\\ e_2 &= R_2 i_2 + L_2\frac{\mathrm{d}i_2}{\mathrm{d}t} + R_3(i_2 - i_1) \end{aligned} \tag{5.5a}$$

and we choose

$$y = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}. \tag{5.5b}$$

Equations (5.5) can be written in the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{5.6a}$$

$$y = C \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}, \tag{5.6b}$$

where

$$A = \begin{bmatrix} -\frac{R_1 + R_3}{L_1} & \frac{R_3}{L_1} \\ \frac{R_3}{L_2} & -\frac{R_2 + R_3}{L_2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}, \quad C = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}. \tag{5.6c}$$

The matrix A defined by (5.6c) is an asymptotically stable Metzler matrix since its characteristic polynomial

$$\det[I_2 s - A] = \begin{vmatrix} s+\frac{R_1+R_3}{L_1} & -\frac{R_3}{L_1} \\ -\frac{R_3}{L_2} & s+\frac{R_2+R_3}{L_2} \end{vmatrix} = s^2 + \left(\frac{R_1+R_3}{L_1} + \frac{R_2+R_3}{L_2}\right) s + \frac{R_1(R_2+R_3)+R_2R_3}{L_1L_2} \tag{5.7}$$

has positive coefficients.

The transfer matrix of the positive electrical circuit has the form:

$$T(s) = C\,[I_2 s - A]^{-1} B = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} s + \frac{R_1 + R_3}{L_1} & -\frac{R_3}{L_1} \\ -\frac{R_3}{L_2} & s + \frac{R_2 + R_3}{L_2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix} =$$
$$= \frac{R_1(R_2 + R_3) + R_2R_3}{s^2 + [(R_1 + R_3)L_1 + (R_2 + R_3)L_2]s + R_1(R_2 + R_3) + R_2R_3}. \tag{5.8}$$

Note that all coefficients of the transfer matrix (5.6b) are positive. This confirm the thesis of Theorem 5.3.

Example 5.2. Consider the positive fractional electrical circuit shown in Fig. 2 with known resistances R_1, R_2, R_3 capacitances C_1, C_2 and source voltage $e = e(t)$.

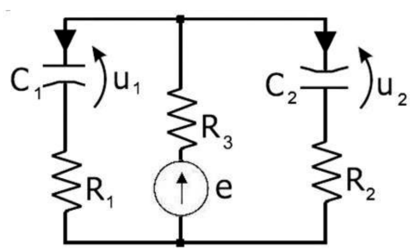

Fig. 2. Positive fractional electrical circuit

Using Kirchhoff's laws we may write the equations:

$$
\begin{aligned}
e' &= R_1 C_1 \frac{\mathrm{d}^\alpha u_1}{\mathrm{d}t^\alpha} + u_1 + R_3 \left(C_1 \frac{\mathrm{d}^\alpha u_1}{\mathrm{d}t^\alpha} + C_2 \frac{\mathrm{d}^\alpha u_2}{\mathrm{d}t^\alpha} \right), \\
e &= R_2 C_2 \frac{\mathrm{d}^\alpha u_2}{\mathrm{d}t^\alpha} + u_2 + R_3 \left(C_1 \frac{\mathrm{d}^\alpha u_1}{\mathrm{d}t^\alpha} + C_2 \frac{\mathrm{d}^\alpha u_2}{\mathrm{d}t^\alpha} \right),
\end{aligned}
\tag{5.9a}
$$

and we choose

$$
y = u_1 + u_2. \tag{5.9b}
$$

Equations (5.9) can be rewritten in the form:

$$
\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = A \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + Be, \tag{5.10a}
$$

$$
y = C \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{5.10b}
$$

where

$$
\begin{aligned}
A &= \begin{bmatrix} -\dfrac{R_2 + R_3}{C_1[R_1(R_2 + R_3) + R_2 R_3]} & \dfrac{R_3}{C_1[R_1(R_2 + R_3) + R_2 R_3]} \\ \dfrac{R_3}{C_2[R_1(R_2 + R_3) + R_2 R_3]} & \dfrac{R_1 + R_3}{C_2[R_1(R_2 + R_3) + R_2 R_3]} \end{bmatrix}, \\
B &= \begin{bmatrix} \dfrac{R_2}{C_1[R_1(R_2 + R_3) + R_2 R_3]} \\ \dfrac{R_1}{C_2[R_1(R_2 + R_3) + R_2 R_3]} \end{bmatrix}, \qquad C = [1 \;\; 1].
\end{aligned}
\tag{5.10c}
$$

The matrix A defined by (5.10c) is an asymptotically stable Metzler matrix since its characteristic polynomial

$$
\begin{aligned}
\det[I_2\lambda - A] &= \begin{vmatrix} \lambda + \dfrac{R_2 + R_3}{C_1[R_1(R_2 + R_3) + R_2 R_3]} & -\dfrac{R_3}{C_1[R_1(R_2 + R_3) + R_2 R_3]} \\ -\dfrac{R_3}{C_2[R_1(R_2 + R_3) + R_2 R_3]} & \lambda + \dfrac{R_1 + R_3}{C_2[R_1(R_2 + R_3) + R_2 R_3]} \end{vmatrix} = \\
&= \lambda^2 + \left(\frac{R_2 + R_3}{C_1[R_1(R_2 + R_3) + R_2 R_3]} + \frac{R_1 + R_3}{C_2[R_1(R_2 + R_3) + R_2 R_3]} \right) \lambda + \\
&\quad + \frac{(R_1 + R_3)(R_2 + R_3) + R_3^2}{C_1 C_2 [R_1(R_2 + R_3) + R_2 R_3]^2}
\end{aligned}
\tag{5.11}
$$

has positive coefficients.

