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On impact of disturbance in the deployment problem of multi-agent system

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The paper is dedicated to the robustness analysis of scalar multi-agent dynamical systems. The open problem we aim to address is the one related to the impact of additive disturbances. Set-theoretic methods are used to achieve the main results in terms of positive invariance and admissible bounds on the disturbances.

Key words: multi-agent systems, deployment problem, robust invariant sets, set-theoretic analysis, disturbance

1. Introduction

The class of Multi-Agent Systems (MASs) covers a generic family of dynamics composed of multiple interacting subsystems called agents. There is a wide spectrum of applications of MASs: formation flight of unmanned air vehicles, clusters of satellites, automated highway systems, self-organized systems and, what follows, enormous number of publications devoted to these issues (see [2, 3, 8, 10–12, 15], for example, and the references therein). Since most of those applications are connected with the networks of communicating dynamic agents

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and represent physical systems such as group of robots or vehicles, one of the main tasks, while dealing with them, is to design the control strategies for a group of agents in view of covering a known, predetermined target area. Then the goal is to obtain a static configuration so that the region of coverage is maximized. This problem is known as the *deployment problem* [1] or *coverage problem* [9].

There are different approaches to the problem of deploying agents according to the local or global information exchanged and the knowledge of the environment.

The present work considers the decentralized deployment using a dynamic Voronoi partition. This method is built on the agents' current position and induces a control policy, which is nonlinear due to the agents arrangement. This means that at each time instant, a bounded convex polyhedron, which is the working environment, is partitioned using a Voronoi algorithm. In these schemes, the polytopic target environment is partitioned into a finite collection of polytopic Voronoi cells as many as there are agents. Moreover, it is necessary to designate internal target points, where agents can reach a static configuration. For this purpose, we consider the Chebyshev centers, which can be expressed geometrically as the centers of the corresponding Voronoi cells.

Within this framework recent results show (see [13]) that nominal closed-loop dynamics are stable and converge to a consensus-like equilibrium. Our aim is to go beyond the state of the art and analyse the impact of additive disturbances on the multi-agent behaviour and the overall coverage problem. Such a robustness analysis is particularly useful in practice given the uncertainties available on the sensing and actuation channels that can be modelled in terms of additive uncertainties.

The main results for the case of a multi-agent dynamical system composed on N scalar subsystems driven by a Voronoi-based coverage control algorithm are twofold:

- for well-defined admissible bounds on the disturbances, one can obtain local robust invariant sets around the nominal equilibrium,
- in the presence of disturbances within the admissible bounds, there exists a unique attractor and its domain of attraction is the entire collection of initial conditions within the bounded environment.

Within this investigation we consider a one-dimensional case $\mathbb{R}$, i.e. the agents' work environment trivializes to an interval, in the discrete-time framework. The main purpose of the paper is to analyze the behaviour of agents for the system in the presence of disturbances. Since disturbance gains have a significant impact on the behaviour of the agents such as switching of their positions, the main problem is to establish a robust invariant set around the equilibrium of the nominal dynamics for a bounded additive uncertainty. Our consideration is conducted in two ways: as a theoretical analysis and a numerical one based on numerous simulations.

The paper is organized in the following manner. Section 2 includes general definitions of geometrical notions. In the subsequent three subsections of this section one finds geometry of the partition based on Chebyshev centers and Voronoi cells, results on decentralized control law applied to MAS without disturbances and stability properties of MAS without disturbances, respectively. Then, Section 3 treats about the impact of disturbances on the deployment of the multi-agent systems. This is the main part of the paper and is divided onto six subsections. After presenting the model with disturbances and a result on ultimate bounds, we provide characterization of MAS behaviour subject to disturbances and illustrate the investigation with an example. Next, the discussion on the use of minimal robust positive invariant sets for the disturbance characterization together with an example is delivered. The investigation on maximal admissible disturbance and a result on the domain of attraction finish this section. Examples illustrating theoretical considerations are given in Section 4. Finally, a section with conclusions closes the paper.

2. Multiagent system: dynamics, geometry and control

Let us start with a few general definitions of geometrical notions.

Definition 1 *[5] Let P be a non-empty polytope in $\mathbb{R}^n$. The depth of a point $x \in P$ is defined as*

$$\operatorname{depth}(x, P) = \min\{\|x - \widetilde{x}\| : \widetilde{x} \in \partial P\}, \tag{1}$$

where $x \in P$, and ∂P denotes the boundary of P.

Definition 2 *The* Chebyshev radius *of P, denoted by $\bar{r}(P)$, is the maximal depth of any $x \in P$,*

$$\bar{r}(P) = \max_{x \in P} \operatorname{depth}(x, P). \tag{2}$$

The *environment* is understood as the space in which the agents move and it is known to them prior to deployment, thus representing a limitation in the selection of the admissible control policies. In the present study, the environment is considered to be static, convex, polytopic, full-dimensional and bounded set $\mathcal{W}$ represented as the intersection of a set of half-spaces

$$\mathcal{W} = \{x \in \mathbb{R}^n \mid Hx \leqslant \theta\} \subset \mathbb{R}^n, \tag{3}$$

where $H \in \mathbb{R}^{m \times n}$ and $\theta \in \mathbb{R}^m$.

A set of $N \in \mathbb{N}$ dynamical subsystems (agents), will be called *Multi-Agent System* (MAS). For each agent $i \in \mathbb{N}_{[1,N]}$[1], a discrete-time linear dynamics is considered

$$x_i(k+1) = x_i(k) + u_i(k), \tag{4}$$

where, in the scalar case, $x_i : \mathbb{N} \to \mathbb{R}$ is the state and $u_i : \mathbb{N} \to \mathbb{R}$ is the control signal for the i-th agent.

Consider the state vector $x(k) = (x_1(k), x_2(k), \ldots, x_N(k))^T$ in which each agent has a position in the environment $\mathcal{W}$ at time k. For each agent x_i, we define, following [5], its *neighborhood* $\mathbb{V}_i$, (related to its position in the state space)

$$\mathbb{V}_i = \{x \in \mathcal{W} \ \mid \ |x_i - x| \leqslant |x_j - x|, \forall i \neq j\}. \tag{5}$$

Equivalently,

$$\mathbb{V}_i = \{x \in \mathbb{R}: \ 2(x_j - x_i)x \leqslant x_j^2 - x_i^2, \forall j \neq i\} \cap \mathcal{W}. \tag{6}$$

According to this definition, the neighborhood $\mathbb{V}_i$ of agent i are those points in the environment $\mathcal{W}$ that are closer in Euclidean distance to agent i than to any other agent j.

