

# Geometry and flatness of $m$ -crane systems

 M. NOWICKI<sup>1,2\*</sup>, W. RESPONDEK<sup>2</sup>, J. PIASEK<sup>1</sup>, and K. KOZŁOWSKI<sup>1</sup>
<sup>1</sup> Poznan University of Technology, Institute of Automatic Control and Robotic, Piotrowo 3a, 61-138 Poznań, Poland

<sup>2</sup> Normandie Université, INSA de Rouen, Laboratoire de Mathématiques, 76801 Saint-Etienne-du-Rouvray, France

**Abstract.** We propose a class of  $m$ -crane control systems that generalizes two- and three-dimensional crane systems. We prove that each representative of the described class is feedback equivalent to the second order chained form with drift. In consequence, we prove that it is differentially flat. Then we investigate its control properties and derive a control law for tracking control problem.

**Key words:** differential flatness, crane, chained systems, feedback equivalence, accessibility, nonlinear control.

## 1. Introduction

Several authors considered crane systems, both two- and three-dimensional, and investigated their structural properties [2], flatness [5], motion planning and tracking [1, 16]. Independently of the dimension, these systems share common properties that can be further generalized. In this article, we propose a class of  $m$ -crane control systems constituting a multidimensional generalization of the above-mentioned systems. We describe this class, investigate its properties and prove that  $m$ -crane systems are feedback equivalent to a normal form, more precisely, to the second order chained form with drift. For that chained form, we prove flatness of differential weight  $5m$  (where  $m$  is the number of controls). This article is organized as follows. In Section 2, we study the  $m$ -crane and derive its equations by calculating the zero dynamics of a constrained system. In Section 3, we show that the system is feedback equivalent to the normal form. Section 4 presents results considering flatness of  $m$ -crane systems. Finally, in Section 5 we use the fact that for a control system to be flat is equivalent to be dynamically linearizable, to derive a control law for the trajectory tracking problem and show, in Section 6, simulation results for the 2-crane system. We stress that the presented dynamic linearization is exact, in the sense that the nonlinearities of the system are fully compensated by a change of coordinates and feedback and should not be confused with the linear approximation, where nonlinearities are neglected.

## 2. Modelling the class of $m$ -crane systems

In this section, a model of  $m$ -crane systems will be proposed and studied. It is a multidimensional generalization of a system that is known in the literature as an overhead crane, see [2, 5], which

have been an inspiration for our study. For better readability, we start with the simplest case of 2-crane<sup>1</sup> and then we extend our considerations to the general case.

**2.1. 2-crane.** Consider a pendulum of mass  $\mu$  attached to a cart (of mass  $M$ ) that moves in the  $X$ -direction. The position of the cart (measured at the point where the rope is connected to the winch) is denoted by  $d$ . We assume that the rope is wound around the winch, therefore the connection point does not change its vertical component and its coordinates are thus  $(d, 0)$ . The forces acting on the mass  $\mu$  are the tension of the rope  $T$  and the gravitational force  $F_g = \mu g$ . The tension can be projected along the horizontal direction  $X$  and the vertical direction  $Z$  as  $T_x = T \sin \theta$  and  $T_z = T \cos \theta$ . The cart is subject to a friction force  $c_d \dot{d}$ , the tension of the rope  $T_x$ , and an external force  $\tilde{\mathcal{F}}$  that is controlled. There is a winch (of radius  $b$  and moment of inertia  $J$ ) on the cart, that changes the length of the rope  $r$ . It is influenced by a friction  $c_r \dot{r}$ , tension  $b^2 T$ , and external torque  $\tilde{\mathcal{C}}$ . We neglect the dynamics of the winch itself, but we take into account the torque that the movement of the winch produces, that serves as a second control in the system. The position of the pendulum is expressed with respect to the global frame  $(X, Z)$  via coordinates  $(x, z)$ . See Fig. 1.

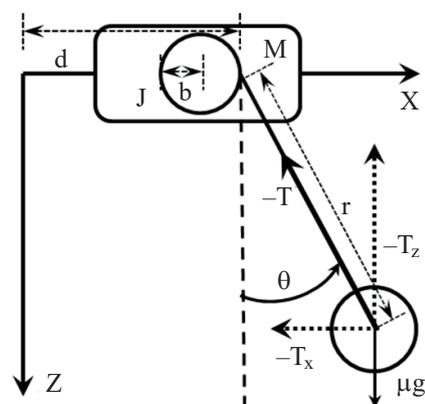


Fig. 1. The overhead two dimensional crane control system

\*e-mail: marcin.s.nowicki@doctorate.put.poznan.pl

<sup>1</sup> see Remark 2

Manuscript submitted 2018-12-18, revised 2019-06-06, initially accepted for publication 2019-06-14, published in October 2019

Based on these simple relations we can write the dynamics of the system:

$$\begin{aligned} \mu \ddot{x} &= -T \sin \theta \\ \mu \ddot{z} &= -T \cos \theta + \mu g \\ M \ddot{d} &= -c_d \dot{d} + \tilde{\mathcal{F}} + T \sin \theta \\ J \ddot{r} &= -c_r \dot{r} - b \tilde{\mathcal{C}} + b^2 T, \end{aligned} \tag{1}$$

that are subject to the following constraints (recall that the connection point of the rope has coordinates  $(d, 0)$ ):

$$\begin{aligned} x &= r \sin \theta + d \\ z &= r \cos \theta. \end{aligned} \tag{2}$$

At first, a feedback is designed to remove dissipative terms, which are irrelevant to this study

$$\begin{aligned} \tilde{\mathcal{F}} &= c_d \dot{d} + \mathcal{F} \\ \tilde{\mathcal{C}} &= \frac{1}{b} (-c_r \dot{r} + \mathcal{C}). \end{aligned}$$

While the above set of differential-algebraic equations (1, 2) is easy to derive, it is obscure when represented as a control system. The reason is that the model is over-represented. First, there is a supplementary configuration variable  $\theta$  that can be eliminated from (1) due to the algebraic equations (2). Second, the variable  $T$  is free (i.e. there is no differential equation for it), so it is a driving variable. Third, it seems like there are 4 degrees of freedom, since there are 4 equations of motion but because of the algebraic (holonomic) constraints (2) the number of degrees of freedom is actually 3. In order to express (1, 2) as a classical mechanical control system of the form  $\ddot{\xi} = F(\xi, \dot{\xi}) + \sum_{i=1}^m G_i(\xi) u_i$ , we will eliminate the holonomic constraints. First, we will get rid of the extra variable  $\theta$ . From the constraints (2), we calculate  $\sin \theta = \frac{x-d}{r}$  and  $\cos \theta = \frac{z}{r}$  and plug them into (1)

$$\begin{aligned} \ddot{x} &= -T \frac{x-d}{r\mu} \\ \ddot{z} &= -T \frac{z}{r\mu} + g \\ \ddot{d} &= \frac{1}{M} \mathcal{F} + T \frac{x-d}{rM} \\ \ddot{r} &= -\frac{1}{J} \mathcal{C} + \frac{b^2}{J} T. \end{aligned} \tag{3}$$

System (3) can be considered as a system that evolves on the tangent bundle  $T\Xi = \{(\xi, \dot{\xi}) : \xi \in \Xi, \dot{\xi} \in T_\xi \Xi\}$  of the 4-dimensional configuration manifold  $\Xi$  with coordinates  $\xi = (x, z, d, r) \in \mathbb{R}^3 \times \mathbb{R}_+ = \Xi$  and three driving variables (free variables, i.e. differentially unconstrained variables) are  $(\mathcal{F}, \mathcal{C}, T)$ , and subject to one holonomic constraint acquired from (2):

$$\rho(\xi) := (x-d)^2 + z^2 - r^2 = 0, \tag{4}$$

which describes a cone in  $\mathbb{R}^2 \times \mathbb{R}_+$ , translated by  $d$  along the variable  $x$ , the apex being excluded by  $r > 0$ .

In this setting, it is immediate to realize what is the role of  $T$  in the system. This variable is controlled by “the Nature” in order to satisfy the constraints of the system. In classical mechanics such variables are called Lagrange multipliers and are well studied [9]. Although for any  $T$  there exists a solution of (3), only particular choices of  $T$  lead to solutions satisfying additionally (4). The driving variable  $T$ , interpreted as a control, forces the solutions of (3) to stay on the submanifold  $Q := \{\xi \in \Xi : \rho(\xi) = 0\}$ . It is natural to find this 3-manifold and to restrict the motion to it. This is the idea behind various methods of “eliminating Lagrange multipliers” [18]. Although there are many natural direct methods, we propose a different approach (similar to the one used in [8], pp. 108). It is our belief that it gives an interesting insight into the nature of the problem.

