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CRITICAL STATE OF SIMPLY SUPPORTED ELASTIC RECTANGULAR PLATE UNDER NON-UNIFORMLY DISTRIBUTED COMPRESSIVE LOAD

The paper is devoted to a simply supported rectangular plate subjected to two types of compressive edge loads. The first load is applied uniformly along a part of two opposite edges, the second one has a non-uniform distribution (defined by a half wave of the \sin^k function). The critical load value of the plate is located between the values for uniformly distributed and concentrated load. Critical value of thickness of the plate is determined. The problem is solved by the orthogonalization method, and the results are compared with those of numerical analysis done by means of the finite element method.

1. Introduction

Thin-walled structures are usually made of rectangular plate elements, which are subjected to complex types of loads. Therefore, conditions of strength and stability must be precisely analysed during designing these structures. The problem of buckling of a rectangular plate due to an uniformly distributed compressive load distributed along two opposite edges of the plate was formulated by Bryan in 1891. Bryan also calculated a critical load for this case. The buckling of rectangular plates for complex cases of load was described (for example) by Volmir (1967) and collected by Woźniak (2001). The problem of buckling of a rectangular plate compressed

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by a concentrated load was solved by Sommerfeld in 1907. A review of the strength and the stability problems of plates and shells subjected to a concentrated load was presented by Łukasiewicz (1976). The buckling problem of non-uniformly compressed cylindrical shells was described by Binkevich and Krasovskii (1973), Teng and Rotter (2004) and also by Błachut (2005). These authors, however, paid less attention to the buckling response induced by a non-uniform compression. Yamaki in 1984 also demonstrated an analysis of compressed shells and the effect of non-uniform compression on the critical load value. The recent analysis of the rectangular plates submitted to a set of complex non-uniform loading cases was provided by Bert and Devarakonda (2003).

The solutions proposed by Bryan (1891) and Sommerfeld (1907) for compressed rectangular plates are concerned with two ideal cases of load. The first one is connected with an uniformly distributed load along the plate edges, the second one – with a concentrated load. In actual structures, usually occur non-uniformly distributed loads. The values of critical loads are located between these two ideal cases. The analysis presented in this paper is concerned with a simply supported rectangular plate subjected to two types of load distribution: an uniform loading on a portion of the plate edge and a non-uniform loading (defined by a half wave of the \sin^k function). This paper provides an analytical and a numerical approach to the problem, and is a continuation of the work by Kurpisz et al. [5].

2. Analytical solution

At the beginning, we consider the plate subjected to a compressive load uniformly distributed along a part of two opposite edges. The scheme of the considered load distribution is presented in figure 1.

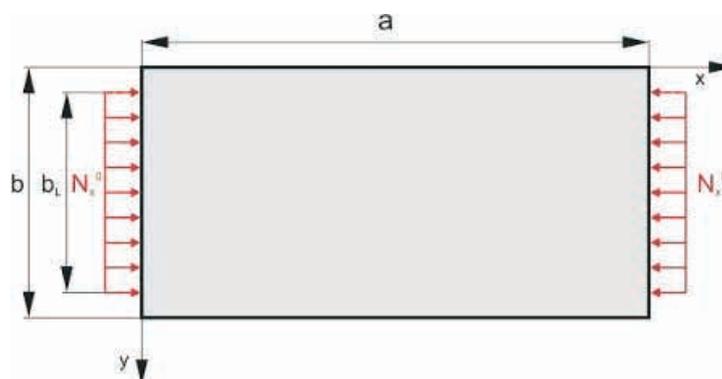


Fig. 1. Plate loading uniformly distributed on two opposite edges

The critical load value of uniaxially and uniformly compressed simply supported rectangular plate [8] occurs for a square plate ($a = b$), and is given by

$$F_{CR}^{(unif)} = b \cdot N_{x,CR} = 4\pi^2 \frac{D}{b},$$

where $D = \frac{E \cdot t^3}{12 \cdot (1 - \nu^2)}$ – plate bending stiffness, a – plate length, b – plate width, t – plate thickness, E – Young's modulus, ν – Poisson ratio.

