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A new extension of the Cayley–Hamilton theorem to fractional different orders linear systems

Tadeusz KACZOREK*

Faculty of Electrical Engineering, Bialystok University of Technology, Wiejska 45D, 15-351 Białystok, Poland

Abstract. The classical Cayley–Hamilton theorem is extended to fractional different order linear systems. The new theorems are applied to different orders fractional linear electrical circuits. The applications of new theorems are illustrated by numerical examples.

Key words: Cayley–Hamilton theorem; electrical circuit; extension; fractional; different order.

1. INTRODUCTION

The classical Cayley–Hamilton theorem [2, 8] says that every square matrix satisfies its own characteristic equation. The Cayley–Hamilton theorem has been extended to rectangular, block matrices and the pairs of block matrices [4, 5]. In [6] the Cayley–Hamilton theorem has been extended to n-dimensional (n-D) real polynomial matrices. The Cayley–Hamilton theorem and its generalizations have been used in control systems, electrical circuits, systems with delays, singular systems, 2-D linear systems, etc. [1, 7–15]. The classical Cayley–Hamilton theorem has been extended to simple fractional order and different fractional order continuous-time linear systems in [3].

In this paper, the Cayley–Hamilton theorem will be extended to fractional different order linear systems and applied to fractional electrical circuits.

The paper is organized as follows. In Section 2 some preliminaries concerning fractional linear single order and two different orders are recalled. An extension of the Cayley–Hamilton theorem to non-square matrices of linear systems is presented in Section 3. The main result of the paper, a new extension of the Cayley–Hamilton theorem to different orders fractional linear systems is given in Section 4. The new theorem is applied to fractional different orders electrical circuits in Section 5. Concluding remarks are given in Section 6.

2. FRACTIONAL DIFFERENT ORDER LINEAR SYSTEMS

Consider the continuous-time fractional different order linear system

$$\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = Ax(t) + Bu(t), \quad 0 < \alpha \le 1, \tag{1}$$

where $x(t) \in \mathfrak{R}^n$ is the state vector, $u(t) \in \mathfrak{R}^m$ is the input vector $A \in \mathfrak{R}^{n\times n}$, $B \in \mathfrak{R}^{n\times m}$, and $\mathfrak{R}^{n\times m}$ is the set of $n \times m$ real matrices.

*e-mail: t.kaczorek@pb.edu.pl

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In this paper, the Caputo definition will be used

$$\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{\dot f(\tau)}{(t-\tau)^\alpha} \,\mathrm{d}\tau, \tag{2}$$

where $0 < \alpha \le 1$, $\alpha \in \mathfrak{R}$ is the order of fractional derivative, $\dot f(\tau) = \dfrac{\mathrm{d} f(\tau)}{\mathrm{d}\tau}$ and $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \,\mathrm{d}t$ is the gamma function.

The solution to equation (1) is given in [8].

Consider the fractional continuous-time linear system with two different fractional orders $\alpha \ne \beta$ described by the equation

$$\begin{bmatrix} \dfrac{\mathrm{d}^\alpha x_1(t)}{\mathrm{d}t^\alpha} \\ \dfrac{\mathrm{d}^\beta x_2(t)}{\mathrm{d}t^\beta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t), \tag{3}$$

and $0 < \alpha, \beta < 1$ where $x_1(t) \in \mathfrak{R}^{n_1}$, $x_2(t) \in \mathfrak{R}^{n_2}$, $u(t) \in \mathfrak{R}^m$ and $y(t) \in \mathfrak{R}^p$ are the state, input, and output vectors respectively, $A_{ij} \in \mathfrak{R}^{n_i\times n_j}$, $B_i \in \mathfrak{R}^{n_i\times m}$; $i, j = 1, 2$.

Initial conditions for (3) have the form

$$x_1(0) = x_{10}, \quad x_2(0) = x_{20} \quad \text{and} \quad x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}. \tag{4}$$

The solution of equation (3) for $0 < \alpha, \beta < 1$ with initial conditions (4) is given in [8].