### 2.2. The constrained system represented as zero dynamics.

For the sake of simplicity, we formulate the following method for the case, when control system has a single constraint. Consider a smooth mechanical system

$$\ddot{\xi} = F(\xi, \dot{\xi}) + \sum_{i=1}^m G_i(\xi) u_i + a(\xi) \lambda \tag{5}$$

where  $\xi \in \Xi \subset \mathbb{R}^n$ , the controls  $u \in \mathbb{R}^m$ , and  $\lambda$  is a Lagrange multiplier to be chosen to fulfill the holonomic constraint

$$\rho(\xi) = 0. \tag{6}$$

We can consider the function  $\rho$  as an  $\mathbb{R}$ -valued output of system (5) and let us assume that its relative degree with respect to  $\lambda$  is well defined and equals two (meaning that the second order time-derivative of  $\rho(\xi(t))$  depends explicitly on  $\lambda$ ). It follows that locally  $\frac{\partial \rho}{\partial \xi_i} \neq 0$  and, without loss of generality, we can suppose that  $\frac{\partial \rho}{\partial \xi_n} \neq 0$  (if not, we permute  $\xi_i$  and  $\xi_n$ ) and we put  $w = \rho(\xi)$  and  $z_j = \xi_j$ ,  $1 \leq j \leq n-1$ . In  $(w, z)$ -coordinates the system reads

$$\begin{aligned} \dot{z}_j &= F_j(w, \dot{w}, z, \dot{z}) + \sum_{i=1}^m G_{ji}(w, z) u_i + a_j(w, z) \lambda \\ \dot{w} &= F_n(w, \dot{w}, z, \dot{z}) + \sum_{i=1}^m G_{ni}(w, z) u_i + a_n(w, z) \lambda, \end{aligned}$$

for  $1 \leq j \leq n-1$ . The constraint equation  $\rho(\xi) = 0$  becomes  $w = 0$  and implies  $\dot{w} = \ddot{w} = 0$ . The relative degree is two, thus  $a_n(w, z) \neq 0$ , and  $\lambda$  can be explicitly calculated as a function of  $z$  and  $u$  as

$$\lambda = -\frac{1}{a_n} \left( F_n + \sum_{i=1}^m G_{ni} u_i \right) \Big|_{w=\dot{w}=0} = \alpha(z, \dot{z}) + \sum_{i=1}^m \beta_i(z) u_i$$

and plugging it into the remaining equations justifies the following.

**Proposition 1.** The zero dynamics of system (5, 6) define the constrained system

$$\ddot{z}_j = F_j + \sum_{i=1}^m G_{ji} u_i + a_j \left( \alpha + \sum_{i=1}^m \beta_i u_i \right) = \tilde{F}_j + \sum_{i=1}^m \tilde{G}_{ji} u_i \quad (7)$$

whose configuration manifold  $Q := \{\xi \in \Xi : \rho(\xi) = 0\}$  is of dimension  $n - 1$ , and is equipped with coordinates  $z$ . For the so obtained control system, the dimension of the state space  $TQ$  is  $2n - 2$  and the number of controls is  $m$ .

To summarize, the method consists of differentiating  $w = \rho(\xi)$  two times along the dynamics and calculating  $\lambda = \alpha(z, \dot{z}) + \sum_{i=1}^m \beta_i(z) u_i$  that forces the system to evolve on the manifold  $Q = \{\xi \in \Xi : \rho(\xi) = 0\}$  and thus to respect the constraint.

**2.3. 2-crane control system.** Based on the method formulated in the previous subsection, the mechanical system of 2-crane will be derived. The constraint is  $w = \rho(\xi) = (x - d)^2 + z^2 - r^2 = 0$  and thus

$$\begin{aligned} \dot{w} &= 2\left((x - d)(\dot{x} - \dot{d}) + z\dot{z} - r\dot{r}\right) = 0 \\ \ddot{w} &= 2\left(\left(\dot{x} - \dot{d}\right)^2 + \dot{z}^2 - \dot{r}^2 + (x - d)(\ddot{x} - \ddot{d}) + \right. \\ &\quad \left. + z\ddot{z} - r\ddot{r}\right) = 0 \end{aligned} \quad (8)$$

implying that the relative degree with respect to  $T$  is, indeed, two because  $\ddot{x}$ ,  $\ddot{z}$  and  $\ddot{r}$  depend explicitly on  $T$ . From (4) and (8) we calculate  $r$  and  $\dot{r}$  and, using (3), put into the above

$$\begin{aligned} \ddot{w} &= 2\left(\frac{\left((x - d)\dot{z} - z(\dot{x} - \dot{d})\right)^2}{(x - d)^2 + z^2} + z\dot{g} - \frac{x - d}{M}\mathcal{F} + \right. \\ &\quad \left. + \frac{\sqrt{(x - d)^2 + z^2}}{J}\mathcal{C} - \frac{\varepsilon_1(x - d)^2 + \varepsilon_2 z^2}{\mu M \sqrt{(x - d)^2 + z^2}}T\right) = \\ &= \eta(x, \dot{x}, z, \dot{z}, d, \dot{d}) + \tau_1(x, d)\mathcal{F} + \tau_2(x, z, d)\mathcal{C} + \\ &\quad + \kappa(x, z, d)T = 0, \end{aligned}$$

where  $\varepsilon_1 = \mu + M + \frac{\mu M b^2}{J}$ ,  $\varepsilon_2 = M + \frac{\mu M b^2}{J}$ , and  $\eta$ ,  $\tau_1$ ,  $\tau_2$ ,  $\kappa$  are smooth functions of the indicated variables. Calculating  $T$  as

$$T = -\frac{\eta + \tau_1 \mathcal{F} + \tau_2 \mathcal{C}}{\kappa} = -(\alpha + \beta_1 \mathcal{F} + \beta_2 \mathcal{C}), \quad (9)$$

and plugging into (3) gives

$$\begin{aligned} \ddot{x} &= (\alpha + \beta_1 \mathcal{F} + \beta_2 \mathcal{C}) \frac{x - d}{\mu \sqrt{(x - d)^2 + z^2}} \\ \ddot{z} &= (\alpha + \beta_1 \mathcal{F} + \beta_2 \mathcal{C}) \frac{z}{\mu \sqrt{(x - d)^2 + z^2}} + g \\ \ddot{d} &= \frac{1}{M}\mathcal{F} - (\alpha + \beta_1 \mathcal{F} + \beta_2 \mathcal{C}) \frac{x - d}{M \sqrt{(x - d)^2 + z^2}}. \end{aligned} \quad (10)$$

Note that since  $\kappa \neq 0$ , the tension  $T$  is well defined, and thus the description is global on  $Q$ . Indeed, from Proposition 1 it follows that the constrained system (10) evolves on the tangent bundle  $TQ = Q \times \mathbb{R}^3$  of the configuration manifold  $Q = \{(x, z, d) \in \mathbb{R}^3 : x - d \neq 0 \text{ and } z \neq 0\}$ . It is a control-affine system of the form (7), with  $m = 2$ , that is, with two inputs.

Now apply to (10) static invertible feedback given by

$$\begin{aligned} u_1 &= \frac{1}{M}\mathcal{F} \\ u_2 &= \frac{1}{\mu \sqrt{(x - d)^2 + z^2}} (\alpha + \beta_1 \mathcal{F} + \beta_2 \mathcal{C}), \end{aligned}$$

which brings the 2-crane system into the form

$$\begin{aligned} \ddot{x} &= u_2(x - d) \\ \ddot{z} &= u_2 z + g \\ \ddot{d} &= u_1 - u_2 \frac{\mu}{M}(x - d). \end{aligned} \quad (11)$$

**2.4.  $m$ -crane.** While it has been reported that the system with one additional degree of freedom, (see [5] and Remark 1 below), the so called three dimensional crane (3-crane, for short) is somehow analogous, it seems that it has never been investigated in details. As it will be shown, the 2-crane and 3-crane belong to a larger class of systems that will be described in this section. Therefore, we deliberately skip modeling the 3-crane and go directly to an arbitrary dimension.

**Remark 1.** In the literature (and the colloquial language), the 2-crane is called the two dimensional crane. Notice, however, that despite the numbers match, they have different meaning. In fact, the name 2-crane refers to the number of controls (the  $m$ -crane has  $m$  controls) while “two-dimensional” refers to a crane that moves in a plane. We avoid the popular name, since it is misleading: the term “two dimensional” brings to mind that the configuration space is 2-dimensional, which is not the case (it is actually 3-dimensional).

