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Coexisting of self-excited and hidden attractors in a new 4D hyperchaotic Sprott-S system with a single equilibrium point

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Coexisting self-excited and hidden attractors for the same set of parameters in dissipative dynamical systems are more interesting, important, and difficult compared to other classes of hidden attractors. By utilizing of nonlinear state feedback controller on the popular Sprott-S system to construct a new, unusual 4D system with only one nontrivial equilibrium point and two control parameters. These parameters affect system behavior and transformation from hidden attractors to self-excited attractors or vice versa. As compared to traditional similar kinds of systems with hidden attractors, this system is distinguished considering it has $(n-2)$ positive Lyapunov exponents with maximal Lyapunov exponent. In addition, the coexistence of multi-attractors and chaotic with 2-torus are found in the system through analytical results and experimental simulations which include equilibrium points, stability, phase portraits, and Lyapunov spectrum. Furthermore, the anti-synchronization realization of two identical new systems is done relying on Lyapunov stability theory and nonlinear controllers strategy. lastly, the numerical simulation confirmed the validity of the theoretical results.

Key words: 3D Sprott S system, New 4D Sprott S system, multiple attractors, anti-synchronization

1. Introduction

In dissipative systems, the attractors play a very important role in the categorization of dynamic systems into two main types: self-excited and hidden attractors. Most of the well-known conventional chaotic/hyperchaotic systems have self-excited attractors starting from Lorenz, 3D Rössler, 4D Rössler, Chen and Liu systems in 1963 1976,1979,1999, and 2004, respectivly [1–3]. In 2010, Kuznetsov et al., introduce the first concept of hidden attractor, but this phe-

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nomenon did not receive great interest until the year 2011 when Leonov et al. developed Chua's circuit [4–6]. Hidden attractors for dynamic systems occur in three categories; (i) just stable equilibrium points [7], (ii) without points [8–12], (iii) Curve / line equilibria [13,14]. As compared to self-excited attractors [15–17], hidden attractors are more complex to detect due to the location of the point not being essential as it is in the other category.

In literature, most of the current work has focused on investigating simple dynamical systems without equilibrium points or stable equilibrium points (system with hidden attractors) [5–14]. Also, most of the mentioned systems were adopted on the three conditions for Sprott in the year 2011 [18]. But, other conditions ought to satisfy in proposed systems:

1. The system should have $(n-2)$ positive Lyapunov exponents.
2. The system should have a maximal Lyapunov exponent (MLE) compared to the original system.

Many research ignored these two conditions, whereas others are concerned with research systems with multiple/coexisting attractors between hidden and self-excited attractors [6]. These reasons motivated us to find a new rare 4D system with $(n-2)$ positive Lyapunov exponents (satisfied the two conditions mentioned above) and have multiple attractors hidden and self-excited. The available 4D dimensions $(n = 4)$ systems with $(n-i,\ i = 1, 2)$ positive Lyapunov exponents and different attractors are listed in Table 1. It is noted from Table 1 that very

Table 1: Categorization of different 4-D chaotic/hyperchaotic systems

No.	System behavior	No. $+veLE_s$	Nature of equilibria	Attractors behavior	Total terms	Nonlinear terms	References
1	Chaotic	$n-3$	stable point	hidden	10	3	2020 [7]
2	Chaotic	$n-3$	no equilibria	hidden	9	2	2017 [8]
3	Hyperchaotic	$n-2$	no equilibria	hidden	7	2	2014 [9]
4	Chaotic	$n-3$	no equilibria	hidden	8	1	2018 [10]
5	Chaotic	$n-3$	no equilibria	hidden	8	1	2017 [11]
6	Hyperchaotic	$n-2$	no equilibria	hidden	9	5	2012 [12]
7	Chaotic	$n-3$	curve of points	hidden	8	1	2017 [13]
8	Chaotic	$n-3$	infinite equilibria	hidden	7	2	2020 [14]
9	Hyperchaotic	$n-2$	unstable point	self-excited	9	4	2012 [15]
10	Hyperchaotic	$n-2$	three equilibria	self-excited	11	3	2012 [16]
11	Hyperchaotic	$n-2$	unstable point	self-excited	9	2	2020 [17]
12	Chaotic	$n-3$	no equilibria	hidden	8	2	2021 [46]
13	Hyperchaotic	$n-2$	stable/unstable	multiple	9	2	this work

few systems are interested in the above-mentioned two conditions as well as to discover with coexisting of hidden and self-excited attractors.

Since Pecora and Carroll presented the first phenomenon in chaos synchronization for the systems with self-excited attractors [19–21]] which witnessed great attention from various areas such as engineering [22,24], chaos control [25], chaos synchronization [26] and encryption [27–30]. Numerous phenomena such as complete synchronization (CS) [1, 2, 31], anti-synchronization (AS) [32, 33], hybrid synchronization (HS) [34, 35], projective synchronization (PS) [36–38] were reported.

Nowadays, CS and AS is the most common phenomenon are used and they are similar in that the master and slave systems converge to zero according to the formula $e_i(t) = y_i(t) - sx_i(t)$, but different in s scaling factor where $s = 1$ and $s = -1$ for CS and AS respectively [39, 40]. Thereafter a few works are interesting to apply these phenomena on systems with hidden attractors. This is another reason that motivated us to do this work.