3. THE CAYLEY–HAMILTON THEOREM FOR NON-SQUARE MATRICES

Consider the non-square matrix $A \in \mathfrak{R}^{m\times n}$ for $n > m$, which can be written in the form

$$A = \begin{bmatrix} A_1 & A_2 \end{bmatrix}, \quad A_1 \in \mathfrak{R}^{m\times m}, \quad A_2 \in \mathfrak{R}^{m\times(n-m)}. \tag{5}$$

Let

$$\det\left[I_m s - A_1\right] = \sum_{i=0}^{m} a_i s^i \quad (a_m = 1) \tag{6}$$

be the characteristic polynomial of the matrix A_1 and I_m is the $m\times m$ identity matrix.

Theorem 1. The matrix (5) for $n>m$ with characteristic polynomial (6) satisfies the equation

$$\sum_{i=0}^{m} a_{m-i}\begin{bmatrix}A_1^{n-i} & A_1^{n-i-1} & A_2\end{bmatrix}=0. \tag{7}$$

Proof. By induction, it is easy to verify that

$$\begin{bmatrix}A_1 & A_2\\ 0 & 0\end{bmatrix}^i=\begin{bmatrix}A_1^i & A_1^{i-1}A_2\\ 0 & 0\end{bmatrix}, \quad i=1,2,\ldots \tag{8}$$

From (8) and (6) we have

$$\det\begin{bmatrix}I_m s-A_1 & -A_2\\ 0 & I_{n-m}s\end{bmatrix}=s^{n-m}\det\begin{bmatrix}I_m s-A_1\end{bmatrix}=\sum_{i=0}^{m} a_{m-i}s^{n-i} \tag{9}$$

and applying the classical Cayley–Hamilton theorem we obtain

$$\sum_{i=0}^{m} a_{m-i}\begin{bmatrix}A_1 & A_2\\ 0 & 0\end{bmatrix}^{n-i}=0. \tag{10}$$

Substituting (8) into (6) we obtain

$$\sum_{i=0}^{m} a_{m-i}\begin{bmatrix}A_1^{n-i} & A_1^{n-i-1}A_2\\ 0 & 0\end{bmatrix}=0 \tag{11}$$

and from (11) the desired equation (7). □

Now let us consider the matrix (5) for

$$A=\begin{bmatrix}A_3 & A_4\end{bmatrix}, \quad A_3\in\Re^{m\times(n-m)}, \quad A_4\in\Re^{m\times m}. \tag{12}$$

Let

$$\det\begin{bmatrix}I_m s-A_4\end{bmatrix}=\sum_{i=0}^{m} a_i s^i \quad (a_m=1) \tag{13}$$

be the characteristic polynomial of the matrix A_4.

Theorem 2. The matrix A with (12) for $n>m$ and the characteristic polynomial (13) satisfies the equation

$$\sum_{i=0}^{n-m} a'_{n-m-i}\begin{bmatrix}A_4^{n-m-i-1}A_3 & A_4^{n-m-i}\end{bmatrix}=0 \quad (a'_{n-m}=1). \tag{14}$$

Proof is similar to Proof of Theorem 1.

4. THE CAYLEY–HAMILTON THEOREM FOR FRACTIONAL DIFFERENT ORDERS LINEAR SYSTEMS

Consider the fractional different orders linear system (3) with the matrix

$$A=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}, \tag{15}$$

where $A_{11}\in\Re^{n_1\times n_1}$, $A_{22}\in\Re^{n_2\times n_2}$.

Theorem 3. Let

$$\det\begin{bmatrix}I_{n_1}s_\alpha-A_{11}\end{bmatrix}=s_\alpha^{n_1}+a_{n_1-1}s_\alpha^{n_1-1}+\ldots+a_1 s_\alpha+a_0 \tag{16a}$$

be the characteristic polynomial of the matrix A_{11} and

$$\det\begin{bmatrix}I_{n_2}s_\beta-A_{22}\end{bmatrix}=s_\beta^{n_2}+b_{n_2-1}s_\beta^{n_2-1}+\ldots+b_1 s_\beta+b_0 \tag{16b}$$

be the characteristic polynomial of the matrix A_{22}.