The precedent analysis can be generalized to any number of dimensions. The model of  $m$ -crane consists of a varying-length rope with a load attached (a pendulum), in an  $m$ -dimensional Euclidean space, hooked on to a platform that moves in the first  $m - 1$  directions (all being controlled). The change of the length of the pendulum is carried out by a winch mounted on the platform. The movement of the pendulum in the  $m - th$  direction is influenced by the gravitational acceleration  $g$ . The configuration of the end-point of the pendulum is in an  $(m - 1)$ -dimensional sphere of radius  $r$  (which can vary) so the system has  $2m - 1$  configuration,  $2(2m - 1)$  states (configurations and velocities) and  $m$  controls. The position of the platform (measured at the point, where the rope is connected to the winch) is  $(d_1, d_2, \dots, d_{m-1})$ , therefore the origin of the sphere has coordinates  $(d_1, d_2, \dots, d_{m-1}, 0)$ . Denote by  $\theta_i$  the angle in the  $(X_i, X_{i+1})$ -plane form  $X_i$  axis with the range  $0 \leq \theta_i < 2\pi$  for  $1 \leq i \leq m - 2$  and  $\theta_{m-1}$  being in the range of  $0 \leq \theta_{m-1} \leq \pi$ .

The configuration of the end-point of the pendulum is described by a Cartesian coordinate system as

$$\begin{aligned} x_1 - d_1 &= r \cos \theta_1 = rS_1 \\ x_2 - d_2 &= r \sin \theta_1 \cos \theta_2 = rS_2 \\ x_3 - d_3 &= r \sin \theta_1 \sin \theta_2 \cos \theta_3 = rS_3 \\ &\vdots \\ x_{m-1} - d_{m-1} &= r \prod_{i=1}^{m-2} \sin \theta_i \cos \theta_{m-1} = rS_{m-1} \\ x_m &= r \prod_{i=1}^{m-2} \sin \theta_i \sin \theta_{m-1} = rS_m, \end{aligned} \tag{12}$$

which describes constraints of the system. The equations of dynamics of the pendulum, the platform, and the winch are:

$$\begin{aligned} \mu_m \ddot{x}_i &= -TS_i && \text{for } 1 \leq i \leq m-1 \\ \mu_m \ddot{x}_m &= -TS_m + \mu_m g \\ \mu_i \ddot{d}_i &= -c_i \dot{d}_i + \tilde{\mathcal{F}}_i + TS_i && \text{for } 1 \leq i \leq m-1 \\ J \ddot{r} &= -c_r \dot{r} - b \tilde{\mathcal{C}} + b^2 T \end{aligned} \tag{13}$$

where  $\mu_m$  is the mass of the end-point (load),  $\mu_1, \dots, \mu_{m-1}$  are mass parameters of the platform in each of  $m-1$  directions (it may vary depending on the construction),  $J$  is the moment of inertia of the winch.

Similarly, as in the case of the 2-crane, the dissipative terms can be compensated by an appropriate feedback

$$\begin{aligned} \tilde{\mathcal{F}}_i &= c_i \dot{d}_i + \mathcal{F}_i \\ \tilde{\mathcal{C}} &= \frac{1}{b} (-c_r \dot{r} + \mathcal{C}). \end{aligned}$$

From (12) we calculate

$$\begin{aligned} S_i &= \frac{x_i - d_i}{r} && \text{for } 1 \leq i \leq m-1 \\ S_m &= \frac{x_m}{r} \end{aligned}$$

and plug into (13), which gives the system

$$\begin{aligned} \mu_m \ddot{x}_i &= -T \frac{x_i - d_i}{r} && \text{for } 1 \leq i \leq m-1 \\ \mu_m \ddot{x}_m &= -T \frac{x_m}{r} + \mu_m g \\ \mu_i \ddot{d}_i &= \mathcal{F}_i + T \frac{x_i - d_i}{r} && \text{for } 1 \leq i \leq m-1 \\ J \ddot{r} &= -\mathcal{C} + b^2 T \end{aligned} \tag{14}$$

that evolves on  $T\Xi$ , where  $\Xi = \mathbb{R}^{2m-1} \times \mathbb{R}_+$  consists of configurations  $(x, d, r) \in \Xi$ , subject to the holonomic constraint

$$\begin{aligned} \rho(\xi) &= (x_1 - d_1)^2 + \dots + (x_{m-1} - d_{m-1})^2 + \\ &+ x_m^2 - r^2 = 0. \end{aligned} \tag{15}$$

We identify the zero dynamics, see section 2.2, defined by above constraint (15), by calculating

$$\begin{aligned} w &= \rho(\xi) = 0 \\ \dot{w} &= \frac{d}{dt} \rho(\xi) = 0 \\ \ddot{w} &= \frac{d^2}{dt^2} \rho(\xi) = 0 \end{aligned}$$

and we express  $r, \dot{r}, T$  (using the first, second, and third equation, respectively) as functions of  $(x_1, \dots, x_m, d_1, \dots, d_{m-1})$  and their derivatives. The solution for  $T$  is of the form

$$T = \alpha + \sum_{i=1}^{m-1} \beta_i \mathcal{F}_i + \beta_m \mathcal{C}$$

and we plug it into (14) to get, for  $1 \leq i \leq m-1$ ,

$$\begin{aligned} \mu_m \ddot{x}_i &= -\left(\alpha + \sum_{i=1}^{m-1} \beta_i \mathcal{F}_i + \beta_m \mathcal{C}\right) \frac{x_i - d_i}{r} \\ \mu_m \ddot{x}_m &= -\left(\alpha + \sum_{i=1}^{m-1} \beta_i \mathcal{F}_i + \beta_m \mathcal{C}\right) \frac{x_m}{r} + \mu_m g \\ \mu_i \ddot{d}_i &= \mathcal{F}_i + \left(\alpha + \sum_{i=1}^{m-1} \beta_i \mathcal{F}_i + \beta_m \mathcal{C}\right) \frac{x_i - d_i}{r}. \end{aligned}$$

We scale the controls by setting  $u_i = \frac{1}{\mu_i} \mathcal{F}_i$  and replace the last control by  $u_m = -\frac{1}{\mu_m r} \left(\alpha + \sum_{i=1}^{m-1} \beta_i \mathcal{F}_i + \beta_m \mathcal{C}\right) = -\frac{1}{\mu_m r} T$ .

Thus the  $m$ -crane system

$$\begin{aligned} \ddot{x}_i &= u_m (x_i - d_i), && \text{for } 1 \leq i \leq m-1, \\ \ddot{x}_m &= u_m x_m + g \\ \ddot{d}_i &= u_i - u_m \frac{\mu_m}{\mu_i} (x_i - d_i), && \text{for } 1 \leq i \leq m-1, \end{aligned} \tag{16}$$

evolves on  $TQ = Q \times \mathbb{R}^{2m-1}$ , where  $Q = \{(x_i, x_m, d_i) \in \mathbb{R}^{2m-1}\}$  is the configuration manifold, equipped with the coordinates  $q = (x_1, \dots, x_m, d_1, \dots, d_{m-1})$ , and consisting of points that respect the constraint (15).

**Remark 2.** One could ask why the simplest considered example is that of the 2-crane. By Section 2.4 it is fairly easy to deduce what would be the 1-crane. It is a pendulum in 0-sphere, which is a pair of points  $\{-r, r\}$ , where  $r$  can vary, and one equation of the dynamics  $J \ddot{r} = -c_r \dot{r} - b \tilde{\mathcal{C}}$ , with one control  $\tilde{\mathcal{C}}$ . This system is simply (static) feedback linearizable to the double integrator.