Let us observe that the collection of $\mathbb{V}_i$ neighborhoods represents a partition of $\mathcal{W}$, i.e.

$$\mathcal{W} = \bigcup_{i=1}^{N} \mathbb{V}_i, \ \mathrm{Int}\{\mathbb{V}_i \cap \mathbb{V}_j\} = \emptyset. \tag{7}$$

Definition 3 *Two agents i, j, $i \neq j$ are* neighbors *provided that $\mathbb{V}_i \cap \mathbb{V}_j \neq \emptyset$.*

In $\mathbb{R}^n$ such a decomposition of the environment $\mathcal{W}$ satisfying the constraints (7) is generally denoted as Voronoi partition. In the case of dynamical system (4), the Voronoi partition is a collection of intervals covering $\mathcal{W}$.

2.1. The geometry of the partition

Consider the Chebyshev measures in the one-dimensional case. For a generic bounded set (interval) $[x_l, x_u]$, the greatest depth that can be reached is the distance from the center of the interval to its edge. Thus, the Chebyshev radius can be explicitly analysed in terms of the length of the interval, and the Chebyshev center is the mid-point of the interval, i.e.

$$\bar{r} = \frac{x_u - x_l}{2}, \qquad \bar{x} = \frac{x_l + x_u}{2}. \tag{8}$$

[1] The restriction of the real or natural numbers will be denoted with a subscript of $\mathbb{R}$ or $\mathbb{N}$, e.g. $\mathbb{N}_{[a,b]} = \mathbb{N} \cap [a,b]$ and $\mathbb{R}_{\geqslant a} = \{x \in \mathbb{R} : x \geqslant a\}$.

The environment represents an interval in itself and, without lost of generality, we consider that $\mathcal{W} = [0, 1]$. Each Voronoi cell $\mathbb{V}_i \subset \mathcal{W}$ is a sub-interval and the initial conditions of N dynamics are such that $x_i(0) \in \mathcal{W} = [0, 1]$. The great advantage of the unidimensional case is the existence of an ordered relationship at any time instant k. In particular, at the initial time step, the indexes of the agents can be considered to be chosen as:

$$x_i(0) \leqslant x_{i+1}(0),\ \forall i \in \mathbb{N}_{[1,N]}\,. \tag{9}$$

The agents' neighborhood can be described explicitly using (8):

- for the extreme (lower and upper) positions:

$$\mathbb{V}_1 = \left[0,\ \frac{x_1 + x_2}{2}\right], \qquad \mathbb{V}_N = \left[\frac{x_{N-1} + x_N}{2},\ 1\right]. \tag{10}$$

- for the agents $i \in \{2, \ldots, N-1\}$:

$$\mathbb{V}_i = \left[\frac{x_{i-1} + x_i}{2},\ \frac{x_i + x_{i+1}}{2}\right]. \tag{11}$$

Further, applying (8) to the equations (10)–(11), we get the linear equations for the Chebyshev centers as follows

$$\begin{aligned}
\bar{x}_1 &= \frac{\frac{x_2 + x_1}{2} + 0}{2} = \frac{x_1}{4} + \frac{x_2}{4}\,,\\
\bar{x}_i &= \frac{\frac{x_{i+1} + x_i}{2} + \frac{x_i + x_{i-1}}{2}}{2} = \frac{x_{i+1}}{4} + \frac{x_i}{2} + \frac{x_{i-1}}{4}\,,\\
\bar{x}_N &= \frac{\frac{x_{N-1} + x_N}{2} + 1}{2} = \frac{x_{N-1}}{4} + \frac{x_N}{4} + \frac{1}{2}\,.
\end{aligned} \tag{12}$$

2.2. Decentralized control design

Recalling the goal of steering each agent towards the Chebyshev center of its Voronoi cell, we apply, following [5], the control law u_i that satisfies the structural constraints:

- *Decentralized feedback*: the control input u_i is a function of the agent's state x_i and the Voronoi cell $\mathbb{V}_i$. Formally,

$$u_i(k) = \mathcal{K}(x_i(k), \mathbb{V}_i(k)) \tag{13}$$

  or further

$$u_i(k) = \mathcal{K}(x_i(k), \bar{x}_i(k), \bar{r}_i(\mathbb{V}_i(k))). \tag{14}$$

- Increase of the *depth* for each cell:

$$\text{depth}(x_i(k), \mathbb{V}_i(k)) \leqslant \text{depth}(x_i(k+1), \mathbb{V}_i(k)). \tag{15}$$

Recalling that $\bar{r}_i(\mathbb{V}_i(k)) \geqslant depth(x_i(k), \mathbb{V}_i(k))$ one can find $\alpha_i \in (0,1]$ satisfying:

$$\begin{aligned}\text{depth}(x_i(k), \mathbb{V}_i(k)) \leqslant \text{depth}&(x_i(k), \mathbb{V}_i(k)) \\ &+ \alpha_i \left[\bar{r}_i(\mathbb{V}_i(k)) - \text{depth}(x_i(k), \mathbb{V}_i(k))\right].\end{aligned}$$

Based on such a selection, a decentralized control scheme fulfilling conditions (14)–(15) can be obtained with deployment properties as summarized by the next statement.

Theorem 1 *[4] A decentralized control law*

$$u_i(k) = \alpha_i(\overline{x}_i(k) - x_i(k)) \tag{16}$$

with an arbitrary time-invariant $\alpha_i \in (0,1]$*, solves the deployment problem with convergence of the multi-agent system to a unique static configuration*

$$x_i^* = (2i-1)\frac{1}{2N}. \tag{17}$$

One of the essential questions to be addressed is whether the agents can swipe their ordering. The next theorem (cf. [13]) provides a negative answer in the disturbance-free case.

Theorem 2 *Let N agents of MAS (4) be described by their positions at each time step:* $x_i(k) \in [0,1], i = 1,\ldots,N$ *with an ordering:*

$$x_i(k) \leqslant x_{i+1}(k),\ \forall i = 2,\ldots,N-1. \tag{18}$$

If the agents' evolution is governed by

$$x_i(k+1) = x_i(k) + \alpha_i\left(\overline{x}_i(k) - x_i(k)\right), \tag{19}$$

with $\alpha_i \in (0,1]$*, then*

$$x_i(k+1) \leqslant x_{i+1}(k+1), \forall i = 2,\ldots,N-1. \tag{20}$$

Applying control law (16) we can rewrite system (4) in the following matrix form

$$x(k+1) = A(\bar{\alpha})x(k) + c(\bar{\alpha}), \tag{21}$$

where $x \in \mathbb{R}^N$ is the state vector, $\bar{\alpha} = [\alpha_1, \ldots, \alpha_N] \in (0, 1]^N$ is the vector of control parameters, while matrix $A(\bar{\alpha})$ and column $c(\bar{\alpha})$ are given by:

$$A(\bar{\alpha}) = \begin{bmatrix} 1-\frac{3\alpha_1}{4} & \frac{\alpha_1}{4} & \ldots & 0 & 0 & 0 \\ \frac{\alpha_2}{4} & 1-\frac{\alpha_2}{2} & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \frac{\alpha_{N-1}}{4} & 1-\frac{\alpha_{N-1}}{2} & \frac{\alpha_{N-1}}{4} \\ 0 & 0 & \ldots & 0 & \frac{\alpha_N}{4} & 1-\frac{3\alpha_N}{4} \end{bmatrix}, \quad c(\bar{\alpha}) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{\alpha_N}{2} \end{bmatrix}. \tag{22}$$

2.3. Stability properties for the multi-agent system

The next result provides structural insight on the closed-loop dynamics.

