10.24425/acs.2025.157146

Archives of Control Sciences
Volume 35(LXXI), 2025
No. 4*, pages 743–769*

On some new classes of functionals and their applications in optimization models

Valeria CÎRLAN and Savin TREANŢĂ

This paper investigates a class of multi-objective variational control problems, named (VCP), characterized by vector-valued integral objective/cost functional and differentiable inequality constraints. More precisely, we formulate the problem in a general framework and, in order to analyze optimality criteria, we associate scalar subproblems with the multi-objective setting and establish conditions under which efficient solutions of the original vector problem correspond to optimal solutions of these scalarized problems. We derive the necessary optimality conditions in the form of Euler–Lagrange-type equations, involving multipliers and piecewise smooth functions as variables. Furthermore, we introduce the new notion of $(\epsilon_0, \epsilon_1) - (c_0, c_1) - type - I$ pair of functionals, adapted to the variational control setting. The introduced generalized convexity concepts are very important in characterizing efficient solutions and in deriving enhanced optimality conditions. These results enrich the theoretical framework of multi-objective variational control and provide a solid basis for future research and applications in mathematical optimization and control theory.

Key words: optimization models, control problem, efficient solution

1. Introduction

Multi-objective variational problems have become one of the most discussed themes in mathematical optimization due to their relevance in diverse applications across control theory, physics, economics, or engineering. These prob-

V. Cîrlan is with Department of Applied Mathematics, National University of Science and Technology Politehnica Bucharest, 060042 Bucharest, Romania and with Ştefan cel Mare University of Suceava, 13 University Street, 720229 Suceava, Romania.

S. Treanţă (corresponding author, e-mail: savin.treanta@upb.ro) is with Department of Applied Mathematics, National University of Science and Technology Politehnica Bucharest, 060042 Bucharest, Romania and Academy of Romanian Scientists, 54 Splaiul Independentei, 050094 Bucharest, Romania and with Fundamental Sciences Applied in Engineering – Research Center (SFAI), National University of Science and Technology Politehnica Bucharest, 060042 Bucharest, Romania.

Received 19.05.2025. Revised 22.10.2025.

lems, which involve optimizing multiple conflicting objectives simultaneously, present real theoretical challenges, especially in the absence of convexity which is conventional basics in optimization theory. Traditional approaches have relied considerably on convexity assumptions to obtain necessary and sufficient optimality conditions as well as duality correlations. However, many real-world problems cannot be solved with convex structures. This limitation has guided to the development of generalized convexity concepts, such as, invexity [23, 25], generalized invexity [13, 26], generalized type-I invexity [20], V-invexity [27], generalized type-I univexity [18], or (F,ρ)-convexity [3], that extend the theoretical framework to include an ample class of functions and control problems. Duality theory fill as a fundamental analytical instrument, enabling the transformation of primal optimization problems into dual problems which are often easier to analyze or solve. This area has been enriched with new forms of dual constructs, such as Wolfe and Mond-Weir type duals, within generalized duality theorems under the newly proposed generalized structure [14]. Bellman [6], Mond and Hanson [24] and Hanson [15] are essential works which have been substantially extended in later research as [2, 7] and many others. The study of duality and optimality in multi-objective variational problems under generalized convexity assumptions has also attracted attention for many researchers. For example, Bhatia and Mehra [8], Khazafi and Rueda [18, 19] and Liu [21] developed duality frameworks for variational and control problems using various generalizations of convexity. Craven [10] established the Kuhn-Tucker necessary conditions for optimality in certain multi-objective variational problems. Furthermore, he demonstrated that these conditions become sufficient when the constraint functions are quasi-convex and the objective functions are pseudoconvex. Fantuzzi [11] studied convex relaxation duality in constrained variational problems, establishing strong duality under specific assumptions offering computationally efficient formulations. In a related direction, Joshi [17] introduced new duality results for vanishing constraint problems, employing generalized invex functions and improved dual formulations in mathematical programming. Recent literature also draw attention to nonconvex and nondifferentiable assumptions by Antczak et al. [4] and Reddy and Mukherjee [31] to find efficient solutions in generalized variational models illustrated by Treanță [33]. The achievements obtained by Xiuhong [37] and Zhian and Qingkai [38] have enriched the knowledge in this field by establishing duality theorems for classes of multiobjective control problems using generalized invexity, highlighting the current evolution and development applicability of these mathematical tools to computational theory. On the other side, control theory combined with variational principles, make easier practical implementations in dynamic's field. The study of multiobjective variational control problems, particularly those that contain system dynamics, constraints and fractional control problems has attracted significant attention. Recently works as those of Antczak et al. [4] or Treanţă [35], provided meticulous analysis on the efficiency and duality of non-differentiable, non-convex and vector fractional problems. Alternative approaches to generalized convexity, nonconvex optimization and vector optimization have been developed by Mishra et al. [22]. These studies emphasize the importance of pseudo-invexity, $V - KT$ conditions and (ρ, b)-quasi-invexity, which are applicable in theoretical research and engineering systems. Also, Treanţă [34] defined strong formulations by developing necessary and sufficient optimality conditions adapted to variational problems exposed to perturbations and uncertainty. New approach on generalized convexity and nonsmooth fractional interval-valued optimization belong to Hung and Van Tuyen [16]. Adly and Kien [1] formulated first and second order KKT conditions for multiobjective control systems with mixed constraints, highlighting sustainable energy management applications. Saadati and Oveisiha [32] explored robust nonsmooth optimization in Asplund spaces, incorporating fuzzy and variational tools to model uncertainty and nonconvexity. These formulations, as seen in the works of Preda [30] and Reddy and Mukherjee [31], allow for a more comprehensive handling of constraint qualifications and solution efficiency, particularly in multi-objective and fractional frameworks. The research of Khazafi and Rueda [19], along with extensions by Nahak and Behera [29], has highlighted the utility of $\rho - (\eta, \theta) - B - type - I$ functions in formulating necessary and sufficient optimality conditions for multiobjective control problems.