### 3. Equivalence of the $m$ -crane to the second order chained form with drift

In this section we will show that the  $m$ -crane (16) can be further simplified and brought to a normal form, that is the second order chained form with a constant drift. For better readability, we first explain the normal form for the 2-crane, and then

we give it for the general case. Consider 2-crane system (11) and introduce a new coordinate  $s = \frac{\mu}{M}x + d$ . The system in  $(s, x, z)$ -coordinates reads

$$\begin{aligned} \dot{s} &= u_1 \\ \dot{x} &= u_2(x - d) = u_2(\bar{M}x - s) \\ \dot{z} &= u_2z + g, \end{aligned} \quad (17)$$

where  $\bar{M} = \frac{\mu + M}{M}$ . Until now all transformations have been global on  $Q \cong \mathbb{R}^n$ . Now, we realize that system (17) exhibits a singularity at  $z = 0$  since the third equation reads  $\dot{z} = u_2z + g$ , is independent of the remaining equations and controls, and its linear approximation at  $z = 0$  is not controllable. Therefore we restrict our consideration assuming  $z \neq 0$ , i.e., now the configuration manifold is  $Q = \{(s, x, z) \in \mathbb{R}^3 : z \neq 0\}$ . We apply the following feedback

$$\begin{aligned} \tilde{u}_1 &= u_1 \\ \tilde{u}_2 &= u_2z, \end{aligned} \quad (18)$$

and the system is

$$\begin{aligned} \dot{s} &= \tilde{u}_1 \\ \dot{x} &= \tilde{u}_2 \frac{\bar{M}x - s}{z} \\ \dot{z} &= \tilde{u}_2 + g, \end{aligned}$$

and we change the coordinates

$$\begin{aligned} z_1 &= \frac{\bar{M}x - s}{z} \\ z_2 &= z \\ z_3 &= x, \end{aligned}$$

which results in the following system

$$\begin{aligned} \dot{z}_1 &= \bar{\alpha} + \bar{\beta}_1 \tilde{u}_1 + \bar{\beta}_2 \tilde{u}_2 \\ \dot{z}_2 &= \tilde{u}_2 + g \\ \dot{z}_3 &= z_1 \tilde{u}_2, \end{aligned}$$

where:

$$\begin{aligned} \bar{\alpha} &= \frac{2\dot{z}}{z^3} \left( (\bar{M}x - s)\dot{z} - (\bar{M}\dot{x} - \dot{s})z \right) - g \frac{\bar{M}x - s}{z^2} \\ \bar{\beta}_1 &= -\frac{1}{z} \\ \bar{\beta}_2 &= (\bar{M} - 1) \frac{\bar{M}x - s}{z^2}, \end{aligned}$$

which, after applying feedback  $v_1 = \bar{\alpha} + \bar{\beta}_1 \tilde{u}_1 + \bar{\beta}_2 \tilde{u}_2$ ,  $v_2 = \tilde{u}_2$ , results in the normal form

$$\begin{aligned} \dot{z}_1 &= v_1 \\ \dot{z}_2 &= v_2 + g \\ \dot{z}_3 &= z_1 v_2. \end{aligned}$$

Throughout this section, the index  $i$  satisfies  $1 \leq i \leq m - 1$ . In order to formulate the general result, define  $Q^+ = \{(x_i, x_m, d_i) \in \mathbb{R}^{2m-1} : x_m > 0\}$  and  $Q^- = \{(x_i, x_m, d_i) \in \mathbb{R}^{2m-1} : x_m < 0\}$ .

**Proposition 2.** The  $m$ -crane, given by (16) on  $Q^+$  and on  $Q^-$ , is globally static feedback equivalent to the second order chained form with a constant drift vector field:

$$\begin{aligned} \dot{z}_i &= v_i \\ \dot{z}_m &= v_m + g \\ \dot{z}_{m+i} &= z_i v_m. \end{aligned} \quad (19)$$

**Proof.** First, introduce new  $m - 1$  global coordinates:

$$s_i = \frac{\mu_m}{\mu_i} x_i + d_i$$

and keep the  $m$  coordinates  $(x_1, \dots, x_m)$ . The transformed system in  $(s, x)$ -coordinates reads

$$\begin{aligned} \dot{s}_i &= u_i \\ \dot{x}_i &= u_m (\bar{\mu}_i x_i - s_i) \\ \dot{x}_m &= x_m u_m + g, \end{aligned}$$

where  $\bar{\mu}_i = \frac{\mu_i + \mu_m}{\mu_i}$ .

The normal form (19) is obtained by applying feedback

$$\begin{aligned} \tilde{u}_i &= u_i \\ \tilde{u}_m &= u_m x_m \quad x_m \neq 0, \end{aligned}$$

which yields the following system

$$\begin{aligned} \dot{s}_i &= \tilde{u}_i \\ \dot{x}_i &= \tilde{u}_m \frac{\bar{\mu}_i x_i - s_i}{x_m} \\ \dot{x}_m &= \tilde{u}_m + g. \end{aligned}$$

Introduce new global coordinates on  $Q^+$  and on  $Q^-$  by

$$\begin{aligned} z_i &= \frac{\bar{\mu}_i x_i - s_i}{x_m} \\ z_m &= x_m \\ z_{m+i} &= x_{i+1} \end{aligned}$$

and the system reads

$$\begin{aligned} \dot{z}_i &= \bar{\alpha}_i + \sum_{j=1}^m \bar{\beta}_{ij} \tilde{u}_j \\ \dot{z}_m &= \tilde{u}_m + g \\ \dot{z}_{m+i} &= z_i \tilde{u}_m, \end{aligned}$$

which, after applying feedback  $v_i = \bar{\alpha}_i + \sum_{j=1}^m \bar{\beta}_{ij} \tilde{u}_j$ ,  $v_m = \tilde{u}_m$ , gives normal form (19).  $\square$

#### 4. Flatness of m-crane

The notion of flatness and of flat systems was proposed and then intensively studied in the '90s by Fliess, Lévine, Martin and Rouchon [3, 4], Jakubczyk [7], Pomet [14, 15], Murray [10], and others. Despite extensive efforts, a complete characterization of flatness is still unknown. Apart from theoretical challenge, it has attracted a lot of attention because of its applications in control design, e.g. constructive controllability problem and trajectory tracking. Among various (equivalent) formulation of the notion of flatness, we will use the following one

**Definition 1.** The system

$$\Sigma : \dot{\zeta} = F(\zeta, u), \quad (20)$$

where  $\zeta \in X \subset \mathbb{R}^N$  and  $u \in U \subset \mathbb{R}^m$ , is (locally) flat at  $(\zeta_0, \bar{u}_0^l)$ , where  $\zeta_0 \in X$  and  $\bar{u}_0^l = (u, \dot{u}, \dots, u^{(l)}) \in U \times \mathbb{R}^{ml}$ , for  $l \geq -1$ , if there exist  $m$  smooth functions (flat outputs)  $\phi_i = \phi_i(\zeta, u, \dot{u}, \dots, u^{(l)})$ , defined in a neighborhood  $\mathcal{O}^l$  of  $(\zeta_0, \bar{u}_0^l)$ , such that the state and the controls can be represented as smooth maps of  $\phi = (\phi_1, \dots, \phi_m)$  and their finite number of derivatives

$$\begin{aligned} \zeta &= \gamma(\phi, \dot{\phi}, \dots, \phi^{(s)}) \\ u &= \delta(\phi, \dot{\phi}, \dots, \phi^{(s)}), \end{aligned}$$

along any trajectory  $\zeta(t)$  given by a control  $u(t)$  that satisfy  $(\zeta(t), u(t), \dot{u}(t), \dots, u^{(l)}(t)) \in \mathcal{O}^l$ . If the functions  $\phi = (\phi_1, \dots, \phi_m)$  and the maps  $\gamma$  and  $\delta$  are defined globally, then the system is called *globally flat*.

In general, flat outputs (if they exist) are not unique and a way to systematize them is by their *differential weight* [17], which is the minimal number of derivatives of a flat output  $\phi$ , needed to express  $\zeta$  and  $u$ . Formally, consider a flat output  $\phi$ , such that

$$\begin{aligned} \zeta &= \gamma(\phi_1, \dot{\phi}_1, \dots, \phi_1^{(s_1)}, \dots, \phi_m, \dot{\phi}_m, \dots, \phi_m^{(s_m)}) \\ u &= \delta(\phi_1, \dot{\phi}_1, \dots, \phi_1^{(s_1)}, \dots, \phi_m, \dot{\phi}_m, \dots, \phi_m^{(s_m)}), \end{aligned}$$

where  $s_m$  stands for the highest order of time-derivatives of  $\phi_i$  present in  $\gamma$  or  $\delta$ . We will call  $\sum_{i=1}^m (s_i + 1) = m + \sum_{i=1}^m s_i$  the differential weight of  $\phi$ . A flat output is called *minimal* if its differential weight is the lowest among all flat output of  $\Sigma$ . We define the *differential weight* of a flat system to be equal to the differential weight of its minimal flat output. What is more, the differential weight is equal to  $N + m + r$ , where  $r$  is the minimal possible dimension of a precompensator defining a dynamic feedback that linearizes the system. Systems that are linearizable by static ( $r = 0$ ) feedback are flat with differential weight  $N + m$ , for details see [11].

For flat mechanical systems, we may distinguish another interesting property that many of them share, namely *configuration flatness* (config-flat). Note that, in the case of mechanical systems, we have  $\zeta = (q, \dot{q})$ , where  $N = 2n$ , with  $q$  denoting

configurations and  $\dot{q}$  are velocities. We say that a mechanical system is *config-flat* if all flat outputs  $\phi_i$  depend on the configuration variables  $q$  only. This property was studied by Murray et al. in [10].

Now we formulate some general results describing the  $m$ -crane system. Recall that system (20) is called strongly accessible at  $\zeta$  if, for any  $T > 0$  the set of points reachable from  $\zeta$  in time  $T$  has nonempty interior, see e.g. [6, 13].