The main contribution of this paper can be summarized in the following points:

- A new 4D dissipative system is suggested from the well-known 3D Sprott-S system.
- The proposed system has a single stable/unstable equilibrium point, it exhibits hidden and self-excited attractors. So it can be considered one of the unfamiliar systems. This point was fulfilled in Table 1.
- It consists of nine terms, including two of nonlinearity and two parameters.
- The system satisfies $(n-2)$ positive Lyapunov exponents with the largest Lyapunov exponents.
- This system is used to realize anti-synchronization.

2. Dynamics and analysis of the new system

Sprott introduced a simple 3D (Sprott S) system [41] that consists of six terms included only one quadratic nonlinear term z^2 without parameters and described by:

$$\begin{aligned} \dot{x} &= -x - 4y, \\ \dot{y} &= x + z^2, \\ \dot{z} &= 1 + x, \end{aligned} \tag{1}$$

where x, y, z are state variables. This system possesses two nontrivial critical points with one positive Lyapunov exponent $LE_1 = 0.188$ and Lyapunov dimension $D_{LE} = 2.151$. In 2021, based on Sprott S system and state feedback control strategy [42–45], a 4D system with $(n-3)$ positive Lyapunov exponent and Lyapunov dimension $D_{LE} = 3.11971$ was suggested by Al-hayali and Al-Azzawi [46], which is described as

$$\begin{aligned}
\dot{x} &= -x - 4y, \\
\dot{y} &= x + z^2 + au, \\
\dot{z} &= 1 + x, \\
\dot{u} &= -by,
\end{aligned} \tag{2}$$

and the control parameters are $(a, b) = (0.01,\ 0.1)$. By introducing nonlinear feedback control to the second equation of 3D Sprott-S system or if adding the feedback term $(-xz)$ to the fourth equation of a system (2), a new 4D system hyperchaotic attractor with $(n-2)$ positive Lyapunov exponent can be proposed as:

$$\begin{aligned}
\dot{x} &= -x - 4y, \\
\dot{y} &= x + z^2 + au, \\
\dot{z} &= 1 + x, \\
\dot{u} &= -by - xz,
\end{aligned} \tag{3}$$

where $x,\ y,\ z,\ u$ are state variables and a and b are both control parameters. Clearly, the new system consists of nine terms, including a single nontrivial equilibrium point with two nonlinearities. This system exhibits a hyperchaotic attractor when $a = 0.0005$, $b = 0.1$, and $x(0) = (1,\ 1,\ 1,\ 1)^T$ as shown in Fig. 1. In addition, the proposed system satisfies one of the Sprott conditions 2011 "system proposed must exhibit some previously unobserved" in creating new systems and our system has rare behavior coexistence of hidden and self-excited attractors

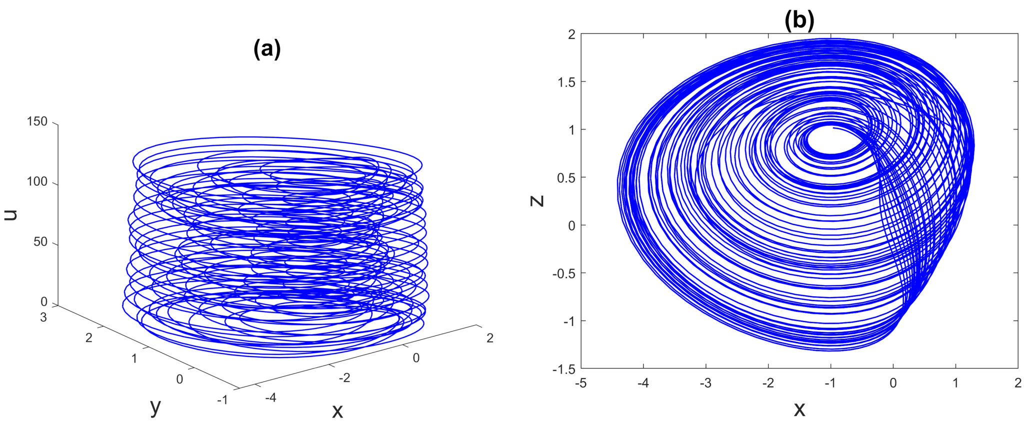

Figure 1: Hyperchaotic attractors of the system (3) in space (a) and plane (b) are illustrated

versus control parameter as illustrates in Fig. 2a–d under the control parameters $b = 50$, $a = 0.7, 1, 1.5, 1.7$ respectively and $x(0) = (0.1,\ 0.1,\ 0.1,\ 0.1)^T$.