Theorem 3 *[6] (Gershgorin Circle Theorem)*
Let $A = [a_{ij}] \in M_{n\times n}$ and let $R_i'(A) = \sum_{j\neq i} |a_{ij}|$, $i = 1, \ldots, n$ denotes the absolute row sums of A_i and consider the n Gershgorin discs $G_i(A) = \{z \in \mathbb{C} : |z - a_{ii}| \leqslant R_i'(A)\}$, $i = 1, \ldots, n$. The eigenvalues of A are in the union of Gershgorin discs

$$G(A) = \bigcup_{i}^{n} \{z \in \mathbb{C} : |z - a_{ii}| \leqslant R_i'(A)\}.$$

For subsequent developments let us introduce the following definition.

Definition 4 *[7] The discrete linear system*

$$x(k+1) = Ax(k) + Bu(k) \tag{23}$$

is called positive if $x(k) \in \mathbb{R}^N_{>0}$ for any initial conditions $x_0 \in \mathbb{R}^N_{>0}$ and all $u(k) \in \mathbb{R}^N_{>0}$ for $k \in \mathbb{Z}_+$.

We recall here the following result on linear positive systems.

Theorem 4 *[7] Discrete–time linear system (23) is positive if and only if $A \in \mathbb{R}^{N\times N}_{>0}$ and $B \in \mathbb{R}^{N\times N}_{>0}$.*

Now we are in the position to formulate the following result.

Theorem 5 *Matrix $A(\bar{\alpha})$ given by formula (22) is positive and Schur stable.*

Proof. First, one can observe that, due to formula (22) and by the construction, all $a_{ij} > 0$ for $A(\bar{\alpha}) = [a_{ij}]$. Further, by Gershgorin Circle Theorem, all eigenvalues $\lambda \in spec(A(\bar{\alpha}))$ satisfy $\lambda \in G_1 \cup G_2$, where $G_1 = \left\{ \left| z - \left(1 - \frac{3\alpha_i}{4}\right)\right| \leqslant \frac{\alpha_i}{4} \right\}$ and $G_2 = \left\{ \left| z - \left(1 - \frac{\alpha_i}{2}\right)\right| \leqslant \frac{\alpha_i}{2} \right\}$ for $i = 1, \dots, N$. Moreover, since $\alpha_i \in (0, 1]$, we get $\frac{1}{2} \leqslant 1 - \frac{\alpha_i}{2} < 1$ and $0 \leqslant 1 - \alpha_i < 1$. Thus $\lambda \in [0, 1) \cup [0, 1] = [0, 1]$ and $\rho(A(\bar{\alpha})) \leqslant 1$. Next we show that the last inequality it strict, i.e. $\rho(A(\bar{\alpha})) < 1$. To prove this let us assume that there exists $\lambda \in \mathrm{spec}(A(\bar{\alpha}))$ and $\lambda = 1$. It implies that there exists a non-zero vector $v \in \mathbb{R}^N$ such that $A(\bar{\alpha}) \cdot v = v$. Therefore we get for $0 < \alpha_i \leqslant 1$:

$$\begin{bmatrix} -\frac{3\alpha_1}{4} & \frac{\alpha_1}{4} & \dots & 0 & 0 & 0 \\ \frac{\alpha_2}{4} & -\frac{\alpha_2}{2} & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ddots & \frac{\alpha_{N-1}}{4} & -\frac{\alpha_{N-1}}{2} & \frac{\alpha_{N-1}}{4} \\ 0 & 0 & \dots & 0 & \frac{\alpha_N}{4} & -\frac{3\alpha_N}{4} \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_{N-1} \\ v_N \end{bmatrix} = 0 \,.$$

Solving the above system of equations we get the unique solution $v = [0, 0, \dots, 0]$ contradicting the fact that $v \neq 0$ is an eigenvector. We conclude that matrix $A(\bar{\alpha})$ is Schur stable, what finishes the proof. □

Corollary 1 *If the multi-agent system shares the same feedback gain $\alpha \in (0, 1]$, then matrix A is positive, Schur stable and symmetric.*

Proof. Since the multi-agent system shares the same feedback gain $\alpha \in (0, 1]$, then it is clear that matrix A given by formula (27) is symmetric. The rest of properties follows directly from Theorem 5. □

Let us denote the set of initial conditions respecting the ordering condition:

$$C = \{x \in \mathbb{R}^n : x_i \leqslant x_{i+1},\ \forall i = 1, \dots, N - 1\} \,.$$

Corollary 2 *In the absence of disturbances, dynamical system (21) admits a unique stable equilibrium point:*

$$\bar{x}^T = \begin{bmatrix} \frac{1}{2N} & \frac{3}{2N} & \dots & \frac{2i-1}{2N} & \dots & \frac{2N-1}{2N} \end{bmatrix} . \tag{24}$$

Moreover, the set $C \cap [0, 1]^N$ is positive invariant with respect to the nominal dynamics.

Proof. The positive Schur matrix A implies there is no switching of positions in system (21) in the absence of disturbance. The positive invariance of the set C follows from the evaluation of the one-step ahead dynamics of the extreme points of this polyhedral set with respect to the resulting linear-time invariant dynamics. □

3. Impact of disturbances on the multi-agent systems

In this section we concentrate on the main problem of this paper: deployment of agents subject to disturbances. For this purpose we need the following definition and notations.

The saturation function $sat_{[0,1]} : \mathbb{R}^n \to [0,1]^n$ is defined as

$$\text{sat}_{[0,1]}(x) = y \quad \text{with} \quad y_i = \max\big(\min(x_i, 1), 0\big).$$

The group of permutation matrices of dimension $n \times n$ is denoted by $\mathbb{S}_n$. The ordering of a finite dimensional vector $x \in \mathbb{R}^n$ is achieved through:

$$y = \Pi(x)x \quad \text{with} \quad y_i \leqslant y_{i+1},\ \forall i = 1, \ldots, n-1,$$

where $\Pi(x) \in \mathbb{S}_n$.