**Proposition 3.** The  $m$ -crane system (16) is strongly accessible at any  $(q, \dot{q}) \in TQ^\pm$ , where  $Q^\pm = \{(x_i, x_m, d_i) \in \mathbb{R}^{2m-1} : x_m \neq 0, 1 \leq i \leq m-1\} = Q^+ \cup Q^-$ .

**Proof.** By Proposition 2, the  $m$ -crane is feedback equivalent to (19) on  $Q^\pm$ . It is well known that the strong accessibility is feedback invariant, therefore we will show strong accessibility for normal form (19). Denote  $\zeta = (z, \dot{z})$  and express system (19) as  $\dot{\zeta} = F(\zeta) + \sum_{i=1}^m v_i G_i(\zeta)$ . It is strongly accessible at  $\zeta \in Z$  if  $\dim \mathcal{L}_0(\zeta) = \dim Z$ , where the Lie ideal  $\mathcal{L}_0$  of the system is the Lie ideal generated by  $G_1, \dots, G_m$  in the Lie algebra  $\mathcal{L} = \{F, G_1, \dots, G_m\}_{LA}$ , see e.g. [6, 13]. In the case of mechanical system (19),  $Z = TQ^\pm$  so the condition for strong accessibility is  $\dim \mathcal{L}_0(\zeta) = 2(2m - 1)$ . We will take the following set of vector fields  $\{G_i, [G_j, ad_F G_m], ad_F G_i, [ad_F G_j, ad_F G_m]\}$ , for  $1 \leq i \leq m$  and  $1 \leq j \leq m - 1$ , that belong to  $\mathcal{L}_0$  and show that they span TZ at every  $\zeta = (z, \dot{z}) = (q, \dot{q})$ . We have

$$F = \sum_{i=1}^{2m-1} \dot{z}_i \frac{\partial}{\partial z_i} + g \frac{\partial}{\partial z_m},$$

and the vector fields

$$\begin{aligned} G_j &= \frac{\partial}{\partial \dot{z}_j} \quad \text{for } 1 \leq j \leq m - 1, \\ G_m &= \frac{\partial}{\partial \dot{z}_m} + \sum_{i=1}^{m-1} z_i \frac{\partial}{\partial \dot{z}_{m+i}}, \\ [G_j, ad_F G_m] &= \frac{\partial}{\partial \dot{z}_{m+j}} \quad \text{for } 1 \leq j \leq m - 1, \\ ad_F G_j &= -\frac{\partial}{\partial z_j} \quad \text{for } 1 \leq j \leq m - 1, \\ ad_F G_m &= -\frac{\partial}{\partial z_m} - \sum_{i=1}^{m-1} z_i \frac{\partial}{\partial z_{m+i}} + \sum_{i=1}^{m-1} \dot{z}_i \frac{\partial}{\partial \dot{z}_{m+i}}, \\ [ad_F G_j, ad_F G_m] &= \frac{\partial}{\partial z_{m+j}} \quad \text{for } 1 \leq j \leq m - 1, \end{aligned} \quad (21)$$

indeed, span TZ at every  $\zeta = (z, \dot{z}) = (q, \dot{q})$ .  $\square$

It is straightforward to see that the  $m$ -crane system is not static feedback linearizable, which we will show for normal form (19) whose Lie brackets are given by (21). For mechanical systems the distribution  $\mathcal{D}^0 = \text{span}\{G_i, 1 \leq i \leq m\}$  is always involutive, but  $\mathcal{D}^1 = \text{span}\{G_i, ad_F G_i, 1 \leq i \leq m\}$  is not involu-

tive since  $[G_j, ad_F G_m]$  and  $[ad_F G_j, ad_F G_m] \notin \mathcal{D}^1$ . The  $m$ -crane is not static feedback linearizable; it is, however, linearizable via dynamic feedback, as asserts the following result.

**Proposition 4.** The  $m$ -crane system (16) is globally config-flat on  $TQ^\pm$ , provided that the control  $u \in \mathbb{R}^m$  satisfies  $u_m \neq 0$ , with a global flat output  $\phi = (\phi_1, \dots, \phi_m) = (x_1, \dots, x_m)$  of minimal differential weight  $5m$ .

**Remark 3.** Notice that  $5m = 2(2m - 1) + m + 2$  implying that the minimal dimension of a linearizing precompensator is 2. It follows from the proof below that, indeed, the 2-dimensional double preintegration  $\ddot{u}_m = v_m$  of the control  $u_m$ , dynamically linearizes the  $m$ -crane.

**Proof.** First, we show that  $\phi = (\phi_1, \dots, \phi_m) = (x_1, \dots, x_m)$  is, indeed, a flat output of (16). On  $Q^\pm$ , we have

$$\begin{aligned} x_i &= \phi_i & x_m &= \phi_m & d_i &= \frac{u_m \phi_i - \ddot{\phi}_i}{u_m} \\ \dot{x}_i &= \dot{\phi}_i & \dot{x}_m &= \dot{\phi}_m & \dot{d}_i &= \frac{u_m \dot{\phi}_i + \phi_i \dot{u}_m - \ddot{\phi}_i}{u_m} - \frac{\dot{u}_m (\phi_i u_m - \dot{\phi}_i)}{u_m^2} \\ u_m &= \frac{\ddot{\phi}_m - g}{\phi_m} & u_i &= \dot{d}_i + u_m \frac{\mu_i}{\mu_i} \left( \phi_i - \frac{u_m \phi_i - \ddot{\phi}_i}{u_m} \right), \end{aligned} \quad (22)$$

which is well defined for  $\phi_m = x_m \neq 0$  and  $u_m \neq 0$ . Although from the above it is not immediately clear that the differential weight is  $5m$ , we will show it in a different way. Since the state space is of dimension  $2(2m - 1)$  it is enough to show that after the two-fold prolongation of a well-chosen input the system is static feedback linearizable. The control to be prolonged is  $d_m := u_m$  (this is just a notation, physically  $d_m$  is not a component of the position), therefore the configuration manifold of the extended system is of dimension  $2m$  with coordinates  $(x_1, \dots, x_m, d_1, \dots, d_m)$  and controls are  $(u_1, \dots, u_{m-1}, v_m)$ , where  $v_m = \ddot{u}_m$ . The dynamics read, for  $1 \leq i \leq m - 1$ ,

$$\begin{aligned} \ddot{x}_i &= d_m(x_i - d_i) \\ \ddot{x}_m &= d_m x_m + g \\ \ddot{d}_i &= u_i - d_m \frac{\mu_m}{\mu_i} (x_i - d_i) \\ \ddot{d}_m &= v_m. \end{aligned} \quad (23)$$

It is straightforward to verify that the linearizability distribution  $\mathcal{D}_e^0 = \text{span}\{G_i^e, 1 \leq i \leq m\}$  and  $\mathcal{D}_e^1 = \text{span}\{G_i^e, ad_{F^e} G_i^e, 1 \leq i \leq m\}$ , the index  $e$  indicating extended system (23), are involutive and of constant rank. Thus the extended system is static feedback linearizable.  $\square$

The  $m$ -crane possess a singularity of the control at which the system ceases to be flat. From (22) we can see that the singular control is  $u_m = 0$ . Since  $u_m$  has a physical interpretation of the tension  $T$ , its singular value  $T = 0$  corresponds to the case of a free-fall of the load (completed by an arbitrary movement

in all  $d_i$ -directions), which is to deduce from the equations of  $m$ -crane system (16) by setting  $u_m = 0$  and thus implying  $\ddot{x}_i = 0$ ,  $\ddot{x}_m = g$ ,  $d_i = u_i$ .

## 5. Derivation of a control law and trajectory generator for the m-crane

In this section, we derive a cascade controller for  $m$ -crane system (16) that solves a trajectory tracking problem. The inner controller is a dynamic feedback controller that linearizes the original system by prolonging it and then transforming the prolonged system into a linear system in the Brunovský canonical form with controllability indices  $(4, 4, \dots, 4)$ , i.e.,  $m$  independent chains of integrators of length 4 each. The outer controller is a simple linear feedback that tracks desired trajectories designed by a generator proposed in Section 5.2. below.