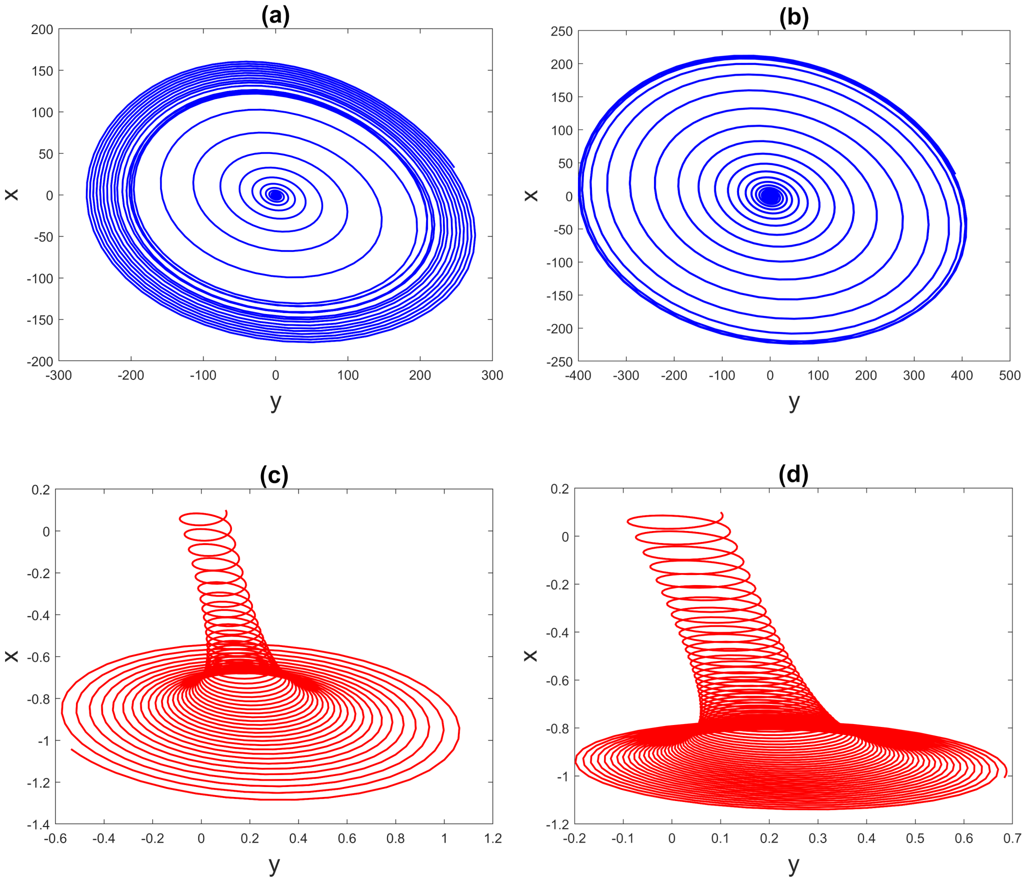

Figure 2: Multiple attractors of the 4D system (3) on the x-y plane with $b = 50$ in: (a) $a = 0.7$, (b) $a = 1$, (c) $a = 1.5$, and (d) $a = 1.7$

2.1. Equilibrium points and stability

Via solving $\dot{x} = \dot{y} = \dot{z} = \dot{u} = 0$, one getting a single nontrivial equilibrium point

$$P_1 = \begin{bmatrix} -1 \\ 0.25 \\ 0.25b \\ \dfrac{(1 - 0.0625b^2)}{a} \end{bmatrix}$$

and the Jacobian matrix of the new system at P_1 and characteristic equation are respectively as

$$J = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} & \frac{\partial \dot{x}}{\partial z} & \frac{\partial \dot{x}}{\partial u} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} & \frac{\partial \dot{y}}{\partial z} & \frac{\partial \dot{y}}{\partial u} \\ \frac{\partial \dot{z}}{\partial x} & \frac{\partial \dot{z}}{\partial y} & \frac{\partial \dot{z}}{\partial z} & \frac{\partial \dot{z}}{\partial u} \\ \frac{\partial \dot{u}}{\partial x} & \frac{\partial \dot{u}}{\partial y} & \frac{\partial \dot{u}}{\partial z} & \frac{\partial \dot{u}}{\partial u} \end{bmatrix} = \begin{bmatrix} -1 & -4 & 0 & 0 \\ 1 & 0 & 2z & a \\ 1 & 0 & 0 & 0 \\ -z & -b & -x & 0 \end{bmatrix}$$

$$\Rightarrow \; J(P_1) = \begin{bmatrix} -1 & -4 & 0 & 0 \\ 1 & 0 & b/2 & a \\ 1 & 0 & 0 & 0 \\ -b/4 & -b & 1 & 0 \end{bmatrix}, \tag{4}$$

$$\lambda^4 + \lambda^3 + (ab+4)\lambda^2 + 2b\lambda + 4a = 0. \tag{5}$$

To determine whether the Eq. (5) is stable, it should use the Routh-Hurwitz criteria to obtain

- $A_1 > 0$, $A_4 > 0$,
- $A_1 A_2 > A_3$,
- $A_1 A_2 A_3 > A_3^2 + A_1^2 A_4$,

where $A_1 = 1$, $A_2 = ab+4$, $A_3 = 2b$ and $A_4 = 4a$. Therefore, the proposed system possesses roots with the negative real part under the condition $a > \frac{2(b-2)}{b}$ and $b \neq 0$. This relationship between both control parameters a and b can be summarized in Fig. 3 and Table 2 with fixed control parameters $b = 4$ and $b = 2$.

It is noted from Table 2 that the equilibrium point P_1 is unstable under the condition $a < \frac{2(b-2)}{b}$ because eigenvalues calculated via solving Eq. (5) contain some with positive real parts. So, the system (2) belongs to self-excited attractors. Whereas, it belongs to hidden attractors under $a > \frac{2(b-2)}{b}$. Thus, we conclude that this system has two types of attractors for the same equilibrium point P_1 based on the relationship between control parameters a and b as:

- Self-excited attractors when $a < \frac{2(b-2)}{b}$.
- Hidden chaotic attractors when $a > \frac{2(b-2)}{b}$.