3.1. Dynamical model with disturbance

In the presence of disturbances, model (21) takes the form:

$$x(k+1) = Ax(k) + c + w(k). \tag{25}$$

Different from the nominal case and properties in Theorem 2, two phenomena can emerge:

- the bounds of the interval $\mathcal{W} = [0,1]$ can be violated;
- the ordering of the agents may not hold along the trajectories of system (25).

In practice, these phenomena are mitigated by a saturation of the positions within $\mathcal{W} = [0,1]$ and the reordering of the agents' position whenever there exists an index i such that $x_i(k) > x_{i+1}(k)$.

With these arguments, and considering the same control gain for each agent ($\alpha_i = \alpha,\ \forall i \in \mathbb{N}_{[1,N]}$), the dynamics of the multi-agent system in the presence of disturbances will be modelled by:

$$x(k+1) = \text{sat}_{[0,1]}\left(\Pi(x(k))(Ax(k) + c + w(k))\right) \tag{26}$$

driven by the evolution matrix A and an affine dependence on the constant vector c both given by

$$A = \begin{bmatrix} 1-\frac{3\alpha}{4} & \frac{\alpha}{4} & \dots & 0 & 0 & 0 \\ \frac{\alpha}{4} & 1-\frac{\alpha}{2} & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ddots & \frac{\alpha}{4} & 1-\frac{\alpha}{2} & \frac{\alpha}{4} \\ 0 & 0 & \dots & 0 & \frac{\alpha}{4} & 1-\frac{3\alpha}{4} \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{\alpha}{2} \end{bmatrix}. \tag{27}$$

Note that the presence of disturbance $w(k)$ is affecting each position, while the permutation matrix $\Pi(x(k))$ accounts for the re-ordering of the agent's position in the case of switching of positions.

3.2. Ultimate Bounds

The LTI framework and the properties underlined in Theorem 5 ensure that the positive, real eigenvalues are positioned inside the unit circle and enable the analysis of the uncertainty by means of the ultimate bounds.

Theorem 6 *Consider multi-agent dynamical system* (25) *with time invariant feedback based on the constant gains* $\alpha_i = \alpha \in (0, 1]$, *allowing the decomposition* $A = V\Lambda V^{-1}$, *where* Λ *is the Jordan canonical form of* A. *In the presence of bounded disturbances*

$$w(k) \in W = \{w \colon |w| \leqslant \bar{w}, \} \quad \text{for all} \quad k \geqslant 0, \tag{28}$$

the polyhedral set

$$\mathcal{U}(\bar{w}) = \left\{x \in \mathbb{R}^n \colon |V^{-1}(x - \bar{x})| \leqslant (I - \Lambda)^{-1}|V^{-1}|\bar{w}\right\} \tag{29}$$

is robust positive invariant with respect to (26) *provided* $\mathcal{U}(\bar{w}) \subset C$.

Proof. Let us denote $z(k) = x(k) - \bar{x}$. For a time-invariant feedback gain $\alpha_i = \alpha$ and exploiting the results of Theorem 5 and Corollary 1, the error dynamics can be represented as:

$$z(k+1) = Az(k) + w(k),$$

where $z = x - \bar{x}$, $A \in \mathbb{R}^{N\times N}$ has all its eigenvalues strictly inside the unit circle and Jordan canonical form $\Lambda = V^{-1}AV$. For this class of dynamical system, the ultimate bounds have been characterized in [8] by means of polyhedral robust invariant set:

$$|V^{-1}z| \leqslant (I - \Lambda)^{-1}|V^{-1}|\bar{w}\,.$$

This set needs to additionally validate the ordering condition $x_i \leqslant x_{i+1}$, $\forall i = 1, \ldots, N-1$, summarized in the statement by the condition $\mathcal{U}(\bar{w}) \subset C$ in order to preserve the linear time-invariant properties of the dynamics (i.e. avoid the saturations and the permutations). It is worth mentioning that $\mathcal{U}$ is not necessarily a subset of $[0, 1]^n$ and thus the saturation mechanism might be activated in (25) in the presence of disturbance, even if there is no switching of position in between the agents. □

3.3. On the maximal admissible disturbance

While Theorem 6 provides a first characterization of the multi-agent system behaviour subject to disturbances, it can be also used to find maximal admissible bounds on the disturbances while guaranteeing the ordering of the agents' position or robust satisfaction of the safety constraints $x(k) \in [0, 1]^N$.

Proposition 1 *Let the robust positive invariant set be $\mathcal{U}(\bar{w}) \subset C$ as in (29) for the system (25) in the presence of bounded disturbances (28). For parameterized disturbance bounds $|w(k)| \leqslant \lambda\bar{w}$, the following results hold:*

1. *The set $\mathcal{U}(\lambda\bar{w})$ is robust positive invariant and preserves the ordering of the MAS for any trajectory with initial conditions $x_0 \in \mathcal{U}(\lambda\bar{w})$ if $0 \leqslant \lambda \leqslant \bar{\lambda}$ with:*

$$\bar{\lambda} = \arg\max_{\lambda(\mathcal{U}(\bar{w}) - \bar{x}) \oplus \bar{x} \subseteq C} \lambda \tag{30}$$

2. *The set $\mathcal{U}(\mu\bar{w})$ is robust positive invariant, preserves the ordering of the MAS for any trajectory with initial conditions $x_0 \in \mathcal{U}(\mu\bar{w})$ and is safe within $[0, 1]^N$ if $0 \leqslant \mu \leqslant \bar{\mu}$ with:*

$$\bar{\mu} = \arg\max_{\mu(\mathcal{U}(\bar{w}) - \bar{x}) \oplus \bar{x} \subseteq \left(C \cap [0,1]^N\right)} \mu \tag{31}$$

It is important to mention that the optimization problems (30) and (31) can be cast in the convex programming framework by means of an extended Farkas' Lemma [14], thus leading to a tractable LP formulation. For example, the optimum of (30) is given by $\bar{\lambda} = 1/\bar{\gamma}$ with:

$$\bar{\gamma} = \arg \min_{F \geqslant 0, \gamma \geqslant 0} \gamma, \tag{32}$$