Then the matrix (15) satisfies the equation

$$\begin{bmatrix}\sum_{i=0}^{n_1} a_{n_1-i}\begin{bmatrix}A_{11}^{n-i} & A_{11}^{n-i-1}A_{12}\end{bmatrix}\\ \sum_{j=0}^{n_2} b_{n_2-j}\begin{bmatrix}A_{22}^{n-j-1}A_{21} & A_{22}^{n-j}\end{bmatrix}\end{bmatrix}=0, \quad n=n_1+n_2. \tag{17}$$

Proof. Applying Theorem 1 to the matrix $\begin{bmatrix}A_{11} & A_{12}\end{bmatrix}$, we obtain

$$\sum_{i=0}^{n_1} a_{n_1-i}\begin{bmatrix}A_{11}^{n-i} & A_{11}^{n-i-1}A_{12}\end{bmatrix}=0. \tag{18}$$

Similarly, applying Theorem 2 to the matrix $\begin{bmatrix}A_{21} & A_{22}\end{bmatrix}$, we obtain

$$\sum_{j=0}^{n_2} b_{n_2-j}\begin{bmatrix}A_{22}^{n-j-1}A_{21} & A_{22}^{n-j}\end{bmatrix}=0. \tag{19}$$

Combining (18) and (19) we obtain (17). This completes the proof. □

Example 1. Consider the matrix (15) with

$$A_{11}=\begin{bmatrix}-3 & 1\\ 0 & -4\end{bmatrix}, \quad A_{12}=\begin{bmatrix}0 & 1\\ 1 & 2\end{bmatrix}, \quad A_{21}=\begin{bmatrix}1 & 0\\ 1 & 2\end{bmatrix}, \quad A_{22}=\begin{bmatrix}-3 & 1\\ 2 & -5\end{bmatrix}. \tag{20}$$

In this case, we have

$$\det\begin{bmatrix}I_{n_1}s_\alpha-A_{11}\end{bmatrix}=s_\alpha^2+7s_\alpha+12 \tag{21}$$

and

$$\det\begin{bmatrix}I_{n_2}s_\beta-A_{22}\end{bmatrix}=s_\beta^2+8s_\beta+13. \tag{22}$$

Using (20), (21) and (22) we obtain

$$\begin{aligned}\sum_{i=0}^{2} a_{2-i}\begin{bmatrix}A_{11}^{4-i} & A_{11}^{3-i}A_{12}\end{bmatrix}&=a_2\begin{bmatrix}A_{11}^4 & A_{11}^3A_{12}\end{bmatrix}\\ &+a_1\begin{bmatrix}A_{11}^3 & A_{11}^2A_{12}\end{bmatrix}+a_0\begin{bmatrix}A_{11}^2 & A_{11}A_{12}\end{bmatrix}\\ &=\begin{bmatrix}81 & -175 & 27 & 47\\ 0 & 256 & -64 & -128\end{bmatrix}\\ &+7\begin{bmatrix}-27 & 27 & -7 & -5\\ 0 & -64 & 16 & 32\end{bmatrix}\\ &+12\begin{bmatrix}9 & -7 & 1 & -1\\ 0 & 16 & -4 & -8\end{bmatrix}=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\end{aligned} \tag{23}$$

and

$$\sum_{j=0}^{2} b_{2-i}\left[A_{22}^{3-j}A_{21} \quad A_{22}^{4-j}\right] = b_2\left[A_{22}^{3}A_{21} \quad A_{22}^{4}\right]$$
$$+b_1\left[A_{22}^{2}A_{21} \quad A_{22}^{3}\right] + b_0\left[A_{22}A_{21} \quad A_{22}^{2}\right]$$
$$= \begin{bmatrix} 2 & 102 & 249 & -304 \\ -49 & -302 & -608 & 857 \end{bmatrix}$$
$$+8\begin{bmatrix} 3 & -16 & -49 & 51 \\ 11 & 54 & 102 & -151 \end{bmatrix}$$
$$+13\begin{bmatrix} -2 & 2 & 11 & -8 \\ -3 & -10 & -16 & 27 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{24}$$

This confirms Theorem 3.