In this paper, our aim is to contribute to this growing field by investigating new duality results and sufficient efficiency conditions for a class of multiobjective variational control problems under (generalized) $(\epsilon_0, \epsilon_1) - (c_0, c_1) - type - I$ assumptions of implied functionals. Based on existing literature, we provide general conditions that summarizes a large presentation of functional forms and constraints. The results presented not only generalize classical duality structures but also offer enhanced analytical tools for addressing complex control systems governed by multiple objectives. To this purpose, we introduce $(\epsilon_0, \epsilon_1) - (c_0, c_1) - type - I$ pairs of functionals in order to derive more general results for sufficient efficiency criteria and duality associated with multi-objective optimization problems.

The paper continues as follows. In Section 2, we define the problem under study, called (VCP), and some preliminaries. In Section 3, we study efficiency criteria associated with (VCP) and ennounce certain theorems to establish several sufficient efficiency conditions. In the next section, we focus on reciprocal results, providing various outcomes of dual type. In the last section, we present our conclusions and future directions, which are worth further investigation.

2. Problem formulation and preliminaries

Consider the following multi-objective variational control problem:

$$(VCP) \quad \min_{(\sigma,\omega)} \left\{ \int_a^b \mu(f,\sigma,\omega)\mathrm{d}f := \left(\int_a^b \mu_1(f,\sigma,\omega)\mathrm{d}f, \ldots, \int_a^b \mu_t(f,\sigma,\omega)\mathrm{d}f \right) \right\}$$

subject to

$$\sigma(a) = \alpha, \quad \sigma(b) = \beta,$$
$$h(f,\sigma,\dot{\sigma},\omega) := g(f,\sigma,\omega) - \dot{\sigma} \leqq 0, \quad f \in I := [a,b].$$

Let the set of feasible pairs associated with (VCP) be denoted as

$$Q = \Big\{ (\sigma,\omega) \in C^1(I,\mathbb{R}^n) \times C\left(I,\mathbb{R}^l\right) \mid \sigma(a) = \alpha,\ \sigma(b) = \beta,$$
$$h(f,\sigma,\dot{\sigma},\omega) \leqq 0,\ f \in I \Big\}.$$