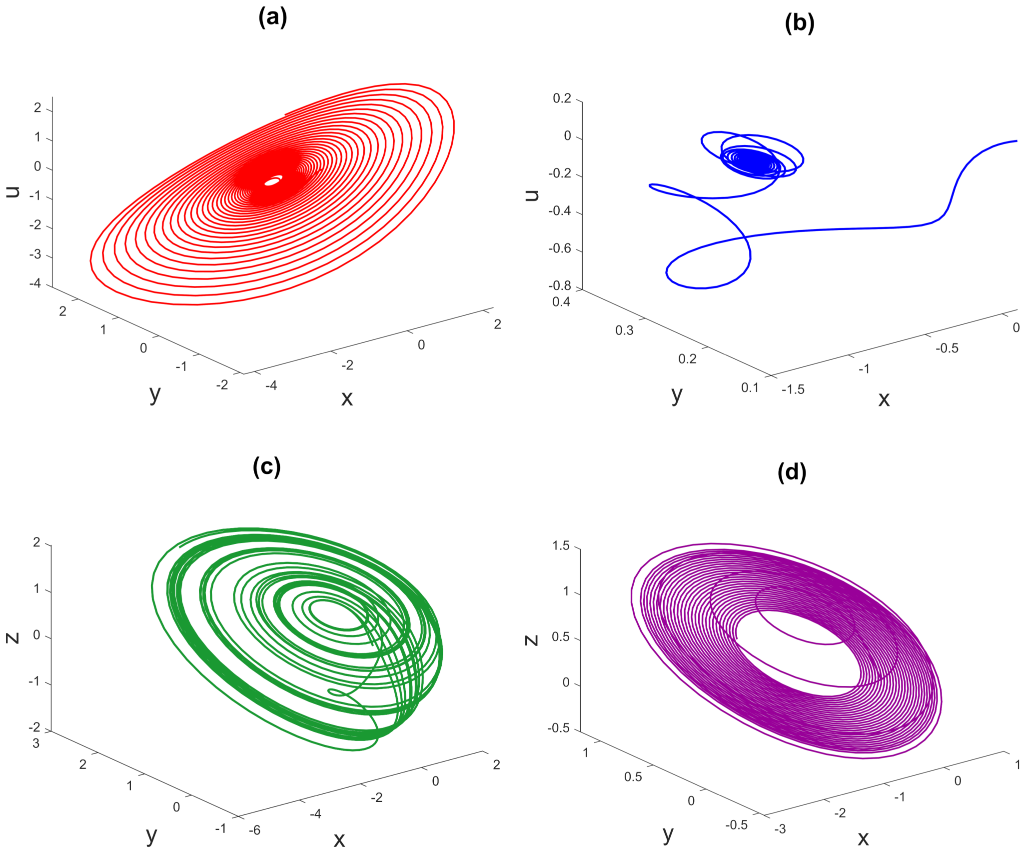

Figure 3: Relationship between both control parameters a and b in new system: (a) $a = 1$, (b) $a = 1.5$ under $b = 4$, (c) $a = -0.009$, and (d) $a = 0.2$ under $b = 2$

Table 2: The relationship between both control parameters a and b

Figure	b	a	Characteristic polynomial	Roots	Stability
Fig. 3(a)	4	1	$\lambda^4 + \lambda^3 + 8\lambda^2 + 8\lambda + 4$	$\lambda_{1,2} = -0.5329 \pm 0.4966i$ $\lambda_{3,4} = 0.0329 \pm 2.7456i$	unstable
Fig. 3(b)		1.5	$\lambda^4 + \lambda^3 + 10\lambda^2 + 8\lambda + 6$	$\lambda_{1,2} = -0.4285 \pm 0.6831i$ $\lambda_{3,4} = -0.0715 \pm 3.036i$	stable
Fig. 3(c)	2	-0.009	$\lambda^4 + \lambda^3 + 3.982\lambda^2 + 4\lambda - 0.036$	$\lambda_1 = 0.0089$ $\lambda_2 = -0.0107$ $\lambda_{3,4} = 0.009 \pm 1.9982i$	unstable
Fig. 3(d)		0.2	$\lambda^4 + \lambda^3 + 4.4\lambda^2 + 4\lambda + 0.8$	$\lambda_1 = -0.2855$ $\lambda_2 = -0.6723$ $\lambda_{3,4} = -0.0211 \pm 2.044i$	stable

2.2. Lyapunov exponents and dissipativity

Lyapunov exponents (LE_S) are tools used to distinguish between attractors' chaotic/hyperchaotic systems. There are many algorithms are available for the computation of the LE_S. Wolf Algorithm is one of the most common algorithms used in calculating LE_S. With control parameters $a = 0.0005$, $b = 0.1$ and $x(0) = (1, 1, 1, 1)^T$, step (sampling time)= 0.5, observation time = 300, the LE_S of the proposed system are numerically found as:

$$\begin{aligned} LE_1 &= 0.2041, \\ LE_2 &= 0.0023, \\ LE_3 &= -0.0008, \\ LE_4 &= -1.2056, \end{aligned} \qquad \sum LE_S = -1. \tag{6}$$

The maximal Lyapunov exponent (MLE) of the 4D system (2) is $LE_1 = 0.2041$, which is greater than the maximal Lyapunov of the original system (1) i.e., $LE_1 = 0.188$. This indicates that the proposed system is highly efficient compared to the original system (1). The corresponding Lyapunov spectrum is shown in Fig. 4. Besides, the Kaplan-Yorke dimension of the new system (2) is calculated as:

$$D_{LE} = j + \frac{1}{|LE_{j+1}|} \sum_{i=1}^{j} LE_i = 3 + \frac{LE_1 + LE_2 + LE_3}{|LE_4|} = 3.1705$$

which will give a good picture of the complexity of the proposed system. It is observed that from Eq. (6), the new system is dissipative due to $\left(\sum LE_S = -1\right)$