5. FRACTIONAL DIFFERENT ORDER LINEAR ELECTRICAL CIRCUITS

In this section, Theorem 3 will be applied to the fractional different order linear electrical circuits. Consider the fractional electrical circuit shown in Fig. 1 with given resistances $R_0 = R_1 = R_2 = R_3 = R_4 = 1$, capacitances $C_1 = C_2 = 1$, inductances $L_1 = 1$, $L_2 = 2$ and source voltage e. As the state variables, the currents i_1, i_2 and the voltages u_{C1}, u_{C2} are chosen.

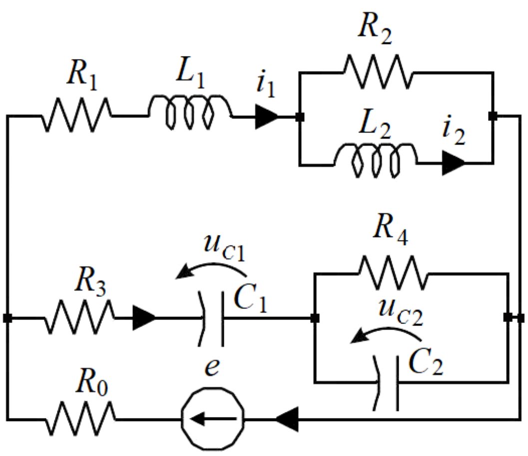

Fig. 1. Fractional electrical circuit

Using Kirchhoff's laws we can write the equations

$$\begin{aligned}
e &= (R_0+R_1+R_2)i_1 - R_2 i_2 + L_1\frac{\mathrm{d}^\alpha i_1}{\mathrm{d}t^\alpha} + R_0 C_1 \frac{\mathrm{d}^\beta u_{C1}}{\mathrm{d}t^\beta},\\
L_2\frac{\mathrm{d}^\alpha i_2}{\mathrm{d}t^\alpha} &= R_2(i_1 - i_2),\\
e &= (R_0+R_3)C_1\frac{\mathrm{d}^\beta u_{C1}}{\mathrm{d}t^\beta} + R_0 i_1 + u_{C1} + u_{C2},\\
C_1\frac{\mathrm{d}^\beta u_{C1}}{\mathrm{d}t^\beta} &= \frac{u_{C2}}{R_4} + C_2\frac{\mathrm{d}^\beta u_{C2}}{\mathrm{d}t^\beta}
\end{aligned} \tag{25}$$

which can be written in the form

$$\begin{bmatrix} \frac{\mathrm{d}^\alpha i_1}{\mathrm{d}t^\alpha} \\ \frac{\mathrm{d}^\alpha i_2}{\mathrm{d}t^\alpha} \\ \frac{\mathrm{d}^\beta u_{C1}}{\mathrm{d}t^\beta} \\ \frac{\mathrm{d}^\beta u_{C2}}{\mathrm{d}t^\beta} \end{bmatrix} = A\begin{bmatrix} i_1 \\ i_2 \\ u_{C1} \\ u_{C2} \end{bmatrix} + Be, \tag{26}$$

where

$$A = \begin{bmatrix} -\frac{(R_1+R_2)(R_0+R_3)+R_0R_3}{L_1(R_0+R_3)} & \frac{R_2}{L_1} & \frac{R_0}{L_1(R_0+R_3)} & \frac{R_0}{L_1(R_0+R_3)} \\ \frac{R_2}{L_2} & -\frac{R_2}{L_2} & 0 & 0 \\ \frac{R_0}{C_1(R_0+R_3)} & 0 & -\frac{1}{C_1(R_0+R_3)} & -\frac{1}{C_1(R_0+R_3)} \\ -\frac{R_0}{C_2(R_0+R_3)} & 0 & -\frac{1}{C_2(R_0+R_3)} & -\frac{R_0+R_3+R_4}{C_2R_1(R_0+R_3)} \end{bmatrix},$$
$$B = \begin{bmatrix} -\frac{R_0}{L_1(R_0+R_3)} \\ 0 \\ \frac{1}{C_1(R_0+R_3)} \\ \frac{1}{C_2(R_0+R_3)} \end{bmatrix}. \tag{27}$$