Regarding (VCP), we formulate the minimization model $\left(P_q^*\right)$, with $q = 1,\ldots,t$, as follows:

$$\left(P_q^*\right) \quad \min_{(\sigma,\omega)} \int_a^b \mu_q(f,\sigma,\omega)\mathrm{d}f$$

$$\text{subject to } \sigma(a) = \alpha,\ \sigma(b) = \beta,$$

$$\int_a^b \mu_p(f,\sigma,\omega)\mathrm{d}f \leqslant \int_a^b \mu_p(f,\sigma^*,\omega^*)\mathrm{d}f,\ p = 1,2,\ldots,t,\ p \neq q,$$

$$h_s(f,\sigma,\dot{\sigma},\omega) \leqslant 0,\ s = 1,\ldots,n,\ f \in I.$$

Next, by considering Geoffrion [12], Nahak and Nanda [28], or Treanţă et al. [36], we formulate the following definitions.

Definition 1. *A pair (σ^*,ω^*) in Q is said to be an efficient pair of (VCP) if, for all $(\sigma,\omega) \in Q$, we have*

$$\int_a^b \mu_p(f,\sigma^*,\omega^*)\mathrm{d}f \geqslant \int_a^b \mu_p(f,\sigma,\omega)\mathrm{d}f, \quad \forall p \in \{1,\ldots,t\}$$

$$\Longrightarrow \int_a^b \mu_p(f,\sigma^*,\omega^*)\mathrm{d}f = \int_a^b \mu_p(f,\sigma,\omega)\mathrm{d}f, \quad \forall p \in \{1,\ldots,t\}.$$

Definition 2. *A pair* (σ^*, ω^*) *in* Q *is said to be a properly efficient pair of* (VCP) *if there exists a real number* $O > 0$ *such that*

$$\int_a^b \mu_p\,(f, \sigma^*, \omega^*)\,\mathrm{d}f - \int_a^b \mu_p(f, \sigma, \omega)\,\mathrm{d}f$$
$$\leqslant O\left(\int_a^b \mu_j(f, \sigma, \omega)\,\mathrm{d}f - \int_a^b \mu_j(f, \sigma^*, \omega^*)\,\mathrm{d}f\right), \quad \forall p \in \{1, \dots, t\},$$

for all $(\sigma, \omega) \in Q$ *and for some* j *such that*

$$\int_a^b \mu_j(f, \sigma, \omega)\,\mathrm{d}f > \int_a^b \mu_j(f, \sigma^*, \omega^*)\,\mathrm{d}f$$

whenever (σ, ω) *is in* Q *and*

$$\int_a^b \mu_p(f, \sigma, \omega)\,\mathrm{d}f < \int_a^b \mu_p(f, \sigma^*, \omega^*)\,\mathrm{d}f.$$

Next, in accordance with Chankong and Haimes [9], we establish the following result.

Lemma 1. *The pair* (σ^*, ω^*) *is an optimal solution of* $\left(P_q^*\right)$, *with* $q = 1, 2, \dots, t$, *is equivalent to* (σ^*, ω^*) *is an efficient pair of* (VCP).

According to Treanţă [34], following Mond and Hanson [24], we state the theorem.

Theorem 1. *Let* (σ^*, ω^*) *be a normal optimal solution of* $\left(P_q^*\right)$, $q = 1, 2, \dots, t$. *Then, there exist the scalars* $\phi_{1q}, \dots, \phi_{tq}$, *where* $\phi_{qq} = 1$, *and* $\zeta_q\colon I \to \mathbb{R}^n$ *(piecewise differentiable function) such that*

$$\mu_{q\sigma}(f, \sigma^*, \omega^*) + \sum_{\substack{p=1 \\ p \neq q}}^{t} \phi_{pq}\mu_{p\sigma}(f, \sigma^*, \omega^*) + \zeta_q(f)^T h_\sigma(f, \sigma^*, \dot{\sigma}^*, \omega^*)$$
$$= \frac{d}{df}\Bigg(\mu_{q\dot{\sigma}}(f, \sigma^*, \omega^*) + \sum_{\substack{p=1 \\ p \neq q}}^{t} \phi_{pq}\mu_{p\dot{\sigma}}\,(f, \sigma^*, \omega^*)$$
$$+\ \zeta_q(f)^T h_{\dot{\sigma}}(f, \sigma^*, \dot{\sigma}^*, \omega^*)\Bigg), \quad f \in I, \tag{1}$$

