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Adaptive observer design for systems with incremental quadratic constraints and nonlinear outputs – application to chaos synchronization

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This work addresses the problem of adaptive observer design for nonlinear systems satisfying incremental quadratic constraints. The output of the system includes nonlinear terms, which puts an additional strain on the design and feasibility of the observer, which is guaranteed under the satisfaction of an LMI, and a set of algebraic constraints. A particular case where the output nonlinearity matches the unknown parameter coefficient is also discussed. The result is illustrated through a numerical example for the chaos synchronization of the Rössler system.

Key words: adaptive observer, incremental quadratic constraints, chaos synchronization

1. Introduction

The problem of state estimation for dynamical systems is among the most well established problems in control theory, since it serves as the basis for feedback control, stabilization, and synchronization, in linear and nonlinear systems. A standard approach in state estimation is observer design [1]. The observer of a

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system is a secondary, 'slave' system, that can estimate the internal states of the original 'master' system, by taking measurements of the master system's input and output. The observer approach is often opted for state estimation, since its feasibility depends on the solution of a Linear Matrix Inequality (LMI), often in combination with some rank or algebraic conditions on the master system matrices [2, 3]. Since tools to solve LMIs are integrated into most programming environments, like MATLAB or Scilab, the observer matrices can easily be computed.

Often in the problem of state estimation, one additional issue that may occur is that although the system's structure may be known, some of the involved parameters may be unidentified. Thus, in addition to the system states, its parameters have to be identified as well. In this case, the observer is termed as adaptive observer [4], and the internal states and the unknown parameters are estimated simultaneously. The problem was addressed initially for linear systems [5], and later the design was expanded to the general nonlinear case.

Initial works on adaptive observers for nonlinear systems were restricted to Lipschitz systems [6–13] with several applications. Chaotic secure communications are established using robust adaptive sliding mode observer [7] and a relaxed notion of persistency of excitation [8]. References [9, 10] applied the observer to chaotic synchronization. Authors in [11] applied their work to a single-link manipulator with a revolute joints actuator. In [12], one-sided Lipschitz systems are considered with an application to a Vertical Take-Off and Landing (VTOL) aircraft system. An adaptive observer was designed for fault estimation [13]. An interesting modulation approach for masking a chaotic signal was combined with adaptive observer design and the application to chaotic secure communications [14]. Adaptive works on nonlinear systems for nonlinearities other than Lipschitz are very limited. An important type of nonlinearity that has recently gained attention for observer design is Incrementally Quadratic Constraints (δQC) [15, 16]. These works have been applied to chaos synchronization [15] and structural control for a wind turbine [16]. It is a notable fact that Lipschitz, one-sided Lipschitz, and various other types of nonlinearities are a particular case of incrementally quadratic constraints, see [17] for more details.

Moreover, it is worth noting some interesting works, where various types of systems are studied. In [18], a stochastic adaptive sliding mode observer is designed and applied to chaotic secure communications. In [19], a Luenberger observer was realized in a microcontroller for a multistable Kapitaniak chaotic system with known parameters. Also, [20, 21] deal with the discrete time case.

One common assumption by most works is to consider the system's output as a linear combination of the states. Yet, many works point out that it is possible for nonlinear terms to appear in the output equations [22–28]. Motivated by the above, our work extends the results of [15] to the case where nonlinear terms appear in the output, coupled with the linear ones. To the best of our knowledge, these types of systems have not been studied in the existing literature. This is a stricter case compared to having the output being decoupled into a purely linear one and a purely nonlinear one, which is a special case of the one considered here. The feasibility of the observer depends on the solution of an LMI, and a set of algebraic conditions, which are required to discard undesired cross error terms that appear in the derivative of the candidate Lyapunov function. These conditions are then relaxed, by exploring a special case where the output nonlinearity is equal to the nonlinear term inside the state equation. This case may be limiting, but it can be of particular interest in applications related to secure communications, where the designer chooses the output. It could also help to simplify the output feedback control problem, which can be investigated in the future. Finally, the design is illustrated through a numerical example of chaos synchronization for the Rössler system.

The rest of the paper is structured as follows: In Section 2, the problem is formulated along with system description and basic assumptions. In Section 3, the main result on observer design is presented. In Section 4, the observer design procedure is illustrated through a numerical example. Finally, Section 5 concludes the work with a discussion on future goals.

We use the following notations. 0 and I stand for appropriate dimensional zero and identity matrices, respectively. $\mathbb{R}^{m\times n}$ represents the $m \times n$ real matrix set. A^T denotes the transpose of a matrix A. $A > 0$ denotes positive definite matrix. The symbol $\star$ inside a symmetric matrix environment corresponds to a symmetric entry. When used inside an equation, it corresponds to symmetric terms in it, for example, $A + B + (A + B)^T := A + B + \star$.

2. Problem formulation

The system under study will be of the form

$$\dot{x} = Ax + B_1\Phi_1(H_1x,\ t) + B_2\Phi_2(H_1x,\ t)\theta, \tag{1a}$$

$$y = Cx + D\Phi_3(H_2x,\ t)\theta, \tag{1b}$$

where $x \in \mathbb{R}^n$ the state, $y \in \mathbb{R}^p$ the output, and $\theta \in \mathbb{R}^q$ a vector of unknown system parameters, assumed to be piece-wise constant. The vector valued functions $\Phi_1 \in \mathbb{R}^r$, $\Phi_2 \in \mathbb{R}^{q\times q}$ represent state nonlinearities and $\Phi_3 \in \mathbb{R}^{q\times q}$ represents output nonlinearity. The system matrices $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times r}$, $B_2 \in \mathbb{R}^{n\times q}$, $H_1 \in \mathbb{R}^{s\times n}$, $H_2 \in \mathbb{R}^{\tau\times n}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times q}$ are known and constant. For simplicity, denote

$$\begin{aligned} B\Phi(q_1,\theta,t) &= B_1\Phi_1(q_1,t) + B_2\Phi_2(q_1,t)\theta, \qquad q_1 = H_1x\,, \\ &= \underbrace{\begin{pmatrix} B_1 & B_2 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} \Phi_1(q_1,t) \\ \Phi_2(q_1,t)\theta \end{pmatrix}}_{\Phi(q_1,\theta,t)} \end{aligned} \tag{2}$$

$$\Psi(q_2,\theta,t)=\Phi_3(q_2,t)\theta, \qquad\qquad q_2=H_2x\,. \tag{3}$$

For the system nonlinearities, we assume that they satisfy incremental quadratic constraints, which are defined below.