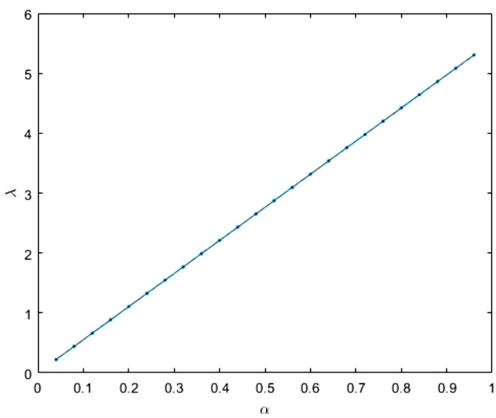
$$\begin{bmatrix} 1 & -1 & 0 & \ldots & 0 \\ 0 & 1 & -1 & \ddots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \ldots & 0 & 1 & -1 \end{bmatrix} = F \begin{bmatrix} V^{-1} \\ -V^{-1} \end{bmatrix},$$

$$F\begin{bmatrix}(I-\Lambda)^{-1}|V^{-1}|\\(I-\Lambda)^{-1}|V^{-1}|\end{bmatrix}\bar{w}\leqslant -\gamma\begin{bmatrix}1 & -1 & 0 & \dots & 0\\ 0 & 1 & -1 & \ddots & 0\\ & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 & -1\end{bmatrix}\bar{x},$$

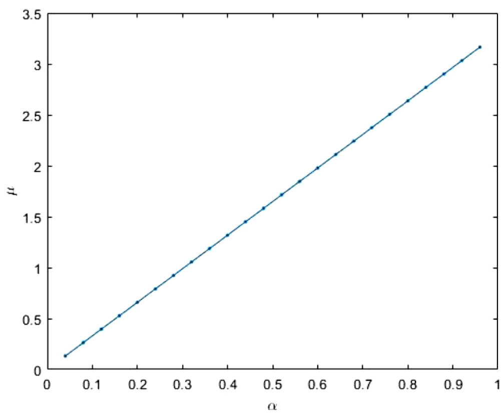
where the constraint $x \geqslant 0$ is interpreted element-wise. The matrix with non-negative elements F plays the role of a parameter related to the duality as detailed in [14].

The convex formulation of (31) has a similar structure by appending the constraints related to the inclusion in $[0, 1]^n$. This is omitted here for brevity.

Example 1 The above theoretical considerations can be illustrated by considering $N = 3$ agents and $\alpha \in [0.04; 0.96]$ with step 0.04. The agents share the same feedback gain α. For each such α, we find the greatest possible λ and μ by solving (30) and (31). The dependence of λ and μ on α is linear as depicted in Figure 1. In conclusion, the faster the convergence towards the Chebysev center, the larger the robustness margin with respect to additive uncertainty as expressed by the admissible bound on the uncertainty.



(a) dependence of λ on α



(b) dependence of μ on α

Figure 1: Dependence of parameters λ and μ on α for $N = 3$

3.4. The use of minimal robust positive invariant sets for the disturbance characterization

The ultimate bounds describe a safe (robust invariant) region in the state space of the multi-agent system in the presence of bounded disturbances. This set includes the equilibrium point in the absence of disturbances and provides a tool to certify an admissible upper bound for the disturbances. The robust positive invariance is preserved by homothety and thus the disturbance bounds can be maximized by scaling out the ultimate bounds set. This can be summarized by the property:

$$\bar{\lambda}A(\mathcal{U}(\bar{w}) - \bar{x}) \oplus \bar{\lambda}W \oplus \bar{x} \subseteq \bar{\lambda}(\mathcal{U}(\bar{w}) - \bar{x}) \oplus \bar{x} \subseteq C\,.$$

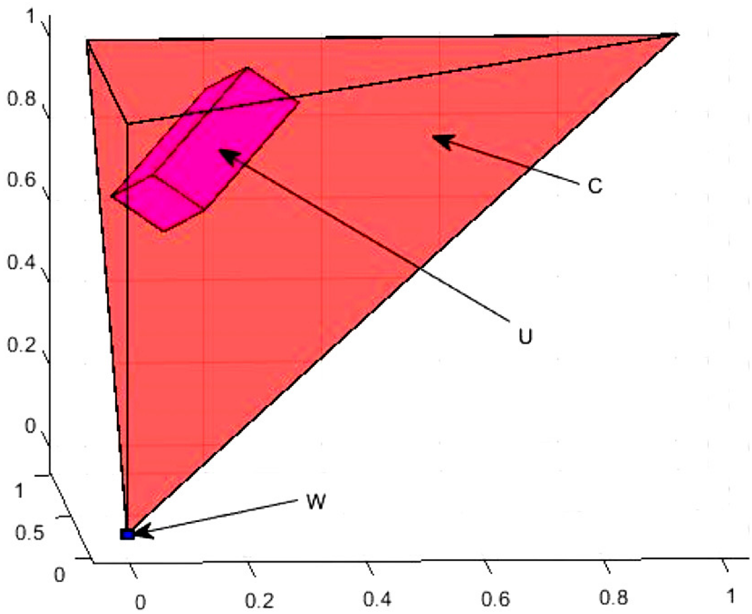


Figure 2: sets C, $U(\bar{w})$ and W

It follows that for any robust positive invariant set Ω satisfying:

$$A\Omega \oplus W \subseteq \Omega \subseteq \mathcal{U}(\bar{w}) - \bar{x} \tag{33}$$

there exists $\delta \geqslant 0$ such that

$$\left(\bar{\lambda} + \delta\right) \Omega \oplus \bar{x} \subseteq C. \tag{34}$$

From (33)–(34), the constraints and positive invariance are satisfied with an improved disturbance bound:

$$A(\bar{\lambda} + \delta)\Omega \oplus (\bar{\lambda} + \delta)W \subseteq (\bar{\lambda} + \delta)\Omega \subseteq C \ominus \bar{x}\,,$$

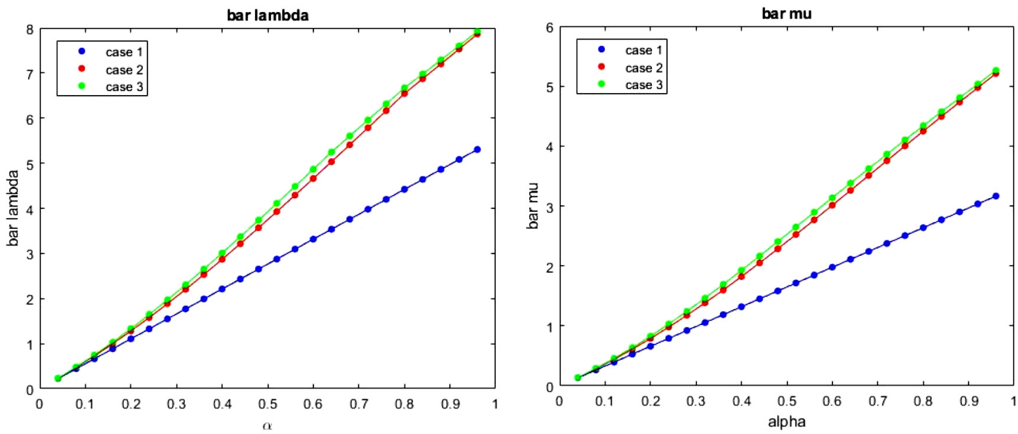
as long as any disturbance within the set $(\bar{\lambda} + \delta)W$ is deemed admissible. By exploiting the partial order of the robust positive invariant sets, the best scaling factor for the disturbance can be obtained using the minimal positive invariant set, which satisfies $A\Omega_\infty \oplus W = \Omega_\infty$. However, this set is generally not finitely determined but iterative procedures can provide its ϵ-approximations all by preserving the robust positive invariance property along the iterations:

$$\Omega_{k+1} = A\Omega_k \oplus W\,. \tag{35}$$

The iterative procedure can be intitialized with $\Omega_0 = \mathcal{U}(\bar{w}) - \bar{x}$.

The chosen Ω set is important in determining the maximal value of λ and μ. We illustrate the above theoretical considerations with two examples.

Example 2 Let us consider $N = 3$ agents and test different choices of $\alpha \in [0.04, 0.96]$ with a granularity 0.04. For each α, we want to find the maximum admissible λ and μ, taking into account three different disturbance bounds as follows: case 1: $\mathcal{U}(\bar{w})$; case 2: Ω_3; case 3: Ω_4 both Ω_3, Ω_4 according to formula (35).



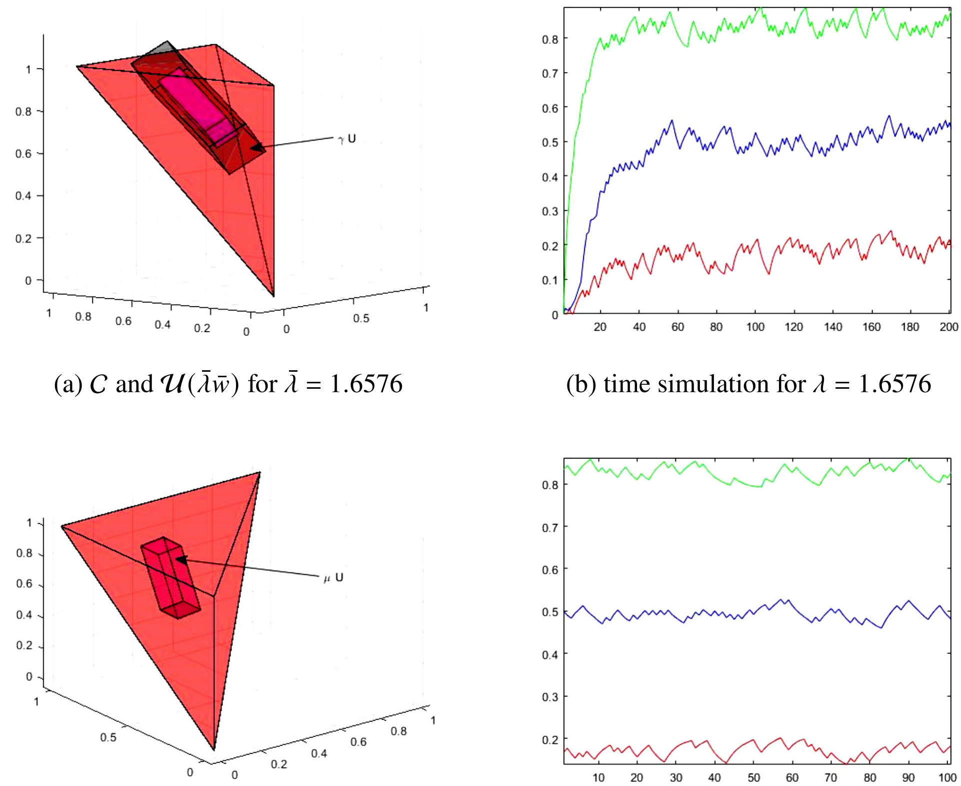
(a) The obtained λ values

(b) The obtained μ values

Figure 3: The obtained λ and μ depending on the considered Ω in three cases described in Example 2

Obviously, the relationship $\Omega_i \supset \Omega_{i+1}$ shows that more iterations of the formula (35) lead to larger admissible disturbances according to the values of λ and μ.

Example 3 Let us consider $N = 3$ agents with $\alpha = 0.3$. We calculate the maximum values of λ and μ. In Figure 4 are shown the sets C and $\mathcal{U}(\bar{w})$ for the obtained values $\lambda = 1.6576$, $\mu = 0.9890$ and time response simulations. Set C is marked in red, and set $\mathcal{U}(\bar{w})$ – in magenta. The simulations show that for the determined values of λ and μ, the agents reach a neighborhood of $\bar{x}$, both when they start the movement from position $X(0) = [0, 0, 0]$ and when we change the initial conditions to equilibrium points $\dfrac{2i-1}{2N}$ for each agent $i \in N$.



(a) C and $\mathcal{U}(\bar{\lambda}\bar{w})$ for $\bar{\lambda} = 1.6576$ (b) time simulation for $\lambda = 1.6576$

(c) C and $\mathcal{U}(\bar{\mu}\bar{w})$ for $\bar{\mu} = 0.9890$ (d) time simulation for $\mu = 0.9890$

Figure 4: The sets C and $\mathcal{U}(\bar{w})$ and time simulations for the obtained λ and μ parameters

3.5. Maximal admissible disturbances

The disturbances analysis with respect to the MAS dynamics:

$$x(k+1) = Ax(k) + c + w(k)$$

can be done globally on the working space $[0, 1]^N$ by imposing the robust satisfaction of the constraints $x_k \in C$.

The robust positive invariance of the set $C = \{x : Cx \leqslant 0\}$ can be transposed by the condition:

$$Cx(k+1) = CAx(k) + c + w(k)) \leqslant 0$$

whenever $Cx(k) \leqslant 0$.

However, as shown in Theorem 2, in the absence of disturbances and whenever $Cx(k) \leqslant 0$,

$$CAx(k) + c \leqslant 0$$

and thus a direct condition on the admissible disturbances can be derived as:

$$Cw(k) \leqslant 0$$

or alternatively:

$$w^1(k) \leqslant w^2(k) \leqslant \cdots \leqslant w^N(k).$$

It is interesting to observe that a similar condition can be obtained after a change of variables $x \mapsto 1-x$, what leads to a reverse order in the interval $[0, 1]$.