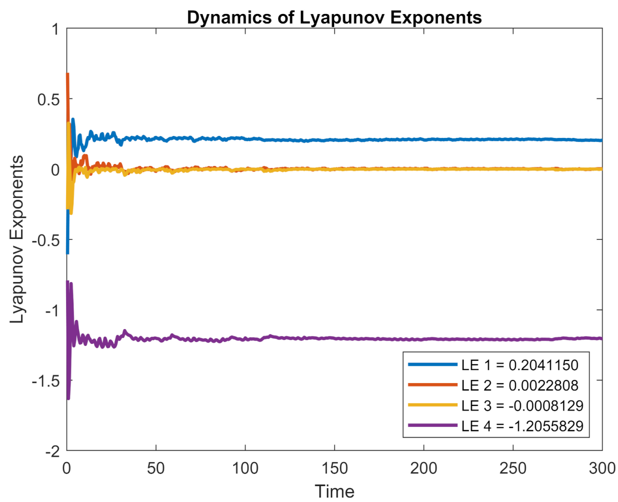

Figure 4: Lyapunov exponents of the system (3) for $a = 0.0005$, $b = 0.1$

is negative and the divergence of a matrix (4) is defined

$$\nabla v = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} + \frac{\partial \dot{u}}{\partial u} = -1.$$

In other words, the trace of a matrix (4) as

$$\operatorname{tr}\left(J\left(P_1\right)\right) = \sum_{i=1}^{4} a_{ii} = a_{11} + a_{22} + a_{33} + a_{44} \quad \Rightarrow \quad \operatorname{tr}\left(J\left(P_1\right)\right) = -1.$$

Clearly, if $\sum LE_S = \nabla v$ then our results are consistent with numerical simulations (Wolf Algorithm).

Tables 3–5 give more details about computing LE_S with various values of parameters and initial condition $x(0) = (1,\ 1,\ 1,\ 1)^T$.

Table 3: LE_S of the new system for $b = 0.1$ with varying the a and $x(0) = (1,\ 1,\ 1,\ 1)^T$

No.	a	LE_1	LE_2	LE_3	LE_4	Sign of the LE_S	Sum of LE_S	Behavior
1	0.001	0.1547	0.0008	−0.0004	−1.1551	$(+, 0, 0, -)$	−1	chaotic 2-torus
2	0.002	0.0971	0.0009	−0.0734	−1.0246	$(+, 0, -, -)$	−0.9999	chaotic
3	0.0005	0.2041	0.0023	−0.0008	−1.2056	$(+, +, 0, -)$	−1	hyperchaotic
4	0.0007	0.1300	0.0018	0.0001	−1.132	$(+, +, 0, -)$	−1.0001	hyperchaotic
5	0.0009	0.1559	0.0003	−0.0008	−1.1553	$(+, 0, 0, -)$	−0.9999	chaotic 2-torus

Table 4: LE_S of the new system for $b = 0.01$ with varying the a and $x(0) = (1, 1, 1, 1)^T$

No.	a	LE_1	LE_2	LE_3	LE_4	Sign of the LE_S	Sum of LE_S	Behavior
1	0.001	0.1008	0.0032	−0.0001	−1.1039	$(+, +, 0, -)$	−1	hyperchaotic
2	0.0003	0.1983	0.0005	−0.0001	−1.1991	$(+, 0, 0, -)$	−1.0001	chaotic 2-torus
3	0.0005	0.1538	−0.0001	−0.0007	−1.1529	$(+, 0, 0, -)$	−1.0001	chaotic 2-torus
4	0.0007	0.1317	0.004	0.0005	−1.1363	$(+, +, 0, -)$	−1.0001	hyperchaotic

It is observed from Tables 3–5, the sum of the LE_S is almost equal to the divergence of the matrix (4). Also, it noticed that from Table 3 (No. 1, 5), Table 4 (No. 2, 3), Table 5 (No. 2, 3) that system (3) has two zero of LE_S i.e., $(+, 0, 0, -)$ this phenomenon is called chaotic with 2-torus and it is considered a rare case.

Table 5: LE_S of the new system for $a = 0.001$ with varying the b and $x(0) = (1, 1, 1, 1)^T$

No.	b	LE_1	LE_2	LE_3	LE_4	Sign of the LE_S	Sum of LE_S	Behavior
1	0.5	0.1263	0.001	−0.0005	−1.1268	$(+, +, 0, -)$	−0.9999	hyperchaotic
2	0.7	0.131	0.0001	−0.0003	−1.1309	$(+, 0, 0, -)$	−1.0001	chaotic 2-torus
3	0.001	0.1558	0.0001	−0.0001	−1.1559	$(+, 0, 0, -)$	−1.0001	chaotic 2-torus
4	0.01	0.1009	0.0032	−0.0001	−1.1039	$(+, +, 0, -)$	−0.9999	hyperchaotic
5	0.03	0.1169	0.0004	−0.0024	−1.115	$(+, 0, -, -)$	−1.0001	chaotic

3. Multistability and bifurcations with initial conditions

Using phase portrait, the coexistence of many attractors of the proposed system is presented in Fig. 5, with the parameters $a = 0.5$ and $b = 50$ and different initial conditions, where the red attractors are corresponding to the initial conditions (IC) $(0.1, 0.1, 0.1, 0.1)$, the blue attractor is corresponding to $(-0.5, -0.3, -0.3, -0.3)$, the green attractors is corresponding to $(-0.3, 0.7, 0.5, -0.6)$, whereas the magenta attractors are corresponding to $(-0.5, 0.4, 0.4, -0.1)$. Figs. 6 and 7 show the coexistence's dynamical behavior of the new system. In addition, there are different behaviors, although they belong to the same control parameters a and b, which indicates the sensitivity of the initial conditions and called for this phenomenon a bifurcation with initial conditions. So, there exist many different self-coexisting attractors in the new system. For illustration, Fig. 8 exhibits various attractors under the control parameters $(a, b) = (0.001,\ 0.01)$ and the corresponding initial conditions are given in Table 6.