Substituting the given values of resistances, inductances and capacitance we obtain

$$A_{11} = \begin{bmatrix} -2.5 & 1 \\ 0.5 & -0.5 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \end{bmatrix},$$
$$A_{21} = \begin{bmatrix} -0.5 & 0 \\ 0.5 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -0.5 & -0.5 \\ -0.5 & -1.5 \end{bmatrix}. \tag{28}$$

Note that $A_{11} \in M_2$ and $-A_{22} \in \Re_+^{2\times 2}$ are asymptotically stable, where M_n is the set of $n\times n$ Metzler matrices (its off-diagonal entries are nonnegative) and $\Re_+^{n\times m}$ is the set of $n\times m$ real matrices with nonnegative entries.

In this case the characteristic polynomials (16) have the forms

$$\det\left[I_2 s_\alpha - A_{11}\right] = \begin{vmatrix} s_\alpha + 2.5 & -1 \\ -0.5 & s_\alpha + 0.5 \end{vmatrix}$$
$$= s_\alpha^2 + 3s_\alpha + 0.75 \tag{29}$$

and

$$\det\left[I_2 s_\beta - A_{22}\right] = \begin{vmatrix} s_\beta + 0.5 & 0.5 \\ 0.5 & s_\beta + 1.5 \end{vmatrix} = s_\beta^2 + 2s_\beta + 0.5 \tag{30}$$

respectively.

Using (29), (30), and (27) we obtain

$$\begin{aligned}\sum_{i=0}^{2} a_{2-i}\left[A_{11}^{4-i} \quad A_{11}^{3-i}A_{12}\right] &= \left[A_{11}^4 \quad A_{11}^3 A_{12}\right] \\ &+ 3\left[A_{11}^3 \quad A_{11}^2 A_{12}\right] + 0.75\left[A_{11}^2 \quad A_{11}A_{12}\right] \\ &= \begin{bmatrix} 50.063 & -22.5 & -9.188 & -9.188 \\ -11.25 & 5.063 & 2.063 & 2.063 \end{bmatrix} \\ &+ 3\begin{bmatrix} -18.375 & 8.25 & 3.375 & 3.375 \\ 4.125 & -1.875 & -0.75 & -0.75 \end{bmatrix} \\ &+ 0.75\begin{bmatrix} 6.75 & -3 & -1.25 & -1.25 \\ -1.5 & 0.75 & 0.25 & 0.25 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\end{aligned} \tag{31}$$

and

$$\begin{aligned}\sum_{j=0}^{2} b_{2-i}\left[A_{22}^{3-j}A_{21} \quad A_{22}^{4-j}\right] &= \left[A_{22}^3 A_{21} \quad A_{22}^4\right] \\ &+ 2\left[A_{22}^2 A_{21} \quad A_{22}^3\right] + 0.5\left[A_{22}A_{21} \quad A_{22}^2\right] \\ &= \begin{bmatrix} -0.5 & 0 & 1.25 & 3 \\ -1.25 & 0 & 3 & 7.25 \end{bmatrix} \\ &+ 2\begin{bmatrix} 0.25 & 0 & -0.75 & -1.75 \\ 0.75 & 0 & -1.75 & -4.25 \end{bmatrix} \\ &+ 0.5\begin{bmatrix} 0 & 0 & 0.5 & 1 \\ -0.5 & 0 & 1 & 2.5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.\end{aligned} \tag{32}$$

Substitution of (31) and (32) into (17) confirms that the fractional different orders linear electrical circuit satisfies Theorem 3.

6. CONCLUDING REMARKS

The classical Cayley–Hamilton theorem has been extended to the different order fractional continuous-time linear systems (Theorem 3). The new theorem has been applied to different order fractional electrical circuits. The considerations can be extended to fractional different order linear systems described by a linear equation with any finite number of sub-vectors of the state vector [6] and to the fractional different order discrete-time linear systems.

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