3.6. About the domain of attraction

In the previous subsections an in-depth analysis of the conditions for the existence of a robust positive invariant (abbr. RPI) set within C has been provided. The arguments were exploiting the contractive properties of the MAS evolution as long as they are governed by a linear dynamics (no switch among the agents and no constraint limitation with respect to the position).

The question we aim to address next is related to the initial conditions that will evolve towards such a RPI set. The main result presented next shows that any initial conditions in C can be chosen for asymptotic convergence towards the RPI set if such a set exists.

Theorem 7 *Consider dynamical system* (26) *and suppose the disturbance set W is such that there exists a robust positive invariant set $\Theta \subseteq C$, which satisfies, for a scalar $0 \leqslant \lambda < 1$, the following property:*

$$A(\Theta - \bar{x}) \oplus \bar{x} \oplus W \subseteq \lambda(\Theta - \bar{x}) \oplus \bar{x}.$$

Then the trajectory $x(k)$ for any initial condition $x_0 \in C$ converges in finite time towards Θ.

Proof. First let us observe that for any $x_0 \in \Theta \subset C$, $x(k) \in \Theta$ due to the RPI properties of the set $\Omega = \Theta - \bar{x}$ with respect to the linear dynamics. On the other hand, for any $x_0 \in C \setminus \Theta$ there exists a scaling factor $\alpha > 1$ such that:

$$x_0 \in \alpha\Omega \oplus \bar{x}\,.$$

We analyse the nonlinear dynamics (26) initiated in $x_0 \in C$ with respect to the positioning of the one-step-ahead transition in the two possible cases:

1. $Ax_0 + c + w_0 \in C$, then $\Pi(0) = I$ and taking into account that $\text{sat}_{[0,1]}(Ax_0 + c + w_0) = Ax_0 + c + w_0$ the LTI properties guarantee that:

$$Ax_0 + c + w_0 \in \lambda(\alpha\Omega) \oplus \bar{x}\,.$$

2. $Ax_0 + c + w_0 \notin C$. For this case, a switch of the positions within the MAS takes place such that:

$$\Pi(k)(Ax_0 + c + w_0) \in C\,.$$

Let us observe that if we are able to prove that:

$$\Pi(k)(Ax_0 + c + w_0) = Ax_0 + c + \widetilde{w}_0$$

for some $\widetilde{w}_0 \in W$, then we found ourselves in the previous case and a contraction factor can be found.

For this purpose let us analyse any arbitrary two agent positions

$$x_0^i \leqslant x_0^{i+1}.$$

In the absence of disturbances we have:

$$Ax_0^i + c \leqslant Ax_0^{i+1} + c\,.$$

Obviously, one can get

$$Ax_0^i + c - \bar{w} \leqslant Ax_0^i + c \leqslant Ax_0^{i+1} + c \leqslant Ax_0^{i+1} + c + \bar{w}\,.$$

but also individualy:

$$Ax_0^i + c - \bar{w} \leqslant Ax_0^i + c \leqslant Ax_0^i + c + \bar{w}\,,$$

$$Ax_0^{i+1} + c - \bar{w} \leqslant Ax_0^{i+1} + c \leqslant Ax_0^{i+1} + c + \bar{w}\,.$$

Now denoting

$$z_1^i = Ax_0^i + c + w_0^i$$
$$z_1^{i+1} = Ax_0^{i+1} + c + w_0^{i+1}$$

we observe that, when there is a switch of position we have:

$$Ax_0^i + c - \bar{w} \leqslant z_1^{i+1} \leqslant z_1^i \leqslant Ax_0^i + c + \bar{w}\,,$$

and thus there exists $\widetilde{w}_0^i$ such that:

$$Ax_0^i + c - \bar{w} \leqslant \underbrace{Ax_0^i + c + \widetilde{w}_0^i}_{z_1^{i+1}} \leqslant \underbrace{Ax_0^i + c + w_0^i}_{z_1^i}\,.$$

Similarly,

$$Ax_0^{i+1} + c - \bar{w} \leqslant z_1^{i+1} \leqslant z_1^i \leqslant Ax_0^{i+1} + c + \bar{w}$$

and there exists $\widetilde{w}_0^{i+1}$ such that:

$$\underbrace{Ax_0^{i+1} + c + w_0^{i+1}}_{z_1^{i+1}} \leqslant \underbrace{Ax_0^{i+1} + c + \widetilde{w}_0^{i}}_{z_1^{i}} \leqslant Ax_0^{i+1} + c + \bar{w}\,.$$

By observing that the permutation of positions in between the agents i and $i+1$ is equivalent to an evolution according to the linear dynamics:

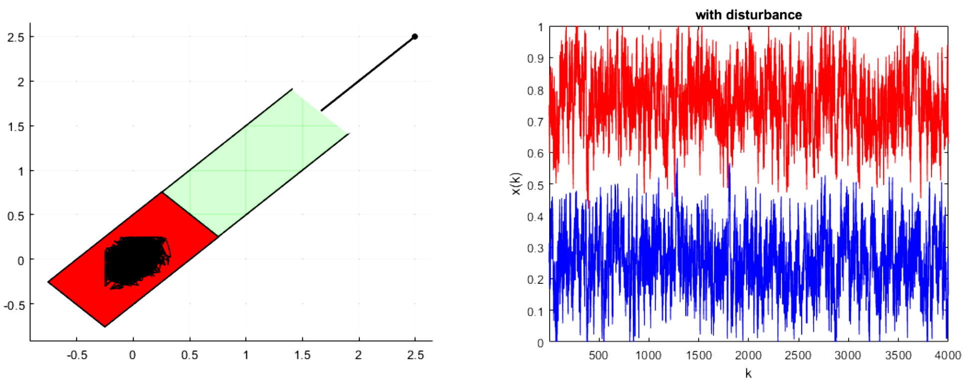
$$x_1^i \leftarrow z_1^{i+1} = Ax_0^i + c + \widetilde{w}_0^i$$

$$x_1^{i+1} \leftarrow z_1^i = Ax_0^{i+1} + c + \widetilde{w}_0^{i+1}\,,$$

for some disturbance realization $-\bar{w} \leqslant w_0^i, w_0^{i+1} \leqslant \bar{w}$, we finish the proof.