**5.1. Control law.** In order to derive the dynamic linearization feedback for system (16), first, we prolong it by  $v_m = \ddot{u}_m$  to obtain system (23), where  $d_m = u_m$  (see the proof of Proposition 4), and then, second, we find a static feedback that linearizes (23). The latter can be done by observing that the flat output  $\phi = (\phi_1, \dots, \phi_m) = (x_1, \dots, x_m)$  is a linearizing output for (23) and therefore a linearizing controller is

$$\begin{aligned} u_i &= a\bar{v}_i + b\bar{v}_m + c_i & \text{for } 1 \leq i \leq m - 1 \\ v_m &= b_m\bar{v}_m + c_m, \end{aligned} \quad (24)$$

where  $a, b, b_m, c_i$ , for  $1 \leq i \leq m$ , are functions of  $\phi, \dot{\phi}, \ddot{\phi}, \phi^{(3)}$  given by

$$\begin{aligned} a &= -\frac{\phi_m}{\ddot{\phi}_m - g} & b &= \frac{\phi_m \ddot{\phi}_i}{(\ddot{\phi}_m - g)^2} \\ c_i &= \frac{2\phi_m^{(3)}(\dot{\phi}_m \ddot{\phi}_i + \phi_m \phi_i^{(3)})}{(g - \ddot{\phi}_m)^2} + \frac{2\phi_i^{(3)}\dot{\phi}_m + \ddot{\phi}_i \ddot{\phi}_m}{g - \ddot{\phi}_m} \\ &+ \frac{2\phi_m \phi_m^{(3)2} \ddot{\phi}_i}{(g - \ddot{\phi}_m)^3} + \left( \frac{\mu_m}{\mu_i} + 1 \right) \ddot{\phi}_i \\ b_m &= \frac{1}{\phi_m} \\ c_m &= \frac{2\phi_m^2(\ddot{\phi}_m - g)}{\phi_m^3} - \frac{\ddot{\phi}_m(\ddot{\phi}_m - g) + 2\phi_m^{(3)}\dot{\phi}_m}{\phi_m^2}. \end{aligned}$$

System (23) (equivalently, system (16) after prolongation), with controller (24), is mapped into the Brunovský canonical form by choosing linear coordinates as  $\phi_i^{(j)}$ , for  $j = 0, 1, 2, 3$ , in which it takes the form

$$\phi_i^{(4)} = \bar{v}_i, \quad \text{for } 1 \leq i \leq m. \quad (25)$$

Now, for this linear system (25), the outer cascade is designed to track desired trajectories  $\phi_d = (\phi_{1d}, \dots, \phi_{md})$  by taking

$$\bar{v}_i = \phi_{id}^{(4)} + k_{i1}(\phi_{id}^{(3)} - \dot{\phi}_i^{(3)}) + k_{i2}(\ddot{\phi}_{id} - \ddot{\phi}_i) + k_{i3}(\dot{\phi}_{id} - \dot{\phi}_i) + k_{i4}(\phi_{id} - \phi_i) \quad \text{for } 1 \leq i \leq m \quad (26)$$

where  $k_{ij}$ , with  $j = 1, 2, 3, 4$ , are control gains. By plugging (26) into (25), the error dynamics of the closed-loop system are obtained as

$$e_i^{(4)} + k_1 e_i^{(3)} + k_2 \ddot{e}_i + k_3 \dot{e}_i + k_4 e_i = 0, \quad \text{for } 1 \leq i \leq m,$$

where  $e_i = \phi_{id} - \phi_i$ . Appropriately chosen gains (for example using the pole placement method) ensure that the error dynamics are asymptotically stable. Note that, as will be apparent later, our reference trajectories are defined on a finite time interval  $[t_0, t_f]$ , therefore our solution requires that the system starts “sufficiently close” to the reference trajectories, in order to be able to almost approach them before the experiment ends, that is, before  $t_f$ . Although this assumption is theoretically limiting, we claim that from practical point of view is easy to be satisfied.

The original control system (13) is controlled by  $\tilde{\mathcal{F}}_i$  and  $\tilde{\mathcal{C}}$  and the control law (26) is expressed in terms for  $\bar{v}_i$ 's. To relate them notice that, first, (24) relates  $\bar{v}_i$ 's with  $u_i$ 's (recall that  $v_m = \ddot{u}_m$ ), second,  $\tilde{\mathcal{F}}_i = c_i \dot{d}_i + \mathcal{F}_i$  and  $\tilde{\mathcal{C}} = \frac{1}{b}(-c_r \dot{r} + \mathcal{C})$ , and third, control inputs  $\mathcal{F}_i$  and  $\mathcal{C}$  can be calculated in terms of  $u_i$ 's as:

$$\begin{aligned} \mathcal{F}_i &= \mu_i u_i \\ \mathcal{C} &= \frac{\mu_m r \dot{d}_m - \alpha - \sum_{i=1}^{m-1} \beta_i \mathcal{F}_i}{\beta_m}, \end{aligned} \quad (27)$$

where  $1 \leq i \leq m - 1$ , with

$$\begin{aligned} r &= \sqrt{\sum_{i=1}^{m-1} (x_i - d_i)^2 + x_m^2}, \\ \alpha &= \frac{\sum_{i=1}^{m-1} (\dot{x}_i - \dot{d}_i)^2 + \dot{x}_m^2 + x_m g}{\kappa} - \frac{(\sum_{i=1}^{m-1} (\dot{x}_i - \dot{d}_i)(x_i - d_i) + x_m \dot{x}_m)^2}{r^2 \kappa}, \\ \beta_i &= -\frac{x_i - d_i}{\mu_i \kappa}, \\ \beta_m &= \frac{r}{J \kappa}, \\ \kappa &= \frac{\sum_{i=1}^{m-1} (x_i - d_i)^2 (\mu_m + \mu_i + b^2 \mu_m \mu_i) + x_m^2 \mu_i (J + b^2 \mu_m)}{r J \mu_m \mu_i}. \end{aligned}$$

**5.2. Trajectory generator.** We apply a method presented in [8] to design reference trajectories  $\phi(t) = (\phi_{1d}(t), \dots, \phi_{md}(t))$  in the space of flat outputs. We state the problem of generating a reference trajectory  $\phi(t)$  as the problem of finding a solution of the reference system  $\phi_{id}^{(4)} = \bar{v}_{id}$ , for  $1 \leq i \leq m$ . In other words,

to find a reference trajectory  $\phi_{id}(t)$  starting, at the initial time  $t_0$ , from a given value  $\phi_{id}(t_0)$ , together with given values of the successive time-derivatives  $\dot{\phi}_{id}(t_0), \ddot{\phi}_{id}(t_0), \phi_{id}^{(3)}(t_0)$  as well as that of the reference control  $\bar{v}_{id}(t_0)$  and arriving, at the final time  $t_f$ , at a given final configuration  $\phi_{id}(t_f)$  (with given values  $\dot{\phi}_{id}(t_f), \ddot{\phi}_{id}(t_f), \phi_{id}^{(3)}(t_f)$ ), and  $\bar{v}_{id}(t_f)$ . We thus, for each component  $\phi_i(t)$  of  $\phi(t)$ , fix 5 initial conditions  $(\phi_{id}(t_0), \dot{\phi}_{id}(t_0), \ddot{\phi}_{id}(t_0), \phi_{id}^{(3)}(t_0), \phi_{id}^{(4)}(t_0))$  and 5 final conditions  $(\phi_{id}(t_f), \dot{\phi}_{id}(t_f), \ddot{\phi}_{id}(t_f), \phi_{id}^{(3)}(t_f), \phi_{id}^{(4)}(t_f))$ , where  $\phi_{id}^{(4)}(t_0) = \bar{v}_{id}(t_0)$ ,  $\phi_{id}^{(4)}(t_f) = \bar{v}_{id}(t_f)$ . We will design each reference trajectory  $\phi_{id}(t)$  in the space of polynomials of degree 9, with 10 coefficients  $a_{ik}$  for  $0 \leq k \leq 9$  calculated based on the initial and final conditions.

The polynomial  $\phi_{id}(t)$  is given by

$$\phi_{id}(t) = \sum_{k=0}^9 a_{ik} \left( \frac{t - t_0}{t_f - t_0} \right)^k, \quad (28)$$

and its derivatives by

$$\phi_{id}^{(k)}(t) = \frac{1}{(t_f - t_0)^k} \sum_{l=k}^9 \frac{l!}{(l-k)!} a_{il} \left( \frac{t - t_0}{t_f - t_0} \right)^{l-k}. \quad (29)$$

The coefficients  $a_{ik}$  are computed by equating the successive derivatives of  $\phi_{id}$  at  $t_0$  and  $t_f$  to the initial and the final conditions, respectively. In order to simplify these calculations, note that, the first 5 coefficients ( $a_{i0}, \dots, a_{i4}$ ) can be calculated directly

$$a_{ij} = \frac{(t_f - t_0)^j}{j!} \phi_{id}^{(j)}(t_0), \quad \text{for } 0 \leq j \leq 4, \quad (30)$$

and the remaining  $a_{i5}, \dots, a_{i9}$  are given by the system of linear equation

$$\phi_{id}^{(k)}(t_f) = \frac{1}{(t_f - t_0)^k} \sum_{l=k}^9 \frac{l!}{(l-k)!} a_{il}, \quad \text{for } 0 \leq k \leq 4. \quad (31)$$

Notice that we solve equations (28–31) independently for each  $i = 1, \dots, m$ . To summarize, there are  $10m$  coefficients  $a_{ik}$  in total,  $5m$  to be calculated from (30) and the remaining  $5m$  can be calculated by solving  $m$  linear systems given by (31).