$$\mu_{q\omega}(f,\sigma^*,\omega^*) + \sum_{\substack{p=1\\ p\neq q}}^{t} \phi_{pq}\mu_{p\omega}(f,\sigma^*,\omega^*)$$
$$+ \ \zeta_q(f)^T h_\omega(f,\sigma^*,\dot{\sigma}^*,\omega^*) = 0, \quad f \in I, \tag{2}$$

$$\zeta_q(f)^T h\,(f,\sigma^*,\dot{\sigma}^*,\omega^*) = 0, \quad f \in I, \tag{3}$$

$$\zeta_q(f) \geqq 0, \quad f \in I, \tag{4}$$

$$\phi_{pq} \geqslant 0, \quad p = 1,2,\ldots,t, \quad p \neq q \tag{5}$$

are satisfied, for all $f \in I$, except at discontinuities.

Definition 3. *Let μ and h be two real-valued functionals. The pair (μ, h) is said to be $(\epsilon_0,\epsilon_1) - (c_0,c_1) - type - I$ at $(G,v) \in Q \subset C^1(I,\mathbb{R}^n) \times C\left(I,\mathbb{R}^l\right)$ with respect to v, ξ, and ϑ if there exist c_0, c_1: $\left(C^1(I,\mathbb{R}^n)\times C\left(I,\mathbb{R}^l\right)\right)^2 \to \mathbb{R}_+$, v, ϑ: $I\times\left(C^1(I,\mathbb{R}^n)\times C\left(I,\mathbb{R}^l\right)\right)^2 \to \mathbb{R}^n$, ξ: $I\times\left(C^1(I,\mathbb{R}^n)\times C\left(I,\mathbb{R}^l\right)\right)^2 \to \mathbb{R}^l$, with*

$$v(f,\sigma,\omega,\sigma,\omega) = v(f,\sigma,\omega,G,v)|_{f=a,b} = \xi(f,\sigma,\omega,\sigma,\omega) = 0, \ \ \epsilon_0,\epsilon_1 \in \mathbb{R},$$

such that, for all $(\sigma,\omega) \in Q$, we have

$$c_0(\sigma,\omega,G,v)\left\{\int_a^b \mu(f,\sigma,\omega)\,\mathrm{d}f - \int_a^b \mu(f,G,v)\,\mathrm{d}f\right\}$$
$$\geqslant \int_a^b \Big[v(f,\sigma,\omega,G,v)^T\mu_\sigma(f,G,v) + \frac{d}{df}v(f,\sigma,\omega,G,v)^T\mu_{\dot{\sigma}}(f,G,v)$$
$$+ \ \xi(f,\sigma,\omega,G,v)^T\mu_\omega(f,G,v) + \epsilon_0\|\vartheta(f,\sigma,\omega,G,v)\|^2\Big]\mathrm{d}f \tag{6}$$

and

$$-\,c_1(\sigma,\omega,G,v)\int_a^b h(f,G,\dot{G},v)\,\mathrm{d}f$$
$$\geqslant \int_a^b \Big[v(f,\sigma,\omega,G,v)^T h_\sigma(f,G,\dot{G},v) + \frac{d}{df}v(f,\sigma,\omega,G,v)^T h_{\dot{\sigma}}(f,G,\dot{G},v)$$
$$+ \ \xi(f,\sigma,\omega,G,v)^T h_\omega(f,G,\dot{G},v) + \epsilon_1\|\vartheta(f,\sigma,\omega,G,v)\|^2\Big]\mathrm{d}f.$$

If in the previous definition, the relation given in (6) is satisfied as a strict inequality, then we say that the pair (μ, h) is semi-strictly - $(\epsilon_0, \epsilon_1) - (c_0, c_1) - type - I$ at $(G, v) \in Q \subset C^1(I, \mathbb{R}^n) \times C(I, \mathbb{R}^l)$ with respect to v, ξ, and ϑ.