4. Numerical examples

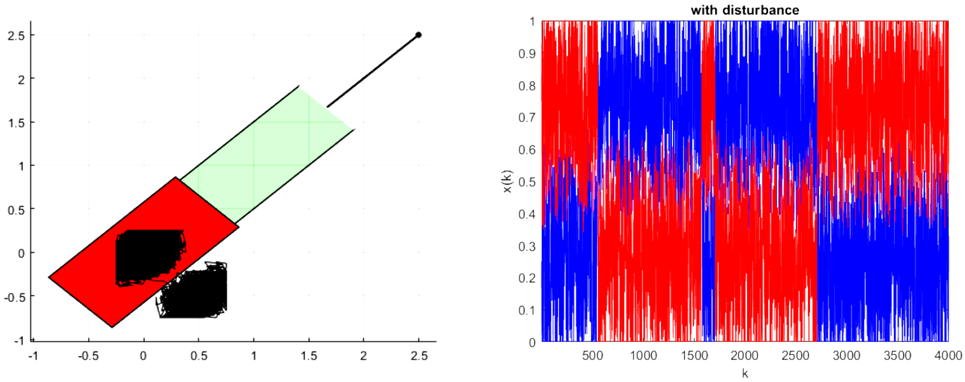
Example 4 Let us consider $N = 2$ agents with $\alpha = 0.7$ and initial conditions $x(0) = [0.25, 0.75]$. Using the LP problem we determine the greatest possible disturbances $\bar{w}$ and constraints – the invariant set is marked in red in Figure 5. Knowing $\bar{w}$, in each of the 4000 iterations, we randomly select the disturbances for each of the agents $w(k) = [w_1(k), w_2(k)]$ such that $|w_1| \leqslant \bar{w}$ and $|w_2| \leqslant \bar{w}$. The result is shown in Figure 5a and 5b. We can observe that the agents trajectories, marked in black, are inside the red constraints set. It is closely related to the results of the time simulation – agents do not change positions with each other. If we increase the disturbance delivered to the agents to be greater than $\bar{w}$, i.e.



(a) $w = 0.175 = \bar{w}$

(b) time simulation for $w = 0.175 = \bar{w}$

Figure 5: (a) and (b)

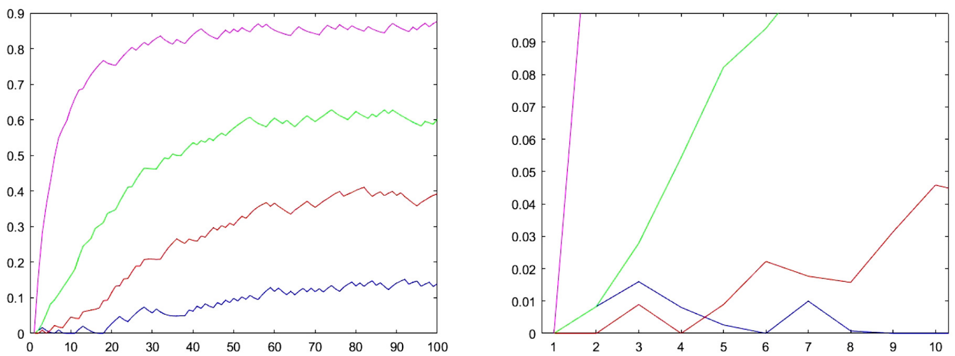


(c) $w > \bar{w}$ (d) time simulation for $w > \bar{w}$

Figure 5: Example for $N = 2$ and $\alpha = 0.7$: invariant set with agent trajectories and time simulations for $\bar{w}$ and for $w > \bar{w}$

$w_1 > \bar{w}$, $w_2 > \bar{w}$, then the trajectories extend beyond the invariant set and the agents change positions among themselves.

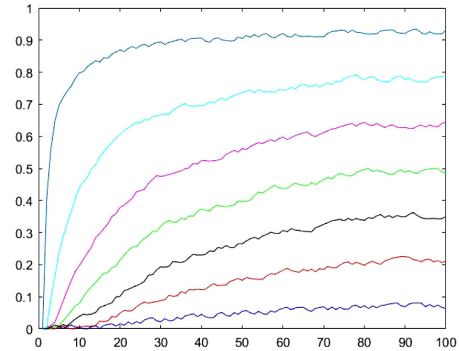
Example 5 In this example, we illustrate Theorem 7. First, let us consider $N = 4$ agents with $\alpha = 0.3$. We choose 0 as the starting position for each of them, i.e. $x(0) = [0, 0, 0, 0, 0, 0, 0]$. Indeed, in Figure 6 it can be seen that the agents change their positions at the beginning, but eventually each of them converges towards a neighborhood of its equilibrium point.



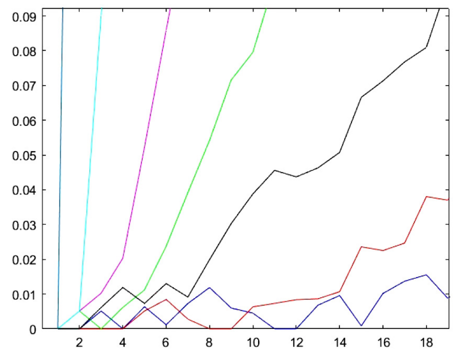
(a) 100 steps (b) first 10 steps

Figure 6: Time simulation for $N = 4$ agents with $\alpha = 0.3$ and $\lambda = 0.2767$

Afterwards, let us consider a group of $N = 7$ agents with $\alpha = 0.8$. Again, as initial conditions we take $x[0] = [0, 0, 0, 0]$. In Figure 7b, analyzing the first 10



(a) 100 steps



(b) first 10 steps

Figure 7: Time simulation for $N = 7$ agents with $\alpha = 0.8$ and $\lambda = 0.1717$

iterations, we can see that the agents change their positions during the transitory phase. However, in Figure 7a we see that once the trajectories reach the RPI sets around the equilibrium switching no longer occur, and the agents remain in the respective sets at all future instants.

5. Conclusion

In the present work the uncertainty impact on the behaviour of the scalar multi-agent system dynamics has been analysed using set-theoretic methods. It has been shown that a robust invariant set can be found around the equilibrium of the nominal dynamics for a bounded additive uncertainty. The admissible bound has been characterized and shown to be maximized by exploiting the homogeneity properties for linear dynamical systems. Furthermore, aside the existence of this local robust invariant set, we have shown that a family of robust invariant candidates can be generated according to the approximations of the minimal RPI set.

Aside the local behaviour, it has been shown that the multi-agent system may exhibit a nonlinear behaviour due to switch of position along the evolution. Those nonlinearities were shown to be equivalent to a different realization of the uncertain linear system. The main consequence of this result was the fact that the safe set was positive invariant and represented a domain of attraction of the mRPI set around the nominal equilibrium. In other words, any safe initial conditions of the multi-agent system resulted in a state trajectory which converged in a finite number of iterations to a neighborhood of the equilibrium. Thus the multiple equilibria or chaotic behaviour were prevented as long as the bounds on the disturbances were fulfilled.

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