## 6. Simulation results

The simulation model corresponds to the small laboratory crane (schematic drawing is presented in Fig. 1), that is characterized by parameters given in Table 1.

Table 1  
Parameters of the simulated 2-crane system

$M = 1.2$ [kg]	$\mu = 0.15$ [kg]
$J = 0.065$ [kgm <sup>2</sup> ]	$b = 0.02$ [m]
$g = 9.81$ [ $\frac{m}{s^2}$ ]	

A control problem considered in the simulations concerns trajectory tracking of the load for the 2-crane case described by (1) or, equivalently, by (11), where the position of the load is given by  $(x, z) = (x_1, x_2)$ . By Proposition 4,  $(\phi_1, \phi_2) = (x, z)$  are the flat outputs of the 2-crane system (11). In order to achieve the trajectory tracking problem, we combine cascade controller (24, 26) and trajectory generator (28–31). Since reference trajectories are  $(\phi_{1d}(t), \phi_{2d}(t)) = (x_d(t), z_d(t))$  and thus describe the time-evolution of the load, we will choose successive derivatives of  $x(t)$  and  $z(t)$ , at  $t_0$  and  $t_f$ , to vanish, which guarantees that the crane will carry the load smoothly with the zero velocity and acceleration at the initial and final position and with the zero initial and final controls (rest-to-rest trajectory).

So for the desired trajectory, initial conditions ( $t_0 = 0$ ) are taken as

$$\begin{aligned} x_d(t_0) &= 0.28 \\ \dot{x}_d(t_0) &= \ddot{x}_d(t_0) = x_d^{(3)}(t_0) = x_d^{(4)}(t_0) = 0 \\ z_d(t_0) &= 0.3 \\ \dot{z}_d(t_0) &= \ddot{z}_d(t_0) = z_d^{(3)}(t_0) = z_d^{(4)}(t_0) = 0 \end{aligned}$$

and final conditions ( $t_f = 30$ ) are taken as

$$\begin{aligned} x_d(t_f) &= 0.78 \\ \dot{x}_d(t_f) &= \ddot{x}_d(t_f) = x_d^{(3)}(t_f) = x_d^{(4)}(t_f) = 0 \end{aligned}$$

$$z_d(t_f) = 1$$

$$\dot{z}_d(t_f) = \ddot{z}_d(t_f) = z_d^{(3)}(t_f) = z_d^{(4)}(t_f) = 0,$$

The reference trajectory is calculated using (28–31) as

$$\begin{aligned} x_d(t) &= x_d(t_0) + D \left( \frac{126t^5}{t_f^5} - \frac{420t^6}{t_f^6} + \frac{540t^7}{t_f^7} - \frac{315t^8}{t_f^8} + \frac{70t^9}{t_f^9} \right) \\ z_d(t) &= z_d(t_0) + Z \left( \frac{126t^5}{t_f^5} - \frac{420t^6}{t_f^6} + \frac{540t^7}{t_f^7} - \frac{315t^8}{t_f^8} + \frac{70t^9}{t_f^9} \right), \end{aligned}$$

where  $D = x_d(t_f) - x_d(t_0)$  and  $Z = z_d(t_f) - z_d(t_0)$ .

We conducted two simulations, with nominal parameters (scenario S1), and with parametrical uncertainties of 20% on the cart and load masses, winch radius and winch's moment of inertia (scenario S2). Results are presented in Fig. 2 and Fig. 3 for, respectively, S1 and S2. In both cases we assume 10% errors on the initial position, that is  $x(t_0) = 0.308$  and  $z(t_0) = 0.33$ .

In the simulation scenario S1, the closed-loop system presents results satisfactory for most practical applications, that is, the tracking error converges around 0 in less than 4 seconds, after that time the reference trajectory is tracked with trajectory tracking error lower than  $10^{-3}$  m, while after 15 s the error reaches  $10^{-10}$  m. It can be seen that the angle  $\theta$  evolves smoothly as expected, although the transient state observed during the initial time of around 5 s is oscillatory with an amplitude of oscillations of circa 0.02 rad. The control force and torque pro-

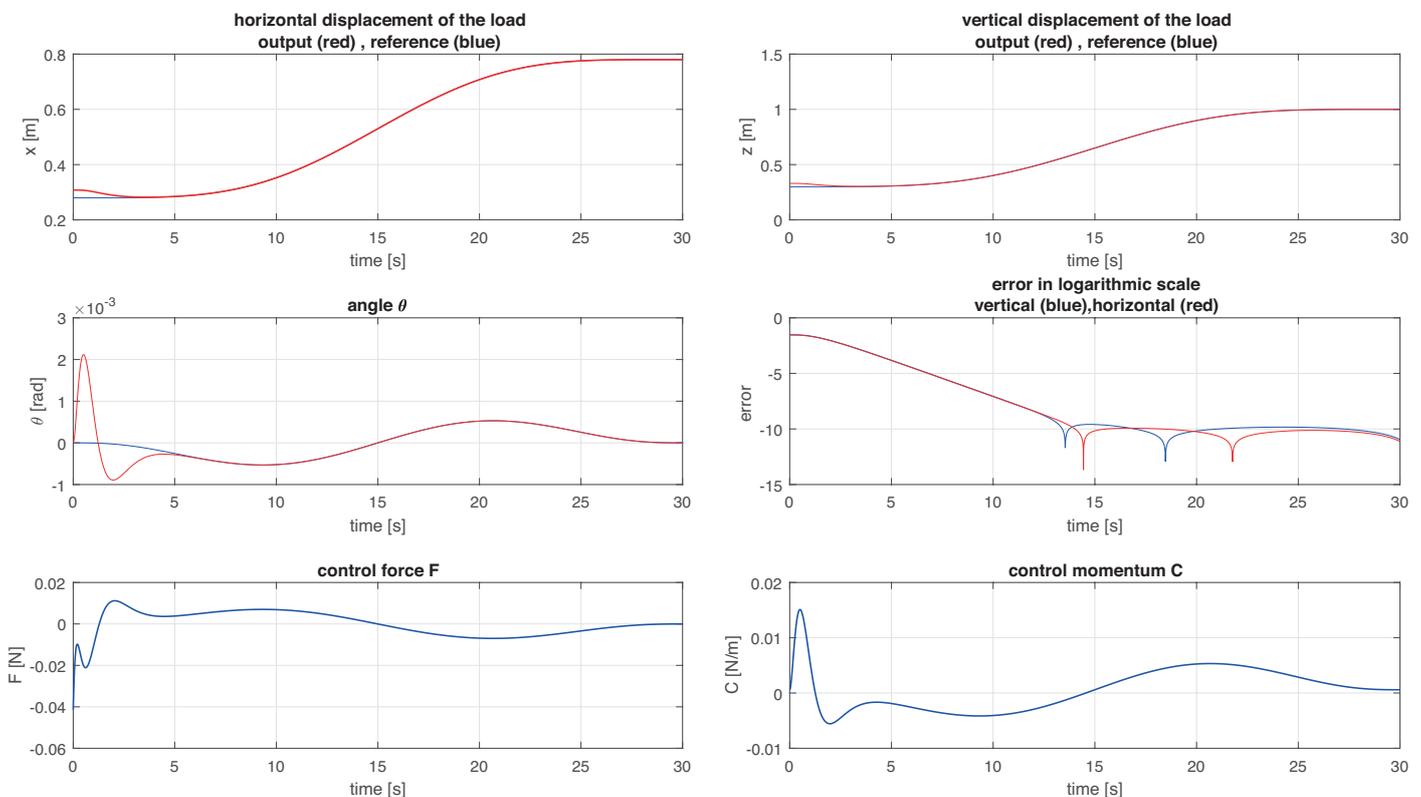


Fig. 2. Simulation results for 2-crane system (S1)