Definition 1 *[28] A symmetric matrix $\mathcal{M}$ is an incremental multiplier matrix (IMM) for $\Phi(q_1,\theta,t)$ if it satisfies the following incremental quadratic constraint (δQC) with respect to its first argument, for all q_1,θ,t:*

$$\begin{pmatrix}\Delta q_1\\ \Delta\Phi\end{pmatrix}^T \mathcal{M}\begin{pmatrix}\Delta q_1\\ \Delta\Phi\end{pmatrix}\geqslant 0, \tag{4}$$

where $\Delta q_1=q_{1,1}-q_{1,2}$ and $\Delta\Phi=\Phi(q_{1,1},\theta,t)-\Phi(q_{1,2},\theta,t)$. Note also that for a given $\mathcal{M}$, any positive scaling $k\mathcal{M}$, for scalar $k>0$ also satisfies the condition (4).

Thus, the assumption on Φ,Ψ is the following:

Assumption 1 *The state and output nonlinearities Φ,Ψ satisfy the δQC condition (4), with symmetric matrices $\mathcal{M}$, $\mathcal{Z}$, given by*

$$\mathcal{M}=\begin{pmatrix}\mathcal{M}_{11} & \mathcal{M}_{12}\\ \mathcal{M}_{12}^T & \mathcal{M}_{22}\end{pmatrix}, \qquad \mathcal{Z}=\begin{pmatrix}\mathcal{Z}_{11} & \mathcal{Z}_{12}\\ \mathcal{Z}_{12}^T & \mathcal{Z}_{22}\end{pmatrix}. \tag{5}$$

In addition, for the feasibility of observer design, the following assumptions on the system matrices are taken.

Assumption 2 *There exist matrices $P>0$, $\mathcal{Y}$, L such that*

$$B_2=LD, \tag{6}$$

$$B_2^TP=\mathcal{Y}C, \tag{7}$$

$$0=\mathcal{Y}D. \tag{8}$$

Equation (6) is solvable if and only if

$$\mathrm{rank}D=\mathrm{rank}\begin{pmatrix}D\\ B_2\end{pmatrix}. \tag{9}$$

From (6), we can compute the matrix L as

$$L=B_2D^++Z_L(I-DD^+), \tag{10}$$

where Z_L is an arbitrary matrix of appropriate dimension and D^+ is any generalized inverse of D satisfying $DD^+D=D$. In addition, a general solution of (8) is given by

$$\mathcal{Y}=Z_{\mathcal{Y}}(I-DD^+), \tag{11}$$

where $Z_{\mathcal{Y}}$ is an arbitrary matrix. Substituting (11) in (7) gives

$$B_2^T P = Z_{\mathcal{Y}}(I - DD^+)C. \tag{12}$$

From the above condition (12), the parameter $Z_{\mathcal{Y}}$ is computed as

$$Z_{\mathcal{Y}} = B_2^T P \mathfrak{D}^+ + Z_{zy}(I - \mathfrak{D}\mathfrak{D}^+), \tag{13}$$

where Z_{zy} is an arbitrary matrix and $\mathfrak{D} = (I - DD^+)C$.

The above conditions will be of use in the determination of the observer system matrices in the next section. At the end of the section, a remark will be added as to the order of which the involved matrices should be computed.

3. Observer design

3.1. General case

In this section, an observer is designed for system (1). The observer has the following form

$$\dot{\hat{x}} = A\hat{x} + B_1\Phi_1(H_1\hat{x}, t) + B_2\Phi_2(H_1\hat{x}, t)\hat{\theta} - L(y - \hat{y}), \tag{14a}$$

$$\hat{y} = C\hat{x} + D\Phi_3(H_2\hat{x}, t)\hat{\theta}\,, \tag{14b}$$

where $\hat{x}$ is the estimate of x, $\hat{y}$ is the output of the observer, and $\hat{\theta}$ is the estimate of the unknown parameter θ. The observer matrices must be determined appropriately, to assure the successful estimation of the system dynamics, that is, to obtain $\|x - \hat{x}\| \to 0$ and $\|\theta - \hat{\theta}\| \to 0$ as $t \to \infty$. The design of the observer is provided in the following main Theorem.

Theorem 1 *Under Assumptions 1 and 2, system (14) is an observer for system (1), if the following linear matrix inequality is solvable for $P > 0$, and scalar $\beta > 0$*

$$\Omega := \begin{pmatrix} W_1 + \beta I & PB + H_1^T \mathcal{M}_{12} & PLD + H_2^T \mathcal{Z}_{12} \\ B^T P + \mathcal{M}_{12}^T H_1 & \mathcal{M}_{22} & 0 \\ D^T L^T P + \mathcal{Z}_{12}^T H_2 & 0 & \mathcal{Z}_{22} \end{pmatrix} \leqslant 0, \tag{15}$$

where $W_1 = P(A + LC) + (A + LC)^T P + {H_1}^T \mathcal{M}_{11} H_1 + {H_2}^T \mathcal{Z}_{11} H_2$, under the adaptive estimation law

$$\dot{\hat{\theta}} = \begin{pmatrix} \Gamma^{-1} & \Gamma^{-1} \end{pmatrix} \begin{pmatrix} \Phi_2(H_1\hat{x}, t)^T \\ \Phi_3(H_2\hat{x}, t)^T \end{pmatrix} \mathcal{Y}(y - \hat{y}), \tag{16}$$

where $\Gamma > 0$ is the adaptive control parameter, that can affect the estimation convergence rate.