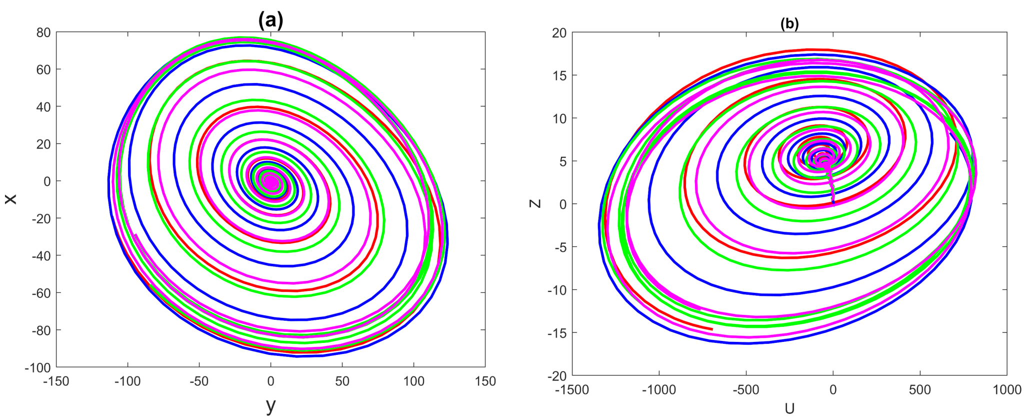

Figure 5: The attractors of a system (3) with different IC for $a = 0.5$, $b = 50$ in: (a) x-y plane, (b) z-u plane

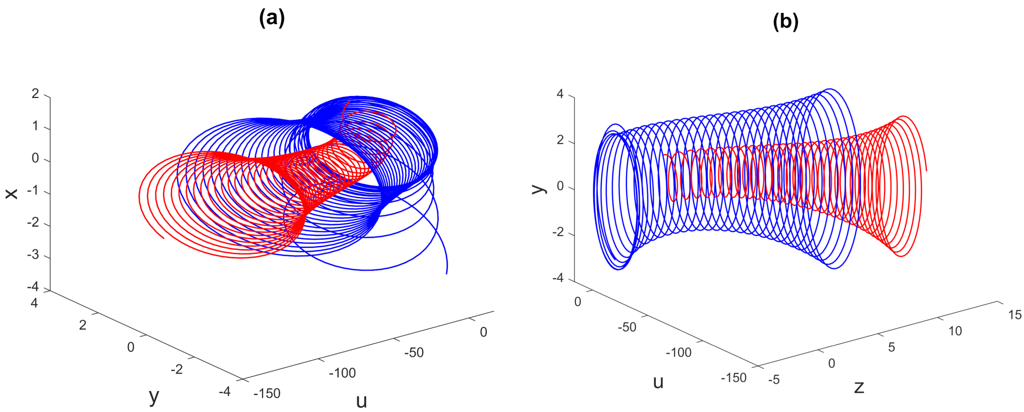

Figure 6: Coexistences of attractors of system (3) with $a = 1.5$, $b = 70$ and IC $(1, 1, 1, 1)$ for red $(-3, -3, -3, 1)$ for blue

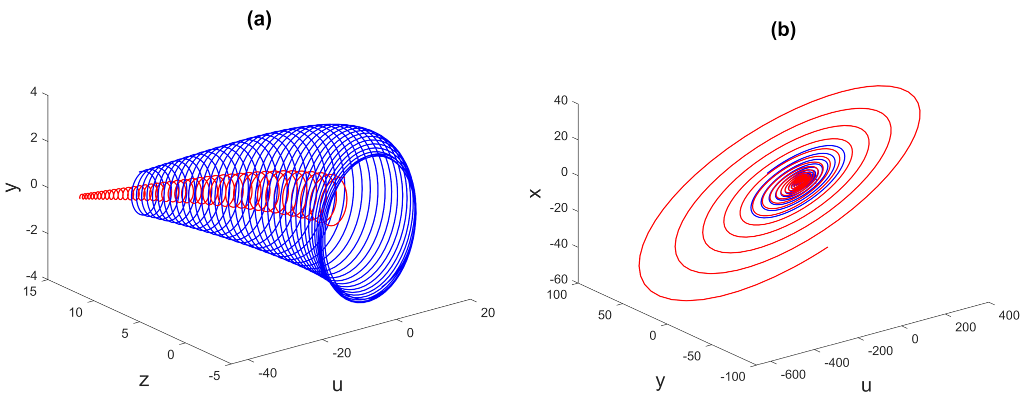

Figure 7: Coexistences of attractors of system (3) with different value of a, $b = 50$ and initial conditions $(1, 1, 1, 1)$ for red $(-3, -3, -3, 1)$ for blue in: (a) $a = 4$, (b) $a = 1.3$

Table 6: The corresponding initial condition of the behavior is shown in Fig. 8 at $a = 0.001$, $b = 0.01$