The critical load value of a simply supported rectangular plate subjected to a concentrated load [6], [8] can be written as

$$F_{CR}^{(con)} = \alpha \pi \frac{D}{b},$$

where

a/b	0,5	0,75	1	1,5	2	3
α	18,80	9,00	6,00	4,46	4,11	4,02

Hence, for square plate, we have

$$F_{CR}^{(con)} = 6\pi \frac{D}{b}.$$

The concentrated load put on the edge can cause a local plastification of the plate. Therefore, it is essential to place this load along a part of the plate edge, so that the critical stresses do not exceed the allowable stresses ($\sigma_{CR} = \sigma_{ALL}$).

The critical load of a simply supported rectangular plate exerted by a compressive load uniformly distributed along a part of two opposite edges is held in the following interval

$$F_{CR}^{(con)} < F_{CR}^{(part-unif)} < F_{CR}^{(unif)},$$

$$\text{thus} \quad \frac{\pi}{2 \cdot (1 - \nu^2)} \frac{E}{\sigma_{ALL}} \left(\frac{t}{b}\right)^2 < \left(\frac{b_L}{b}\right)_{CR} < \frac{\pi^2}{3 \cdot (1 - \nu^2)} \frac{E}{\sigma_{ALL}} \left(\frac{t}{b}\right)^2, \quad (1)$$

where

$$F_{CR}^{(part-unif)} = b_L \cdot t \cdot \sigma_{CR} = b_L \cdot t \cdot \sigma_{ALL}.$$

The stability equation of the longitudinal compressed plate is given by the following formula

$$D \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) = t \sigma_x \frac{\partial^2 w}{\partial x^2}, \quad (2)$$

where σ_x – compression stress of the axially loaded plate.

The load uniformly distributed along a part of two opposite edges is defined by the following formula

$$F^{(part-unit)} = b_L \cdot t \cdot \sigma_x = b_L \cdot N_x^0,$$

where

$$N_x^0(y) = \begin{cases} c \cdot N_x & \text{for } y \in \left[\frac{b-b_L}{2}, \frac{b+b_L}{2} \right]; b_L \in (0, b) \\ 0 & \text{other cases} \end{cases}$$

is the intensity of load.

It is assumed that $\int_0^b N_x^0(y) dy = b \cdot N_x$, therefore $c = \frac{b}{b_L}$.

The deflection function takes the form

$$w(x, y) = w_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \text{ for } x \in [0, a]; y \in [0, b]; n, m = 1, 2, 3, \dots \quad (3)$$

Substituting $w(x, y)$ into equation (2), we have

$$D \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right]^2 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) = N_x^0(y) \left(\frac{m\pi}{a} \right)^2 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right). \quad (4)$$

Using the Galerkin's method for equation (4), we obtain

$$D \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right]^2 = N_x \left(\frac{m\pi}{a} \right)^2 \left[1 - \frac{b \cdot (-1)^n}{n\pi b_L} \sin\left(\frac{n\pi b_L}{b}\right) \right].$$

Hence

$$N_x = \frac{D \left[\frac{m\pi}{a} + \frac{n^2 \pi a}{b^2 m} \right]^2}{1 - \frac{b \cdot (-1)^n}{n\pi b_L} \sin\left(\frac{n\pi b_L}{b}\right)}. \quad (5)$$

The critical intensity of load $N_{x,CR}$ is the minimum value of equation (5). Let $\alpha = \frac{a}{b}$, then equation (5) takes the following form

$$N_x = \frac{D\frac{\pi^2}{b^2}}{1 - \frac{b \cdot (-1)^n}{n\pi b_L} \sin\left(\frac{n\pi b_L}{b}\right)} \cdot \left(\frac{m}{\alpha} + \frac{n^2\alpha}{m}\right)^2. \quad (6)$$

In order to find minimum value of equation (6), let us assume that $n = 1$ and $f(\alpha) = \left(\frac{m}{\alpha} + \frac{\alpha}{m}\right)^2$. It is easy to see that $\frac{df}{d\alpha} = 0$ and $\frac{d^2f}{d\alpha^2} > 0$ for $\alpha = m$, therefore $f_{\min} = f(\alpha = m) = 4$.