Proof. First, define the following error terms

$$e_x = x - \hat{x}\,, \tag{17}$$

$$\Delta q_1 = H_1 x - H_1 \hat{x} = H_1 e_x\,, \tag{18}$$

$$\Delta q_2 = H_2 x - H_2 \hat{x} = H_2 e_x\,, \tag{19}$$

$$\Delta y = y - \hat{y}\,, \tag{20}$$

$$\Delta\Phi = \Phi(q_1, \theta, t) - \Phi(\hat{q}_1, \theta, t), \tag{21}$$

$$\Delta\Psi = \Psi(q_2, \theta, t) - \Psi(\hat{q}_2, \theta, t), \tag{22}$$

$$e_\theta = \theta - \hat{\theta}\,. \tag{23}$$

Since the nonlinearity $\Phi(q_1, \theta, t)$ satisfies (4), it holds that

$$\begin{pmatrix} \Delta q_1 \\ \Delta\Phi \end{pmatrix}^T \mathcal{M} \begin{pmatrix} \Delta q_1 \\ \Delta\Phi \end{pmatrix} \geqslant 0 \Rightarrow \begin{pmatrix} e_x \\ \Delta\Phi \end{pmatrix}^T \begin{pmatrix} {H_1}^T \mathcal{M}_{11} H_1 & \star \\ \mathcal{M}_{12}^T H_1 & \mathcal{M}_{22} \end{pmatrix} \begin{pmatrix} e_x \\ \Delta\Phi \end{pmatrix} \geqslant 0. \tag{24}$$

Similarly, for the nonlinearity $\Psi(q_2, \theta, t)$ it holds that

$$\begin{pmatrix} e_x \\ \Delta\Psi \end{pmatrix}^T \begin{pmatrix} {H_2}^T \mathcal{Z}_{11} H_2 & \star \\ \mathcal{Z}_{12}^T H_2 & \mathcal{Z}_{22} \end{pmatrix} \begin{pmatrix} e_x \\ \Delta\Psi \end{pmatrix} \geqslant 0. \tag{25}$$

First, consider the output error term

$$\begin{aligned}
\Delta y &= y - \hat{y} \\
&= Cx + D\Phi_3(H_2 x, t)\theta - C\hat{x} - D\Phi_3(H_2\hat{x}, t)\hat{\theta} \\
&= Ce_x + D\left(\Phi_3(H_2 x, t)\theta - \Phi_3(H_2\hat{x}, t)\theta + \Phi_3(H_2\hat{x}, t)\theta - \Phi_3(H_2\hat{x}, t)\hat{\theta}\right) \\
&= Ce_x + D\left(\Delta\Psi + \Phi_3(H_2\hat{x}, t)e_\theta\right).
\end{aligned} \tag{26}$$

Taking the derivative of e_x, considering (26), the error dynamics are

$$\begin{aligned}
\dot{e}_x &= \dot{x} - \dot{\hat{x}} \\
&= Ax + B_1\Phi_1(H_1 x, t) + B_2\Phi_2(H_1 x, t)\theta \\
&\quad - A\hat{x} - B_1\Phi_1(H_1\hat{x}, t) - B_2\Phi_2(H_1\hat{x}, t)\hat{\theta} + LCe_x + LD\left(\Delta\Psi + \Phi_3(H_2\hat{x}, t)e_\theta\right) \\
&= (A + LC)e_x + B_1\Phi_1(H_1 x, t) + B_2\Phi_2(H_1 x, t)\theta - B_1\Phi_1(H_1\hat{x}, t) \\
&\quad - B_2\Phi_2(H_1\hat{x}, t)\theta + B_2\Phi_2(H_1\hat{x}, t)\theta - B_2\Phi_2(H_1\hat{x}, t)\hat{\theta} \\
&\quad + LD\left(\Delta\Psi + \Phi_3(H_2\hat{x}, t)e_\theta\right) \\
&= (A+LC)e_x + B\Delta\Phi + LD\Delta\Psi + \left(B_2\Phi_2(H_1\hat{x}, t) + LD\Phi_3(H_2\hat{x}, t)\right)e_\theta\,.
\end{aligned} \tag{27}$$

Now, consider the following candidate Lyapunov function $V(t) = e_x^T P e_x + e_\theta^T \Gamma e_\theta$.

Differentiating this function, taking into account (16), (26), (27), and also the fact that $\dot{\theta} = 0 \Rightarrow \dot{e}_\theta = -\dot{\hat{\theta}}$ yields

$$\begin{aligned}
\dot{V} &= \dot{e}_x^T P e_x + e_x^T P \dot{e}_x + \dot{e}_\theta^T \Gamma e_\theta + e_\theta^T \Gamma \dot{e}_\theta \\
&= \left((A+LC)e_x + B\Delta\Phi + LD\Delta\Psi + \begin{pmatrix} B_2 & LD \end{pmatrix} \begin{pmatrix} \Phi_2(H_1\hat{x},t) \\ \Phi_3(H_2\hat{x},t) \end{pmatrix} e_\theta \right)^T Pe_x \\
&\quad + e_x^T P \left((A+LC)e_x + B\Delta\Phi + LD\Delta\Psi + \begin{pmatrix} B_2 & LD \end{pmatrix} \begin{pmatrix} \Phi_2(H_1\hat{x},t) \\ \Phi_3(H_2\hat{x},t) \end{pmatrix} e_\theta \right) \\
&\quad - \left(\begin{pmatrix} \Gamma^{-1} & \Gamma^{-1} \end{pmatrix} \begin{pmatrix} \Phi_2(H_1\hat{x},t)^T \\ \Phi_3(H_2\hat{x},t)^T \end{pmatrix} \mathcal{Y} \left(Ce_x + D\left(\Delta\Psi + \Phi_3(H_2\hat{x},t)e_\theta\right)\right) \right)^T \Gamma e_\theta \\
&\quad - e_\theta^T \Gamma \left(\begin{pmatrix} \Gamma^{-1} & \Gamma^{-1} \end{pmatrix} \begin{pmatrix} \Phi_2(H_1\hat{x},t)^T \\ \Phi_3(H_2\hat{x},t)^T \end{pmatrix} \mathcal{Y} \left(Ce_x + D\left(\Delta\Psi + \Phi_3(H_2\hat{x},t)e_\theta\right)\right) \right).
\end{aligned}$$