The transfer function of the fractional positive electrical circuit has the form:

$$T(s) = C[I_2\lambda - A]^{-1}B =$$
$$= [1\ \ 1]\begin{bmatrix} \lambda + \frac{R_2+R_3}{C_1[R_1(R_2+R_3)+R_2R_3]} & -\frac{R_3}{C_1[R_1(R_2+R_3)+R_2R_3]} \\ -\frac{R_3}{C_2[R_1(R_2+R_3)+R_2R_3]} & \lambda + \frac{R_1+R_3}{C_2[R_1(R_2+R_3)+R_2R_3]} \end{bmatrix}^{-1} \begin{bmatrix} \frac{R_2}{C_1[R_1(R_2+R_3)+R_2R_3]} \\ \frac{R_1}{C_2[R_1(R_2+R_3)+R_2R_3]} \end{bmatrix} = \tag{5.12}$$
$$= \frac{[R_2C_2[R_1(R_2+R_3)+R_2R_3]+R_1C_1[R_1(R_2+R_3)+R_2R_3]]\,\lambda+(R_2+R_3)R_1+(R_1+R_3)R_2+R_3R_1+R_3R_2}{C_1C_2[R_1(R_2+R_3)+R_2R_3]^2\lambda^2+[((R_2+R_3)C_2+(R_1+R_3))\,C_1[R_1(R_2+R_3)+R_2R_3]\,\lambda+(R_2+R_3)(R_1+R_3)-R_3^2}.$$

All coefficients of the transfer matrix (5.12) are positive and this confirm the thesis of Theorem 5.3.

Example 5.3. Consider the positive electrical circuit shown in Fig. 3 with known resistances R_1, R_2, R_3, R_4 inductances L_2, L_3 capacitances C_1, C_4 and source voltages e_1, e_2, e_3.

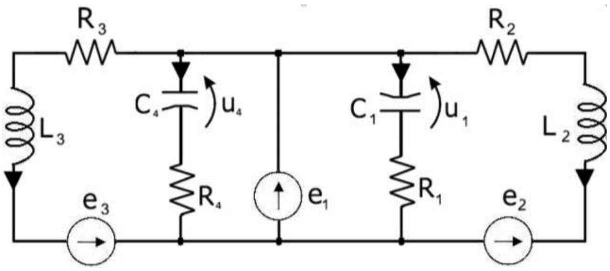

Fig. 3. Positive electrical circuit

Using Kirchhoff's laws we may write the equations:

$$\begin{aligned} e_1 &= u_j + R_jC_j\frac{\mathrm{d}u_j}{\mathrm{d}t}, \quad j = 1.4, \\ e_1 + e_k &= R_ki_k + L_k\frac{\mathrm{d}i_k}{\mathrm{d}t}, \quad k = 2.3 \end{aligned} \tag{5.13a}$$

and we choose

$$y = u_1 + i_2 + i_3\,. \tag{5.13b}$$

Equations (5.13) can be written in the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u_1 \\ u_4 \\ i_2 \\ i_3 \end{bmatrix} = A\begin{bmatrix} u_1 \\ u_4 \\ i_2 \\ i_3 \end{bmatrix} + B\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \qquad y = C\begin{bmatrix} u_1 \\ u_4 \\ i_2 \\ i_3 \end{bmatrix}, \tag{5.14a}$$

where

$$A = \operatorname{diag}\begin{bmatrix} -\frac{1}{R_1C_1} & -\frac{1}{R_4C_4} & -\frac{R_2}{L_2} & -\frac{R_3}{L_3} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{R_1C_1} & 0 & 0 \\ \frac{1}{R_4C_4} & 0 & 0 \\ \frac{1}{L_2} & \frac{1}{L_2} & 0 \\ \frac{1}{L_3} & 0 & \frac{1}{L_3} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}. \tag{5.14b}$$

The diagonal matrix A is asymptotically stable since its characteristic polynomial

$$\begin{aligned} \det[I_4 s - A] &= \det\left[\operatorname{diag}\begin{bmatrix} s + \frac{1}{R_1C_1} & s + \frac{1}{R_4C_4} & s + \frac{R_2}{L_2} & s + \frac{R_3}{L_3} \end{bmatrix}\right] = \\ &= s^4 + \frac{(C_1R_1 + C_4R_4)L_2L_3 + (L_2R_3 + L_3R_2)C_1R_1C_4R_4}{R_1C_1R_4C_4L_2L_3}s^3 + \\ &+ \frac{L_2L_3 + (L_2R_3 + L_3R_2)(C_1R_1 + C_4R_4) + C_1R_1C_4R_4R_2R_3}{R_1C_1R_4C_4L_2L_3}s^2 + \\ &+ \frac{(C_1R_1 + C_4R_4)R_2R_3}{R_1C_1R_4C_4L_2L_3}s + \frac{R_2R_3}{R_1C_1R_4C_4L_2L_3} \end{aligned} \tag{5.15}$$

has positive coefficients.