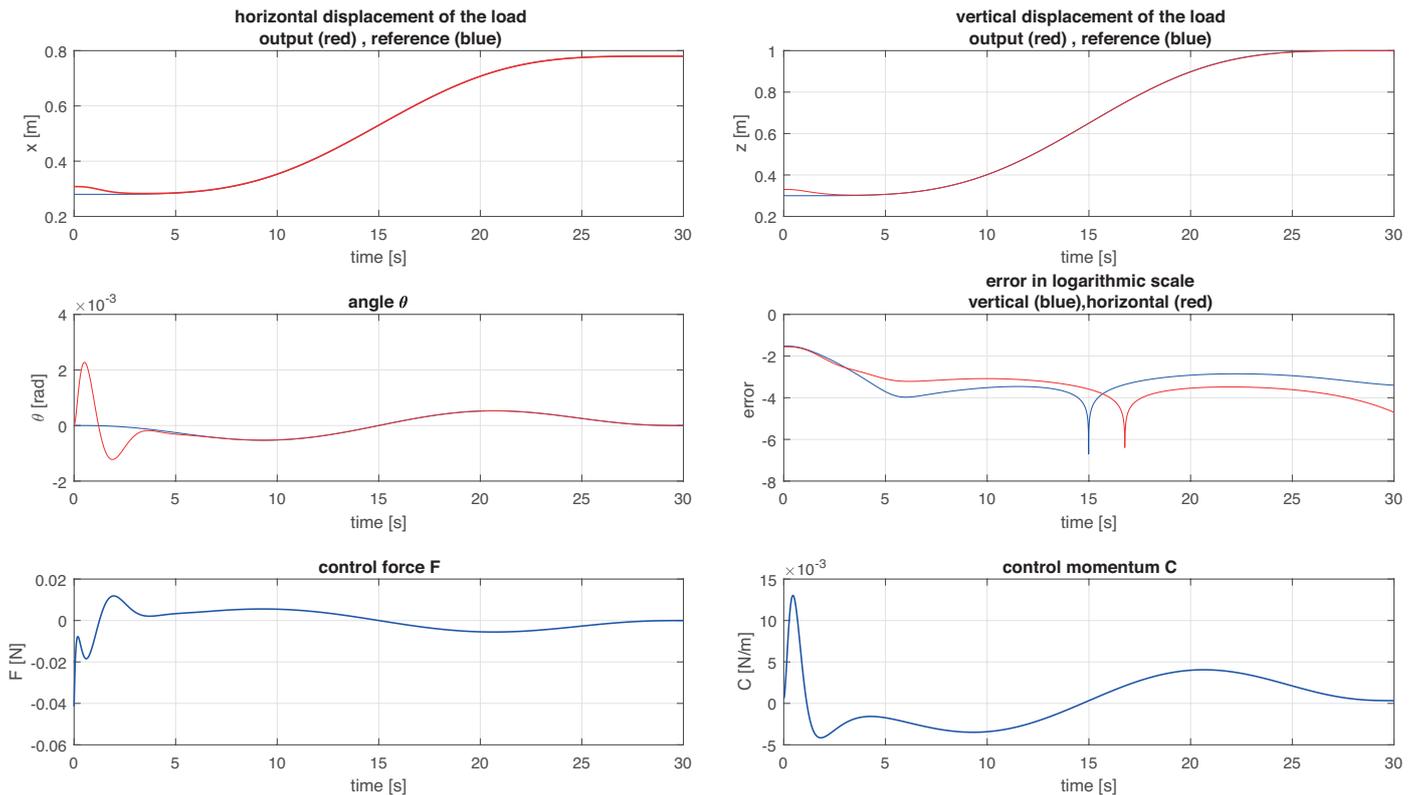


Fig. 3. Simulation results for 2-crane system with parametrical uncertainties (S2)

duce smooth and bounded control signals. It is significant in the context of implementation in a real-life system. What is more, the control force  $\mathcal{F}$  tends to 0, while  $\mathcal{C}$  converges to a constant value that defying the gravitational force. In simulation scenario  $S2$ , the system also shows fast error convergence, but comparing to  $S1$ , it is slower of around 3 seconds, what can be observed especially in the graph of the angle  $\theta$ . It is worth noting that as a result of parametric uncertainties being present in simulation scenario  $S2$ , the errors converge to values around  $10^{-4}$  m. Oscillations of higher amplitude may be observed on the plot of the control torque  $\mathcal{C}$ , while the plot of the control force  $\mathcal{F}$  remains very similar to  $S1$ . For analyzed scenarios  $S1$  and  $S2$ , the presented controller shows robustness with respect to uncertainties of coefficients and, simultaneously, it can also handle uncertain measurements of the cart position as well as of the cable length.

## 7. Summary

In this work, a generalization of crane control systems, called  $m$ -crane, was proposed. This generalization has allowed us to explore many control properties that are shared by representatives of this class. Among others, it was proven, that  $m$ -crane systems are strongly accessible and, which is especially important in applications, differentially flat (with differential weight  $5m$ ). Next we used this last property to solve

trajectory tracking problem, that was illustrated by simulations. The geometrical approach to characterize the class of control systems presented in this article can be extended further to classification of systems that are linearizable via a two-fold prolongation.

**Acknowledgements.** The authors are grateful to anonymous reviewers whose suggestions and remarks improved the presentation of the final version of the paper.

This work is partially supported by the Statutory Grant no. 09/93/SBAD/0911.

## REFERENCES

- [1] C. Cheng and C. Chen, "Controller design for an overhead crane system with uncertainty", *Control Engineering Practice* 4(5), 645–653, 1996.
- [2] B. d'Andréa-Novel and J. Lévine, "Modelling and Nonlinear Control of an Overhead Crane", In: Kaashoek M.A., van Schuppen J.H., Ran A.C.M. (eds) *Robust Control of Linear Systems and Nonlinear Control. Progress in Systems and Control Theory*, vol 4. Birkhäuser Boston, 1990.
- [3] M. Fliess, J. Lévine, P. Martin, and P. Rouchon, "Sur les systemes non linéaires différentiellement plats", *C. R. Acad. Sci. Paris Sér. I Math.*, 315, pp. 619–624, 1992.
- [4] M. Fliess, J. Lévine, P. Martin, and P. Rouchon, "Flatness and defect of non-linear systems: introductory theory and examples", *International Journal of Control* (61)6, 1327–1361, 2007.

- [5] M. Fliess, J. Lévine, and P. Rouchon, “A simplified approach of crane control via a generalized state-space model”, *Decision and Control, Proceedings of the 30th IEEE Conference on*, 736–741 vol. 1. 10.1109/CDC.1991.261409, 1992.
- [6] A. Isidori, *Nonlinear Control Systems*, 3rd edition, Springer-Verlag, London, 1995.
- [7] B. Jakubczyk, “Invariants of dynamic feedback and free systems”, in *Proc. ECC*, pp. 1510–1513, 1993.
- [8] J. Lévine, *Analysis and Control of Nonlinear Systems: A Flatness-Based Approach*, Springer Science&Business Media, p. 184–186, 2009.
- [9] J.E. Marsden and T.S. Ratiu, *Introduction to Mechanics and Symmetry: A Basic Exposition of Classical Mechanical Systems*, Springer Publishing Company, Incorporated, 10.1016/0967-0661(96)00046-9, 2010.
- [10] R.M. Murray, M. Rathinam, and W. Sluis, “Differential Flatness of Mechanical Control Systems: A Catalog of Prototype Systems”, *Proceedings of the 1995 ASME International Congress and Exposition*, 1995.
- [11] F. Nicolau and W. Respondek, “Flatness of multi-input control-affine systems linearizable via one-fold prolongation”, *SIAM Journal on Control and Optimization, Society for Industrial and Applied Mathematics* 55(5), 3171–3203, 2017.
- [12] F. Nicolau and W. Respondek, “Flatness of mechanical systems with 3 degrees of freedom”, *IFAC-PapersOnLine* 48(13), 19–24, 2015.
- [13] H. Nijmeijer and A.J. van der Schaft, “Nonlinear Dynamical Control Systems”, Springer-Verlag, New York, 1990.
- [14] J. Pomet, “A differential geometric setting for dynamic equivalence and dynamic linearization”, *Banach Center Publ.*, (32), 319–339, 1995.
- [15] J. Pomet, “On dynamic feedback linearization of fourdimensional affine control systems with two inputs”, *ESAIM Control Optim. Calc. Var*, 2, 151–230, 1997.
- [16] J. Smoczek, “Interval arithmetic-based fuzzy discrete-time crane control scheme design”, *Bull. Pol. Ac.: Tech.* (61)4, 863–870, 2013.
- [17] W. Respondek, “Symmetries and minimal flat outputs of nonlinear control systems”, In *New Trends in Nonlinear Dynamics and Control and their Applications*, volume LNCIS 295, pp. 65–86. Springer, 2003.
- [18] S.S. Vlase, “A Method of Eliminating Lagrangian Multipliers from the Equation of Motion of Interconnected Mechanical Systems”, *ASME. J. Appl. Mech.* 54(1), 235–237. DOI: 10.1115/1.3172969, 1987.