Under the conditions given in Assumption 2, first using $\mathcal{Y}D = 0$ we obtain

$$\begin{aligned}
\dot{V} = &\left((A+LC)e_x + B\Delta\Phi + LD\Delta\Psi + \begin{pmatrix} B_2 & LD \end{pmatrix} \begin{pmatrix} \Phi_2(H_1\hat{x},t) \\ \Phi_3(H_2\hat{x},t) \end{pmatrix} e_\theta \right)^T Pe_x \\
&- e_\theta^T \Gamma \left(\begin{pmatrix} \Gamma^{-1} & \Gamma^{-1} \end{pmatrix} \begin{pmatrix} \Phi_2(H_1\hat{x},t)^T \\ \Phi_3(H_2\hat{x},t)^T \end{pmatrix} \mathcal{Y} Ce_x \right) + \star.
\end{aligned}$$

Furthermore, using $B_2 = LD$ gives

$$\begin{aligned}
\dot{V} = &\left((A+LC)e_x + B\Delta\Phi + LD\Delta\Psi + B_2\left(\Phi_2(H_1\hat{x},t) + \Phi_3(H_2\hat{x},t)\right) e_\theta\right)^T Pe_x \\
&- e_\theta^T \left(\left(\Phi_2(H_1\hat{x},t) + \Phi_3(H_2\hat{x},t)\right)^T \mathcal{Y} Ce_x \right) + \star
\end{aligned}$$

and using $B_2^T P = \mathcal{Y}C$ the derivative $\dot{V}$ is simplified to

$$\begin{aligned}
\dot{V} &= \left((A+LC)e_x + B\Delta\Phi + LD\Delta\Psi\right)^T Pe_x + e_x^T P \left((A+LC)e_x + B\Delta\Phi + LD\Delta\Psi\right) \\
&= \begin{pmatrix} e_x \\ \Delta\Phi \\ \Delta\Psi \end{pmatrix}^T \begin{pmatrix} P(A+LC) + (A+LC)^T P & PB & PLD \\ B^T P & 0 & 0 \\ D^T L^T P & 0 & 0 \end{pmatrix} \begin{pmatrix} e_x \\ \Delta\Phi \\ \Delta\Psi \end{pmatrix}.
\end{aligned} \tag{28}$$

Finally, taking into account (24), (25), the above yields

$$\begin{aligned}
\dot{V} &\leqslant \begin{pmatrix} e_x \\ \Delta\Phi \\ \Delta\Psi \end{pmatrix}^T \begin{pmatrix} W_1 & \star & \star \\ B^T P + \mathcal{M}_{12}^T H_1 & \mathcal{M}_{22} & \star \\ D^T L^T P + \mathcal{Z}_{12}^T H_2 & 0 & \mathcal{Z}_{22} \end{pmatrix} \begin{pmatrix} e_x \\ \Delta\Phi \\ \Delta\Psi \end{pmatrix} \\
&\leqslant \Omega - \beta e_x^T e_x\,.
\end{aligned} \tag{29}$$

Thus, under the condition that $\Omega \leqslant 0$, it holds that

$$\dot{V} \leqslant -\beta e_x^T e_x \Rightarrow \tag{30}$$

$$V - V(0) \leqslant -\beta \int_0^t e_x(s)^T e_x(s) ds \Rightarrow \tag{31}$$

$$V + \beta \int_0^t e_x(s)^T e_x(s) \mathrm{d}s \leqslant V(0), \tag{32}$$

since V is positive definite, we obtain

$$\beta \int_0^t e_x(s)^T e_x(s) \mathrm{d}s \leqslant V(0). \tag{33}$$

The above means that the integral $\int_0^t e_x(s)^T e_x(s) \mathrm{d}s$ exists and is bounded. Since $e_x(s)^T e_x(s) \geqslant 0$ is a positive function with a bounded integral, it holds that $\lim_{t\to\infty} e_x(t) = 0$. □

Now, the design procedure is summarized in the form of following algorithm.

Algorithm 1 Computational steps to design observer (14) for system (1).

1. Solve (10) to determine matrix L.
2. Solve the LMI (15) to determine P.
3. Solve (13) to find $Z_{\mathcal{Y}}$, and replace in (11) to find a suitable $\mathcal{Y}$.

3.2. Special case of output nonlinearity

In this subsection, we consider a particular case where the output nonlinearity has the same structure as the unknown parameter coefficient, that is

$$\Phi_2(H_1 x, t) = \Phi_3(H_2 x, t), \quad H_1 = H_2 \,. \tag{34}$$

This case is considered since it leads to the relaxation of the algebraic conditions (6), (7), and the simplification of the LMI (15). In this case, Assumption 2 is now converted to the following condition.

Assumption 3 *There exist matrices $P > 0$, $\mathcal{Y}$, L such that*

$$(B_2 + LD)^T P = 2\mathcal{Y} C \,, \tag{35}$$

$$0 = \mathcal{Y} D. \tag{36}$$

The solution of (36) is given by (11). Replacing in (35), results in

$$
\begin{aligned}
PB_2 + PLD &= \underbrace{(2Z_{\mathcal{Y}}(I - DD^{+})C)^T}_{\mathcal{F}} \Rightarrow \\
PLD &= \mathcal{F} - PB_2 \Rightarrow \\
L &= P^{-1}\mathcal{F}D^{+} - B_2D^{+}.
\end{aligned} \tag{37}
$$

For the special case considered, the following result on observer design is provided.