The transfer matrix of the electrical circuit has the form:

$$\begin{aligned} T(s) &= C[I_2 s - A]^{-1}B = \\ &= \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}\left\{\operatorname{diag}\begin{bmatrix} s + \frac{1}{R_1C_1} & s + \frac{1}{R_4C_4} & s + \frac{R_2}{L_2} & s + \frac{R_3}{L_3} \end{bmatrix}\right\}^{-1}\begin{bmatrix} \frac{1}{R_1C_1} & 0 & 0 \\ \frac{1}{R_4C_4} & 0 & 0 \\ \frac{1}{L_2} & \frac{1}{L_2} & 0 \\ \frac{1}{L_3} & 0 & \frac{1}{L_3} \end{bmatrix} = \\ &= \begin{bmatrix} \frac{1}{sR_1C_1 + 1} + \frac{1}{sL_2 + R_2} + \frac{1}{sL_3 + R_3} & \frac{1}{sL_2 + R_2} & \frac{1}{sL_2 + R_2} \end{bmatrix}. \end{aligned} \tag{5.16}$$

All coefficients of the transfer matrix (5.16) are positive. This confirm the thesis of Theorem 5.3.

6. Realization problem for positive asymptotically stable electrical circuits

The realization problem for positive asymptotically stable linear electrical circuits can be stated as follows. Given the transfer matrix $T(s)$ with positive coefficients, find positive asymptotically stable linear electrical circuit with matrices (4.2) satisfying (5.1).

Methods for computation of positive realizations of linear systems for a given transfer matrix have been proposed in [8, 9, 15, 16, 28].

Example 6.1. Find the values of resistances R_1, R_2, R_3 and inductances L_1, L_2 of the positive electrical circuit shown in Fig. 1 with the transfer matrix

$$T(s) = \begin{bmatrix} \dfrac{11}{s^2 + 13s + 11} & \dfrac{11}{s^2 + 13s + 11} \end{bmatrix}. \tag{6.1}$$

The comparison of the coefficients (5.8) and (6.1) yields the equalities:

$$\begin{aligned} R_1(R_2 + R_3) + R_2R_3 &= 11, \\ (R_1 + R_3)L_2 + (R_2 + R_3)L_1 &= 13. \end{aligned} \tag{6.2}$$

It is easy to check if the equalities (5.15) are satisfied by

$$R_1 = 1, \quad R_2 = 2, \quad R_3 = 3 \quad \text{and} \quad L_1 = 1, \quad L_2 = 2. \tag{6.3}$$

The desired positive and asymptotically stable electrical circuit shown in Fig. 1 has the resistances and inductances given by (6.3).

The problem can be also solved by the method given in [26].

Theorem 6.1. There exists a positive asymptotically stable realization (5.2) of (6.1) only if its coefficients are positive.

Proof is given in [26].

Similar results can be obtained for the positive fractional linear systems.

Theorem 6.2. If the matrix $A \in M_n$ is Hurwitz and $B \in \Re+^{n\times m}$, $C \in \Re_+^{p\times n}$ of the positive fractional system then all coefficients of its transfer matrix are positive.

Theorem 6.3. [26] There exists a positive asymptotically stable realization of (6.1) only if its coefficients are positive.

Therefore, from the above considerations we have the following important conclusion.

Conclusion 6.1. The positivity of the coefficients of the transfer matrices of the positive linear continuous-time systems is invariant under their (integer and fractional) orders.

Note that the above considerations can be easily extended to the standard and positive when the output equation has the form:

$$y(t) = Cx(t) + Du(t), \tag{6.4}$$

where $D \in \Re_+^{p\times m}$.

7. Concluding remarks

The invariant properties of the stability, reachability, observability and transfer matrices of positive linear electrical circuits with integer and fractional orders have been investigated. It has been shown that:

1. The stability of positive linear circuits is invariant under their integer and fractional orders (Theorems 2.2 and 2.4).
2. The reachability of positive linear electrical circuits is invariant under their integer and fractional orders (Theorem 3.4 and Collorary 3.1).
3. The observability of positive linear electrical circuits is invariant under their integer and fractional orders (Theorem 4.6).
4. The transfer matrix of positive linear systems is invariant under their integer and fractional orders (Theorems 6.3 and Conclusion 6.1).

The considerations can be extended to positive linear discrete-time systems.

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