Thus $N_{x,CR} = \frac{4D\frac{\pi^2}{b^2}}{1 + \frac{b}{\pi b_L} \sin\left(\frac{\pi b_L}{b}\right)}$, and therefore the critical intensity of non-uniformly distributed load is given by

$$N_{CR}^{(part-unif)} = \frac{b}{b_L} \cdot N_{x,CR} = \frac{4D\frac{\pi^2}{bb_L}}{1 + \frac{b}{\pi b_L} \sin\left(\frac{\pi b_L}{b}\right)}.$$

The critical compressive non-uniformly distributed load value has the following form

$$F_{CR}^{(part-unif)} = \frac{4D\frac{\pi^2}{b}}{1 + \frac{b}{\pi b_L} \sin\left(\frac{\pi b_L}{b}\right)}.$$

It is obvious that if $b_L \rightarrow b$ then $F_{CR}^{(part-unif)} \rightarrow 4D\frac{\pi^2}{b} = F_{CR}^{(unif)}$, and if $b_L \rightarrow 0$ then $F_{CR}^{(part-unif)} \rightarrow 2D\frac{\pi^2}{b} = 6.28D\frac{\pi}{b} \approx F_{CR}^{(con)}$. This small difference is due to a different method of deriving the formula for the critical concentrated load. The critical stress can be written as

$$\sigma_{CR}^{(part-unif)} = \frac{4D\frac{\pi^2}{tbb_L}}{1 + \frac{b}{\pi b_L} \sin\left(\frac{\pi b_L}{b}\right)}.$$

By transformation of inequality (1), we obtain

$$\sqrt{\frac{3 \cdot (1 - \nu^2)\sigma_{ALL}}{\pi^2 E} \frac{b_L}{b}} < \frac{t}{b} < \sqrt{\frac{2 \cdot (1 - \nu^2)\sigma_{ALL}}{\pi E} \frac{b_L}{b}}, \quad (7)$$

where

$$\frac{t}{b} = \sqrt{\frac{3 \cdot (1 - \nu^2) \sigma_{ALL}}{\pi^3 E} \left[\frac{\pi b_L}{b} + \sin \left(\frac{\pi b_L}{b} \right) \right]} \quad (8)$$

Now, let us consider a simply supported rectangular plate subjected to an uniaxially, non-uniformly distributed compressive load (defined by a half wave of the \sin^k function). Figure 2 presents this kind of load distribution.

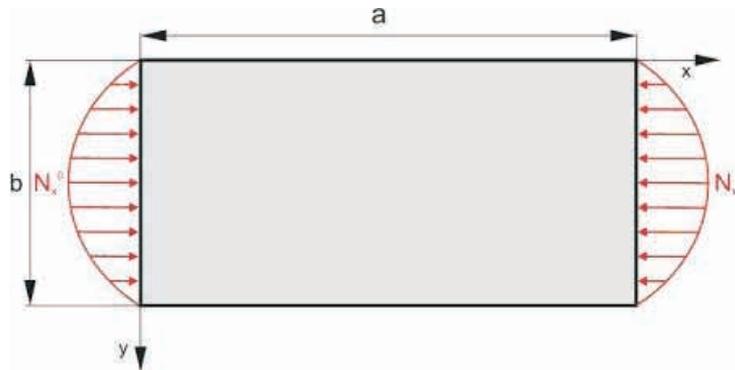


Fig. 2. The load scheme for a non-uniformly distributed load and $k = 1$

The critical load of a simply supported rectangular plate exerted by an uniaxially but non-uniformly distributed compressive load is held in the following interval

$$F_{CR}^{(con)} < F_{CR}^{(non-unif)} < F_{CR}^{(unif)},$$

where

$$F_{CR}^{(non-unif)} = b \cdot t \cdot \sigma_{CR} = b \cdot t \cdot \sigma_{ALL}.$$

The intensity of load is defined by the following formula

$$N_x^0(y) = c_k \cdot N_x \cdot \sin^k \left(\frac{\pi y}{b} \right) \quad \text{for } y \in [0, b]; k = 1, 3, 5, \dots \quad (9)$$

It is also assumed that $\int_0^b N_x^0(y) dy = b \cdot N_x$, therefore

$$c_k = \begin{cases} \frac{\pi \cdot k!}{2^k \cdot \left[\left(\frac{k-1}{2}\right)!\right]^2} & \text{for odd } k \\ \frac{2^k \cdot \left[\left(\frac{k}{2}\right)!\right]^2}{k!} & \text{for even } k \end{cases}$$

First, we analyze the intensity of load (9) for odd k .