Theorem 2 *Under Assumptions 1, 3, system (14) is an observer of system (1) with condition (34), if the following linear matrix inequality is solvable for $P > 0$, and scalar $\beta > 0$*

$$
\hat{\Omega} := \begin{pmatrix} \hat{W}_1 + \beta I & \star & \star \\ B^T P + \mathcal{M}_{12}^T H_1 & \mathcal{M}_{22} & \star \\ W_2^T & 0 & \mathcal{Z}_{22} \end{pmatrix} \leqslant 0, \tag{38}
$$

where

$$
\begin{aligned}
\hat{W}_1 &= P(A - B_2D^{+}C) + (A - B_2D^{+}C)^T P + \mathcal{F}D^{+}C + (\mathcal{F}D^{+}C)^T \\
&\quad + {H_1}^T \mathcal{M}_{11} H_1 + {H_1}^T \mathcal{Z}_{11} H_1 \\
W_2 &= \mathcal{F}D^{+}D - PB_2D^{+}D + H_1^T \mathcal{Z}_{12}
\end{aligned}
$$

under the adaptive estimation law

$$
\dot{\hat{\theta}} = 2\Gamma^{-1}\Phi_2(H_1\hat{x}, t)^T \mathcal{Y}(y - \hat{y}), \tag{39}
$$

where $\Gamma > 0$ is a control parameter.

Proof. Following a procedure similar to the proof of Theorem 1, from equation (27) along with the assumption $\Phi_1 = \Phi_2$, the error dynamics is reduced to

$$
\dot{e}_x = (A + LC)e_x + B\Delta\Phi + LD\Delta\Psi + (B_2 + LD)\,\Phi_2(H_1\hat{x}, t)e_\theta\,. \tag{40}
$$

Then, considering again the candidate Lyapunov function $V(t) = e_x^T P e_x + e_\theta^T \Gamma e_\theta$, and differentiating yields

$$
\begin{aligned}
\dot{V} &= ((A + LC)e_x + B\Delta\Phi + LD\Delta\Psi + (B_2 + LD)\,\Phi_2(H_1\hat{x}, t)e_\theta)^T\, Pe_x \\
&\quad + e_x^T P\,((A + LC)e_x + B\Delta\Phi + LD\Delta\Psi + (B_2 + LD)\,\Phi_2(H_1\hat{x}, t)e_\theta) \\
&\quad - \left(2\Gamma^{-1}\Phi_2(H_1\hat{x}, t)^T \mathcal{Y}\,(Ce_x + D\,(\Delta\Psi + \Phi_2(H_1\hat{x}, t)e_\theta))\right)^T \Gamma e_\theta \\
&\quad - e_\theta^T \Gamma \left(2\Gamma^{-1}\Phi_2(H_1\hat{x}, t)^T \mathcal{Y}\,(Ce_x + D\,(\Delta\Psi + \Phi_2(H_1\hat{x}, t)e_\theta))\right).
\end{aligned}
$$

Considering conditions (35), (36), of Assumption 3 the derivative $\dot{V}$ above is simplified to

$$\dot{V} = ((A+LC)e_x + B\Delta\Phi + LD\Delta\Psi)^T Pe_x + e_x^T P((A+LC)e_x + B\Delta\Phi + LD\Delta\Psi)$$

and considering L as in (37), and $\hat{\Omega} \leqslant 0$, the error dynamics can be shown to converge to zero following the same procedure in the proof of Theorem 1. □

The design procedure is summarized in the following algorithm.

Algorithm 2 Computational steps to design observer (14) for system (1), under (34)

1. Solve (11) to determine matrix $\mathcal{Y}$.
2. Solve the LMI (38) to determine P.
3. Determine L from (37).

The following Corollary gives a simplification of the LMI (38).

Corollary 1 *Considering*

$$B\Delta\Phi + LD\Delta\Psi = B_1\Delta\Phi_1 + (B_2 + LD)\Delta\Phi_2\,, \tag{41}$$

where $\Delta\Phi_1 = \Phi_1(q_1,t) - \Phi_1(\hat{q}_1,t)$, $\Delta\Phi_2 = \Phi_2(q_1,t)\theta - \Phi_2(\hat{q}_1,t)\theta$. *The derivative* $\dot{V}$ *in the proof of Theorem (2) can be written as*

$$\dot{V} = \begin{pmatrix} e_x \\ \Delta\Phi_1 \\ \Delta\Phi_2 \end{pmatrix}^T \begin{pmatrix} P(A+LC) + (A+LC)^T P & PB_1 & P(B_2+LD) \\ B_1^T P & 0 & 0 \\ (B_2^T + D^T L^T)P & 0 & 0 \end{pmatrix} \begin{pmatrix} e_x \\ \Delta\Phi_1 \\ \Delta\Phi_2 \end{pmatrix}. \tag{42}$$

Further decomposing the matrix $\mathcal{M}$ *as*

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{121} & \mathcal{M}_{122} \\ \mathcal{M}_{121}^T & \mathcal{M}_{221} & \mathcal{M}_{222} \\ \mathcal{M}_{122}^T & \mathcal{M}_{222}^T & \mathcal{M}_{223} \end{pmatrix} \tag{43}$$

condition (24) is written as

$$\begin{pmatrix} \Delta q_1 \\ \Delta\Phi \end{pmatrix}^T \mathcal{M} \begin{pmatrix} \Delta q_1 \\ \Delta\Phi \end{pmatrix} \geqslant 0 \Rightarrow \begin{pmatrix} e_x \\ \Delta\Phi_1 \\ \Delta\Phi_2 \end{pmatrix}^T \begin{pmatrix} H_1{}^T \mathcal{M}_{11} H_1 & \star & \star \\ \mathcal{M}_{121}^T H_1 & \mathcal{M}_{221} & \star \\ \mathcal{M}_{122}^T H_1 & \mathcal{M}_{222}^T & \mathcal{M}_{223} \end{pmatrix} \begin{pmatrix} e_x \\ \Delta\Phi_1 \\ \Delta\Phi_2 \end{pmatrix}. \tag{44}$$