Figure	Initial conditions	LE_i	Sign of LE_S	Attractor
Fig. 8(a)	$(1, 1, 1, 1)$	$(0.1008, 0.0032, 0.0001, -1.1039)$	$(+, +, 0, -)$	hyperchaotic attractor
Fig. 8(b)	$(0.1, 0.1, 1, 1)$	$(0.0907, 0.0007, -0.0023, -1.0892)$	$(+, 0, -, -)$	chaotic attractor
Fig. 8(c)	$(0.6, 0.6, 0.6, 0.65)$	$(0.1265, 0.0006, 0.0004, -1.1275)$	$(+, 0, 0, -)$	chaotic 2-torus attractor

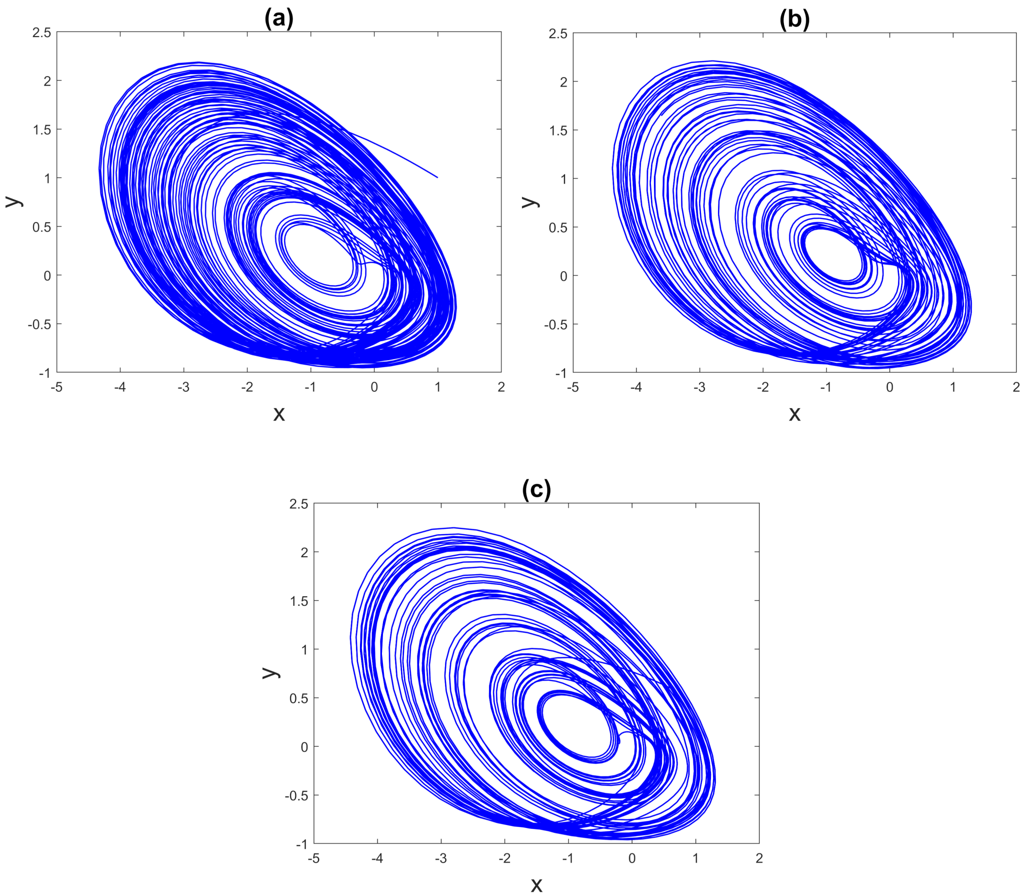

Figure 8: Self-excited attractors: (a) hyperchaotic attractors, (b) chaotic attractors, (c) chaotic 2-torus attractors

4. Anti-synchronization

According to the anti-synchronization, a new system can be modeled to drive and response systems and represented in Eq. (7) and Eq. (8), respectively.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \underbrace{\begin{bmatrix}-1 & -4 & 0 & 0\\ 1 & 1 & 0 & a\\ 1 & 0 & 0 & 0\\ 0 & -b & 0 & 0\end{bmatrix}}_{A}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \underbrace{\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_3^2\\ 1\\ -x_1x_3\end{bmatrix}}_{f(x)}, \tag{7}$$

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\\ \dot{y}_4\end{bmatrix} = \underbrace{\begin{bmatrix}-1 & -4 & 0 & 0\\ 1 & 1 & 0 & a\\ 1 & 0 & 0 & 0\\ 0 & -b & 0 & 0\end{bmatrix}}_{B}\begin{bmatrix}y_1\\ y_2\\ y_3\\ y_4\end{bmatrix} + \underbrace{\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}y_3^2\\ 1\\ -y_1y_3\end{bmatrix}}_{g(y)} + \underbrace{\begin{bmatrix}u_1\\ u_2\\ u_3\\ u_4\end{bmatrix}}_{U}. \tag{8}$$

It is clear that the matrix A is equal to matrix B due to the two systems being identical and U is the control to design. define the error dynamics for anti-synchronization between the above systems as $e_i(t) = y_i(t) - sx_i(t)$, where $s = -1$ is constant scaling factor and satisfied the condition (9)

$$\lim_{t\to\infty} e_i(t) = 0. \tag{9}$$

Adding system (7) from the system (8) leads to an error dynamics system in (10)

$$\begin{aligned}\dot{e}_1 &= -e_1 - 4e_2 + u_1\,,\\ \dot{e}_2 &= e_1 + ae_4 + e_3^2 - 2x_3y_3 + u_2\,,\\ \dot{e}_3 &= 2 + e_1 + u_3\,,\\ \dot{e}_4 &= -be_2 - e_1e_3 + x_1y_3 + y_1x_3 + u_4\,.\end{aligned} \tag{10}$$