The substitution of formulas (3) and (9) into (4) gives

$$\begin{aligned} D \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right]^2 \sin \left(\frac{m\pi x}{a} \right) \sin \left(\frac{n\pi y}{b} \right) &= \\ = \frac{N_x \cdot \pi \cdot k!}{2^k \cdot \left[\left(\frac{k-1}{2}\right)!\right]^2} \left(\frac{m\pi}{a} \right)^2 \sin^k \left(\frac{n\pi y}{b} \right) \sin \left(\frac{m\pi x}{a} \right) \sin \left(\frac{n\pi y}{b} \right) \end{aligned}$$

from which, using the Galerkin's method and taking advantage of the fact that

$$\int_0^b \sin^k \left(\frac{\pi y}{b} \right) dy = \frac{2^k \cdot b \cdot \left[\left(\frac{k-1}{2}\right)!\right]^2}{\pi \cdot k!} \quad \text{for odd } k,$$

we obtain

$$D \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right]^2 = 2 \cdot \left(\frac{m\pi}{a} \right)^2 \frac{k+1}{k+2} N_x.$$

Thus

$$N_x = \frac{D \left[\frac{m\pi}{a} + \frac{n^2 \pi a}{b^2 m} \right]^2}{2 \cdot \frac{k+1}{k+2}} \quad (10)$$

In a similar way like in the case of non-uniformly distributed load one can find the minimum value (N_{CR}) of equation (10). This value is obtained for $n = 1$ and $m = \frac{a}{b}$.

Hence, the critical intensity of load takes the form

$$N_{CR}^{(non-unif)} = \frac{2D \pi^2}{b^2} \cdot \frac{k+2}{k+1}.$$

The critical compressive load has the following form

$$F_{CR}^{(non-unif)} = N_{CR}^{(non-unif)} \cdot b = \frac{2D \pi^2}{b} \cdot \frac{k+2}{k+1}. \quad (11)$$

It is easy to see that the load from formula (11) appears between two extreme cases of the critical load – the uniformly distributed (for $k \rightarrow 0$) and the concentrated one (for $k \rightarrow +\infty$). The critical stress can be written as

$$\sigma_{CR}^{(non-unif)} = \frac{F_{CR}^{(non-unif)}}{t b} = \frac{2D \pi^2}{t b^2} \cdot \frac{k+2}{k+1}.$$

From inequality (1), one obtains the following inequality

$$\sqrt{\frac{3 \cdot (1 - \nu^2) \sigma_{ALL}}{\pi^2 E}} < \frac{t}{b} < \sqrt{\frac{2 \cdot (1 - \nu^2) \sigma_{ALL}}{\pi E}} \quad (12)$$

Obviously, inequality (12) is satisfied by

$$\frac{t}{b} = \sqrt{\frac{6(1 - \nu^2) \sigma_{ALL}}{\pi^2 E} \cdot \frac{k+1}{k+2}}$$

for $\sigma_{ALL} = \sigma_{CR}$.

Due to the fact that $\int_0^b \sin^k \left(\frac{\pi y}{b} \right) dy = \frac{b \cdot k!}{2^k \cdot \left[\left(\frac{k}{2} \right)! \right]^2}$ for even k , we can

derive, in a similar way, the formulas for $N_{CR}^{(non-unif)}$, $F_{CR}^{(non-unif)}$, $\sigma_{CR}^{(non-unif)}$, for even k . These formulas are the same as those for odd k .