So, the LMI (38) *can be simplified to*

$$\hat{\Omega} := \begin{pmatrix} \hat{W}_1 + \beta I & \star & \star \\ B_1^T P + \mathcal{M}_{121}^T H_1 & \mathcal{M}_{221} & \star \\ W_2^T & \mathcal{M}_{222}^T & \mathcal{M}_{223} \end{pmatrix} \leqslant 0, \tag{45}$$

where

$$\hat{W}_1 = P(A - B_2 D^+ C) + (A - B_2 D^+ C)^T P + \mathcal{F} D^+ C + (\mathcal{F} D^+ C)^T + H_1{}^T \mathcal{M}_{11} H_1 \,,$$
$$W_2 = P B_2 (I - D^+ D) + \mathcal{F} D^+ D + H_1^T \mathcal{M}_{122} \,.$$

Notice that with this simplification, the output nonlinearity is integrated in the term of $\Delta\Phi$*, so the IMM matrix* $\mathcal{Z}$ *is discarded in the resulting LMI.*

4. Numerical example

In this section, the problem of chaos synchronization will be expounded through the proposed observer design. Consider the well known Rössler system [8, 10, 15, 29]:

$$\dot{x}_1 = -x_2 - x_3 \,, \tag{46a}$$
$$\dot{x}_2 = x_1 + \theta x_2 \,, \tag{46b}$$
$$\dot{x}_3 = b + x_3 x_1 - \gamma x_3 \,, \tag{46c}$$

where, $b = 2$, $\gamma = 4$ are system parameters, and $\theta \in [0.4, 0.5]$ is the unknown parameter. The system can be rewritten in the form of (1) as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (2 + x_3 x_1) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} x_2 \theta \,, \tag{47}$$

where

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & -4 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \Phi_1(H_1 x, t) = 2 + x_3 x_1, \quad H_1 = I_3 \,,$$
$$B_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \Phi_2(H_1 x, t) = x_2, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Phi(q_1, \theta, t) = \begin{pmatrix} 2 + x_3 x_1 \\ x_2 \theta \end{pmatrix}$$

Also, let the output be

$$y = Cx + D x_2 \theta \tag{48}$$

so

$$\Phi_3(H_2 x, t) = \Phi_2(H_1 x, t) = x_2\,, \qquad H_2 = H_1\,. \tag{49}$$

Assume that the initial conditions of the master system (46) are $(x_1(0), x_2(0), x_3(0)) = (1, 1, 1)$. To determine the IMM for the nonlinearity Φ, since the parameter θ is unknown, but in the interval $\theta \in [0.4, 0.5]$, we take $\max(\theta) = 0.5$ and follow the procedure in [27]. The resulting matrix is

$$k\mathcal{M} = k\begin{pmatrix} 50.9051 & 0 & 0 & -1.6744 & 0 \\ 0 & 47.8576 & 0 & 0 & 0.3247 \\ 0 & 0 & 50.8846 & 0 & 0 \\ -1.6744 & 0 & 0 & -51.1882 & 0 \\ 0 & 0.3247 & 0 & 0 & -45.313 \end{pmatrix}. \tag{50}$$

Here, the scaling factor is chosen as $k = 0.1$. This choice was made after some trial and error, to guarantee the satisfaction of the LMI (45). The output matrices are taken as

$$C = \begin{pmatrix} 12 & 1 & 3 \\ 7 & 2 & 2 \end{pmatrix}, \qquad D = \begin{pmatrix} 2 \\ 3 \end{pmatrix}. \tag{51}$$

The scalar parameter in the LMI is taken as $\beta = 0.2$. The matrix $\mathcal{Y}$ is

$$\mathcal{Y} = \begin{pmatrix} -0.2308 & 0.1538 \end{pmatrix} \tag{52}$$

and after the LMI is solved, the observer matrix is computed as

$$P = \begin{pmatrix} 91.5116 & 70.1739 & 20.9021 \\ 70.1739 & 54.1125 & 16.0674 \\ 20.9021 & 16.0674 & 32.5555 \end{pmatrix}, \qquad L = \begin{pmatrix} -1.0828 & -1.6242 \\ 1.2513 & 1.8770 \\ -0.0020 & -0.0029 \end{pmatrix}. \tag{53}$$

The adaptive control parameter is taken as $\Gamma = 1$, and the initial conditions for the observer and the parameter estimation are $(\hat{x}_1(0), \hat{x}_2(0), \hat{x}_3(0)) = (-5, -5, -5)$ and $\hat{\theta}(0) = 0$ respectively, while the real parameter value is $\theta = 0.4$. The system is simulated in MATLAB 2018, using *ode45*, with a relative and absolute tolerance set to 10^{-5}. The simulation result is shown in Fig. 1. The observer system requires around 5 seconds to synchronize with the master system.

As an additional simulation, since the initial assumption was that the unknown parameter is piecewise constant, a simulation is performed for the case where $\theta = 0.45+0.05\mathrm{sign}(\cos(0.5t))$. So the parameter θ is a pulse signal that oscillates between the values 0.4 and 0.5. The simulation is shown in Fig. 2. Again, the observer system requires around 5 seconds to synchronize to the master system. Also, as the parameter goes through step changes, it is estimated correctly, with a small overshoot.