We design the following nonlinear controller:

$$\begin{aligned}u_1 &= -e_3 - 36e_2 + e_3e_4 \qquad ,\\ u_2 &= 3e_1 - e_2 + 2x_3y_3\,,\\ u_3 &= -2 - e_3 - 10e_2e_3\,,\\ u_4 &= -x_1y_3 - x_3y_1 - e_4\,.\end{aligned} \tag{11}$$

Inserted controller (11) into (10) we get:

$$\begin{aligned}\dot{e}_1 &= -e_1 - 40e_2 - e_3 + e_3e_4 \qquad ,\\ \dot{e}_2 &= 4e_1 - e_2 + ae_4 + e_3^2\,,\\ \dot{e}_3 &= e_1 - e_3 - 10e_2e_3\,,\\ \dot{e}_4 &= -be_2 - e_4 - e_1e_3\,.\end{aligned} \tag{12}$$

To check whether the proposed control is suitable for the error system (10), we find the characteristic equations and the corresponding eigenvalues for each equation before and after adding the control i.e. for systems (9) and (12) respectively, and Table 7 summarizes these results.

System (12) possesses eigenvalues with negative real parts. therefore, the proposed controller (11) fulfilled the anti-synchronization between systems (7) and (8).

Table 7: The characteristic equation and roots for systems (10) and (12) with $a = 0.01$ and $b = 0.1$

Syst.	characteristic equation	$\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4$	Sign of the real λ_i	Stability
(10)	$\lambda^4 + \lambda^3 + (ab+4)\lambda^2 + ab\lambda = 0$	$\lambda_1 = 0$ $\lambda_2 = -0.0002$ $\lambda_{3,4} = -0.4999 \pm 1.9367i$	$(0, -, -, -)$	unstable
(12)	$\lambda^4+\lambda^3+(ab+23)\lambda^2+(2ab+38)\lambda$ $+2ab+18=0$	$\lambda_{1,2} = -1 \pm 3.9660i$ $\lambda_{3,4} = -1 \pm 1.1276i$	$(-, -, -, -)$	stable

According to another method, construct Lyapunov function as

$$V(e_i) = e_i^T P e_i \quad \Rightarrow \quad V(e_i) = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \end{bmatrix} \underbrace{\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{20} \end{bmatrix}}_{P} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \tag{13}$$

and

$$\dot{V}(e_i) = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4\,, \tag{14}$$

$$\begin{aligned} \dot{V}(e) &= e_1\left(-e_1 - 40e_2 - e_3 + e_3e_4\right) + e_2\left(4e_1 - e_2 + ae_4 + e_3^2\right) \\ &\quad + e_3\left(e_1 - e_3 - 10e_2e_3\right) + e_4\left(-be_2 - e_4 - e_1e_3\right), \end{aligned} \tag{15}$$

$$\begin{aligned} \dot{V}(e_i) &= -e_1^2 - 10e_2^2 - e_3^2 - e_4^2 \\ &= -\begin{bmatrix} e_1 & e_2 & e_3 & e_4 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{Q} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = -e_i^T Q e_i\,, \end{aligned} \tag{16}$$

where $Q = \mathrm{diag}(1, 10, 1, 1)$, yields to $Q > 0$. As a result, $\dot{V}(e_i) < 0$ (negative definite) on R^4. The proposed controller (11) achieves the anti-synchronization.

Figure 9 illustrates numerically the anti-synchronization that takes place between (7) and (8) with the IC as $(2, 7, 10, -5)$ and $(10, 1, 3, 8)$ which confirms the validity of the theoretical results.

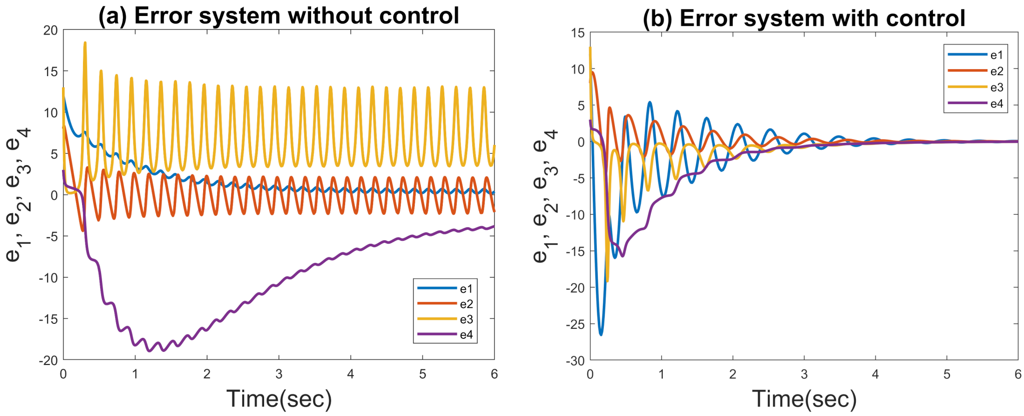

Figure 9: Anti-synchronization errors with controllers (11)

5. Conclusions

A novel 4D hyperchaotic system with $(n-2)+veLE_s$ is reported from the 4D Sprott-S system. The new system has nine terms including single nonlinearity with two (bifurcation) parameters and coexistence of multi-attractors which means it belongs to the rare system compared with other existing systems and has maximal Lyapunov exponent. Further, several phenomena of this system have been discovered such as chaotic and chaotic with 2-torus behaviors. Finally, relying on the nonlinear control strategy and Lyapunov stability theory, anti-synchronization for the proposed system is realized.

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