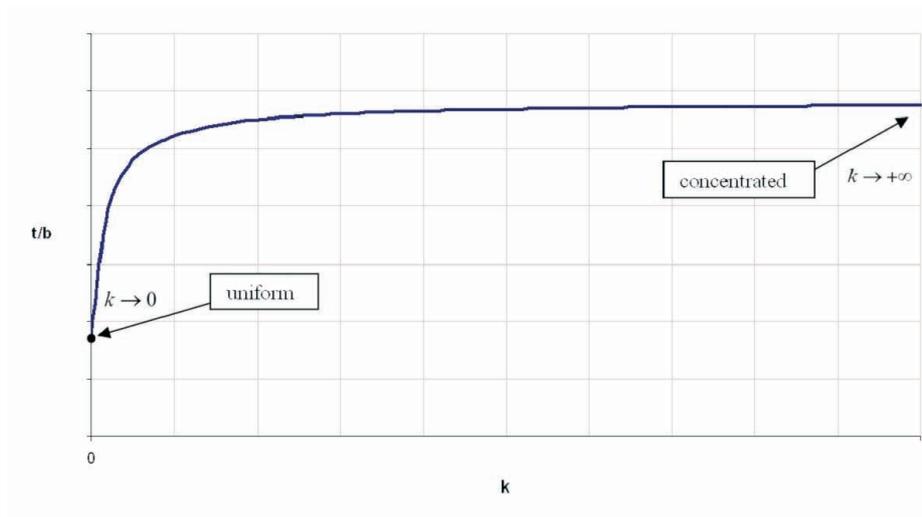


Fig. 3. The graph of the ratio t/b for fixed k

The region below the curve in figure 3 pertains to the case in which plate load is too high, and the stresses evoked by this load exceed the critical value. A plate in this state can buckle and lose its stability. It is then necessary to avoid this area, and take the ratios $\frac{t}{b}$ (for fixed k) which lie above the curve.

3. Numerical analysis

The numerical analysis is carried out by means of the FE code ABAQUS/Standard. Quadrilateral shell elements SR4 are used in the FE model. Numerical analysis is performed only for a load uniformly distributed on the part of the edge of length b_L . The value of the considered load is equal to $F = 1000$ N. The material properties are $E = 2,05 \cdot 10^5$ MPa, $\nu = 0,3$, $\sigma_{ALL} = 200$ MPa. This analysis is performed for the square plate family with the dimensions $a = b = 100$ mm and the ratio $\frac{b_L}{b}$ taken from the interval $[0,05; 1]$. The results shown below are calculated for $\frac{b_L}{b} = 0,5$. The critical value of plate thickness t in this case is obtained from formula (8) and equals 1.486 mm.

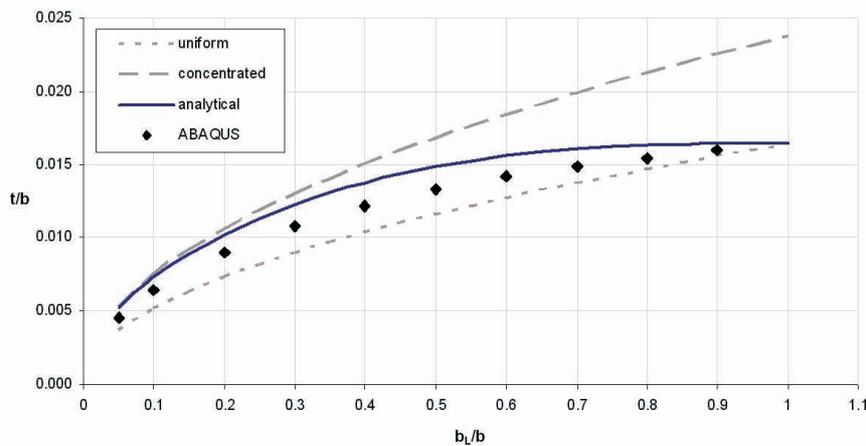


Fig. 4. The critical ratios of plate sizes

The curves marked as “concentrated” and “uniformly” in figure 4 correspond to formula (7). The analytical solution associated with formula (8) is represented by the curve “analytical” and the results from ABAQUS by “ABAQUS”. The region above the “analytical” curve represents safety ratios of $\frac{t}{b}$. It means that if the $\frac{t}{b}$ value is not taken from this region, then the plate can buckle.

4. Conclusion

Many structures are usually submitted to non-uniformly or partly uniformly distributed loads, therefore these two types of loads has been investigated in this paper. The considered plate is in the critical state, so it can easy buckle. The formula for safety value of $\frac{t}{b}$ for each type of the analysed loads has been determined. The FEM analysis has shown a good agreement with the analytical solution.

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Stan krytyczny przegubowo podpartej sprężystej płyty prostokątnej ściskanej nierównomiernie rozłożonym obciążeniem

Streszczenie

Przedmiotem badań jest stan krytyczny nierównomiernie ściskanej prostokątnej płyty podpartej przegubowo-przesuwnie na czterech brzegach. Rozwiązania dla płyt równomiernie ściskanych i

obciążonych siłą skupioną są dobrze znane. Badany stan krytyczny występuje pomiędzy stanem krytycznym dla równomiernie obciążonej płyty, a stanem krytycznym dla płyty obciążonej siłą skupioną. Krytyczne wymiary płyty są ustalone w wyniku badań. Problem rozwiązano przy pomocy metody ortogonalizacji, wyniki porównano z obliczeniami z programu ABAQUS.