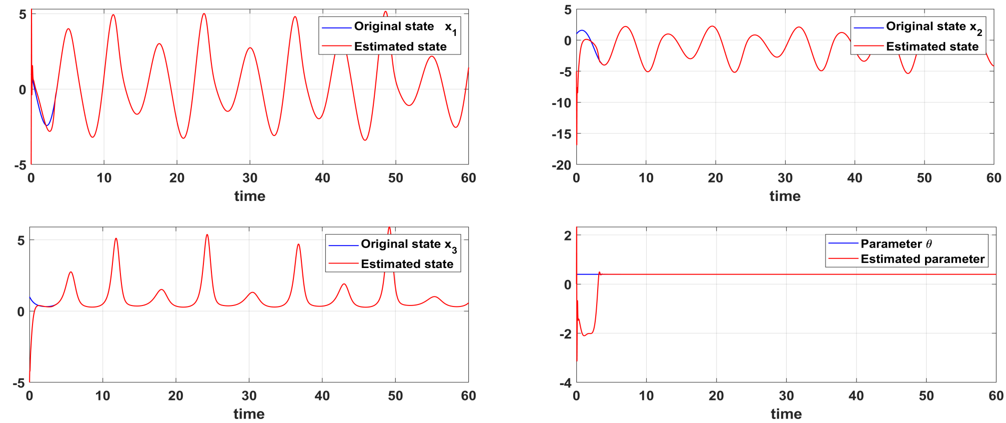

Figure 1: Simulation of the observer design for the Rössler system (46), for $\theta = 0.4$

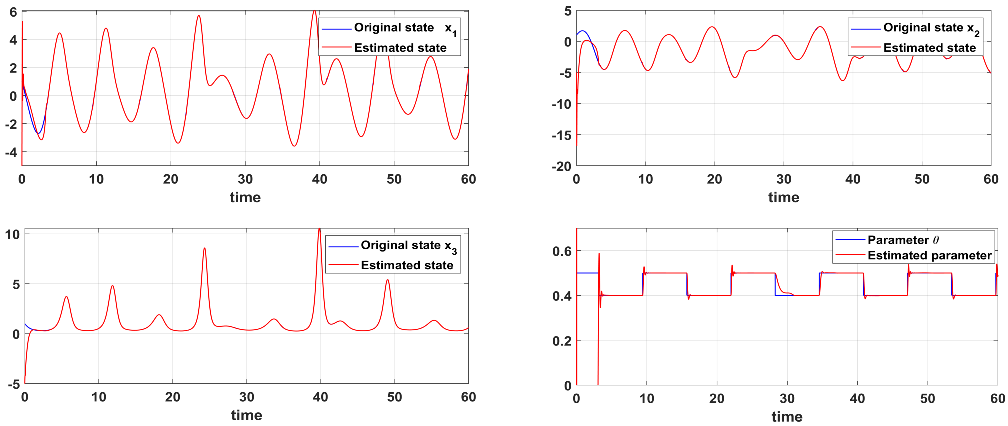

Figure 2: Simulation of the observer design for the Rössler system (46), for $\theta = 0.45 + 0.05\text{sign}(\cos(0.5t))$.

5. Conclusions

This work studied the problem of adaptive observer design, for systems with nonlinearities satisfying incremental quadratic constraints. A restricting case was considered, where the output of the system includes nonlinear combinations of the states and unknown parameter. The observer design is feasible under the feasibility of a linear matrix inequality, and a set of algebraic conditions on the system matrices. Numerical simulations were performed to address the problem of chaos synchronization.

Future works will consider different types of nonlinearities that cannot be described by incremental quadratic constraints. The problem of observer output feedback can also be addressed. Moreover, systems with hidden attractors can be studied, which constitute a trending topic of research for nonlinear analysis [30]. Additionally, the adjustment of the proposed observer to the case of discrete time systems is of interest, since continuous systems will often be discretized first, before applying control to them. Finally, the extension to the general case of descriptor systems is of interest [31], since they can be used to model a wider family of dynamical systems that are described by algebraic and differential equations.

References

[1] R. Rajamani: Observers for Lipschitz nonlinear systems. *IEEE transactions on Automatic Control*, **43**(3), (1998), 397–401. DOI: 10.1109/9.661604.

[2] M. Kumar Gupta, N, Kumar Tomar and S. Bhaumik: Full- and reduced-order observer design for rectangular descriptor systems with unknown inputs. *Journal of the Franklin Institute*, **352**(3), (2015), 1250–1264. DOI: 10.1016/j.jfranklin.2015.01.003.

[3] M. Kumar Gupta, N. Kumar Tomar, V. Kumar Mishra and S. Bhaumik: Observer design for semilinear descriptor systems with applications to chaos-based secure communication. *International Journal of Applied and Computational Mathematics*, **3**(1), (2017), 1313–1324. DOI: 10.1007/s40819-017-0419-0.

[4] Ch. Hua, X. Guan, X. Li and P. Shi: Adaptive observer-based control for a class of chaotic systems. *Chaos, Solitons & Fractals*, **22**(1), (2004), 103–110. DOI: 10.1016/j.chaos.2003.12.072.

[5] R. Carroll and D. Lindorff: An adaptive observer for single-input single-output linear systems. *IEEE Transactions on Automatic Control*, **18**(5), (1973), 428–435. DOI: 10.1109/TAC.1973.1100367.

[6] Y. Man Cho and R. Rajamani: A systematic approach to adaptive observer synthesis for nonlinear systems. *IEEE Transactions on Automatic Control*, **42**(4), (1997), 534–537. DOI: 10.1109/9.566664.

[7] S. Bowong and J-J. Tewa: Unknown inputs adaptive observer for a class of chaotic systems with uncertainties. *Mathematical and Computer Modelling*, **48**(11-12), (2008), 1826–1839. DOI: 10.1016/j.mcm.2007.12.028.

[8] H. Dimassi and A. Loría: Adaptive unknown-input observers-based synchronization of chaotic systems for telecommunication. *IEEE Transactions on Circuits and Systems I: Regular Papers*, **58**(4), (2010), 800–812. DOI: 10.1109/TCSI.2010.2089547.

[9] A. Loría, E. Panteley, and A. Zavala-Río: Adaptive observers with persistency of excitation for synchronization of chaotic systems. *IEEE Transactions on Circuits and Systems I: Regular Papers*, **56**(12), (2009), 2703–2716. DOI: 10.1109/TCSI.2009.2016636.

[10] H. Dimassi, A. Loria, and S. Belghith: A new secured transmission scheme based on chaotic synchronization via smooth adaptive unknown-input observers. *Communications in Nonlinear Science and Numerical Simulation*, **17**(9), (2012), 3727–3739. DOI: 10.1016/j.cnsns.2012.01.024.

[11] M. Ekramian, F. Sheikholeslam, S. Hosseinnia, and M.J. Yazdanpanah: Adaptive state observer for Lipschitz nonlinear systems. *Systems & Control Letters*, **62**(4), (2013), 319–323. DOI: 10.1016/j.sysconle.2013.01.002.

[12] M. Yang, J. Huang, L. Yang, and M. Zhang: A note on adaptive observer design method for one-sided Lipschitz systems. *Circuits, Systems, and Signal Processing*, **40** (2021), 1021–1039. DOI: 10.1007/s00034-020-01505-8.

[13] N. Oucief, M. Tadjine, and S. Labiod: Adaptive observer-based fault estimation for a class of Lipschitz nonlinear systems. *Archives of control sciences*, **26**(2), (2016), 245–259. DOI: 10.1515/acsc-2016-0014.

[14] W-D. Chang, Sh-P. Shih, and Ch-Y. Chen: Chaotic secure communication systems with an adaptive state observer. *Journal of Control Science and Engineering*, **2015**, Art. ID 471913, (2015). DOI: 10.1155/2015/471913.

[15] H. Zhang, W. Zhang, Y. Zhao, M. Ji, and L. Huang: Adaptive state observers for incrementally quadratic nonlinear systems with application to chaos synchronization. *Circuits, Systems, and Signal Processing*, **39**(3) (2020), 1290–1306. DOI: 10.1007/s00034-019-01207-w.

[16] C. Cong: Observer-based robust control of uncertain systems via an integral quadratic constraint approach. *International Journal of Dynamics and Control*, **7**(3), (2019), 926–939. DOI: 10.1007/s40435-018-00507-4.

[17] L. Moysis, M. Kumar Gupta, V. Mishra, M. Marwan, and Ch. Volos: Observer design for rectangular descriptor systems with incremental quadratic constraints and nonlinear outputs – Application to secure communications. *International Journal of Robust and Nonlinear Control*, **30**(18), (2020), 8139–8158. DOI: 10.1002/rnc.5233.

[18] M. Ayati and H. Khaloozadeh: A stable adaptive synchronization scheme for uncertain chaotic systems via observer. *Chaos, Solitons & Fractals*, **42**(4), (2009), 2473–2483. DOI: 10.1016/j.chaos.2009.03.108.

[19] J.H. Pérez-Cruz, J.M. Allende Peña, Ch. Nwachioma, J. de Jesus Rubio, J. Pacheco, J.A. Meda-Campaña, D. Ávila-González, O. Guevara Galindo, I.A. Romero, and S.I. Belmonte Jiménez: A Luenberger-like observer for multistable Kapitaniak chaotic system. *Complexity*, **2020** Art. ID 9531431, (2020). DOI: 10.1155/2020/9531431.

[20] W. Zhang, Y. Zhao, M. Abbaszadeh, and M. Ji: Full-order and reduced-order exponential observers for discrete-time nonlinear systems with incremental quadratic constraints. *Journal of Dynamic Systems, Measurement, and Control*, **141**(4), (2019). DOI: 10.1115/1.4041712.

[21] A. Chakrabarty, S.H. Żak, and S. Sundaram: State and unknown input observers for discrete-time nonlinear systems. In *2016 IEEE 55th Conference on Decision and Control (CDC)*, (2016), 7111–7116.

[22] Y. Zhao, W. Zhang, W Guo, S. Yu and F. Song: Exponential state observers for nonlinear systems with incremental quadratic constraints and output nonlinearities. *Journal of Control, Automation and Electrical Systems*, **29**(2), (2018), 127–135. DOI: 10.1007/s40313-018-0369-8.

[23] A. Zulfiqar, M. Rehan, and M. Abid: Observer design for one-sided Lipschitz descriptor systems. *Applied Mathematical Modelling*, **40**(3), (2016), 2301-2311. DOI: 10.1016/j.apm.2015.09.056.

[24] M. Kumar Gupta, N. Kumar Tomar, and M. Darouach: Unknown inputs observer design for descriptor systems with monotone nonlinearities. *International Journal of Robust and Nonlinear Control*, **28**(17), (2018), 5481–5494. DOI: 10.1002/rnc.4331.

[25] X. Fan and M. Arcak: Observer design for systems with multivariable monotone nonlinearities. *Systems & Control Letters*, **50**(4), (2003), 319–330. DOI: 10.1016/S0167-6911(03)00170-1.

[26] A. Chakrabarty and M. Corless: Estimating unbounded unknown inputs in nonlinear systems. *Automatica*, **104** (2019), 57–66. DOI: 10.1016/j.automatica.2019.02.050.

[27] Y. Zhao, W. Zhang, H. Su, and J. Yang: Observer-based synchronization of chaotic systems satisfying incremental quadratic constraints and its application in secure communication. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, **50**(12), (2018), 5221–5232. DOI: 10.1109/TSMC.2018.2868482.

[28] B. Açıkmeşe and M. Corless: Observers for systems with nonlinearities satisfying incremental quadratic constraints. *Automatica*, **47**(7), (2011), 1339–1348. DOI: 10.1016/j.automatica.2011.02.017.

[29] T-L. Liao and Sh-H. Tsai: Adaptive synchronization of chaotic systems and its application to secure communications. *Chaos, Solitons & Fractals*, **11**(9), (2000), 1387–1396. DOI: 10.1016/S0960-0779(99)00051-X.

[30] G. Grassi, A. Ouannas, and V-T Pham: A general unified approach to chaos synchronization in continuous-time systems (with or without equilibrium points) as well as in discrete-time systems. *Archives of Control Sciences*, **28**(1), (2018), 135–154. DOI: 10.24425/119082.

[31] M. Kumar Gupta, N. Kumar Tomar, and Sh. Bhaumik: Observer design for descriptor systems with Lipschitz nonlinearities: An LMI approach. *Nonlinear Dynamics and System Theory*, **14**(3), (2014), 291–301.