Definition 4. *The pair (μ, h) is said to be quasi - $(\epsilon_0, \epsilon_1) - (c_0, c_1) - type - I$ at $(G, v) \in Q \subset C^1(I, \mathbb{R}^n) \times C(I, \mathbb{R}^l)$ with respect to v, ξ, and ϑ, if there exist $c_0, c_1 : \left(C^1(I, \mathbb{R}^n) \times C(I, \mathbb{R}^l)\right)^2 \to \mathbb{R}_+$, $v, \vartheta : I \times \left(C^1(I, \mathbb{R}^n) \times C(I, \mathbb{R}^l)\right)^2 \to \mathbb{R}^n$, $\xi : I \times \left(C^1(I, \mathbb{R}^n) \times C(I, \mathbb{R}^l)\right)^2 \to \mathbb{R}^l$, with $v(f, \sigma, \omega, \sigma, \omega) = v(f, \sigma, \omega, G, v)|_{f=a,b} = \xi(f, \sigma, \omega, \sigma, \omega) = 0$, $\epsilon_0, \epsilon_1 \in \mathbb{R}$, such that, for all $(\sigma, \omega) \in Q$, the inequality*

$$c_0(\sigma, \omega, G, v) \left\{ \int_a^b \mu(f, \sigma, \omega) \mathrm{d}f - \int_a^b \mu(f, G, v) \mathrm{d}f \right\} \leqslant 0$$

implies

$$\int_a^b \Big[v(f, \sigma, \omega, G, v)^T \mu_\sigma(f, G, v) + \frac{d}{df} v(f, \sigma, \omega, G, v)^T \mu_{\dot{\sigma}}(f, G, v)$$
$$+ \xi(f, \sigma, \omega, G, v)^T \mu_\omega(f, G, v) + \epsilon_0 \|\vartheta(f, \sigma, \omega, G, v)\|^2 \Big] \mathrm{d}f \leqslant 0,$$

and

$$-c_1(\sigma, \omega, G, v) \int_a^b h(f, G, \dot{G}, v) \mathrm{d}f \leqslant 0$$

involves

$$\int_a^b \Big[v(f, \sigma, \omega, G, v)^T h_\sigma(f, G, \dot{G}, v) + \frac{d}{df} v(f, \sigma, \omega, G, v)^T h_{\dot{\sigma}}(f, G, \dot{G}, v)$$
$$+ \xi(f, \sigma, \omega, G, v)^T h_\omega(f, G, \dot{G}, v) + \epsilon_1 \|\vartheta(f, \sigma, \omega, G, v)\|^2 \Big] \mathrm{d}f \leqslant 0.$$

Definition 5. *The pair (μ, h) is said to be strongly - pseudo - $(\epsilon_0, \epsilon_1) - (c_0, c_1) - type - I$ at $(G, v) \in Q \subset C^1(I, \mathbb{R}^n) \times C(I, \mathbb{R}^l)$ with respect to v, ξ, and ϑ, if there exist $c_0, c_1 : \left(C^1(I, \mathbb{R}^n) \times C(I, \mathbb{R}^l)\right)^2 \to \mathbb{R}_+$, $v, \vartheta : I \times \left(C^1(I, \mathbb{R}^n) \times C(I, \mathbb{R}^l)\right)^2 \to \mathbb{R}^n$, $\xi : I \times \left(C^1(I, \mathbb{R}^n) \times C(I, \mathbb{R}^l)\right)^2 \to \mathbb{R}^l$, with $v(f, \sigma, \omega, \sigma, \omega) =$*

$\nu(f,\sigma,\omega,G,v)|_{f=a,b} = \xi(f,\sigma,\omega,\sigma,\omega) = 0$, $\epsilon_0,\epsilon_1 \in \mathbb{R}$, *such that, for all* $(\sigma,\omega) \in Q$, *the following relation*

$$\int_a^b \Big[\nu(f,\sigma,\omega,G,v)^T \mu_\sigma(f,G,v) + \frac{d}{df}\nu(f,\sigma,\omega,G,v)^T \mu_{\dot\sigma}(f,G,v)$$
$$+\,\xi(f,\sigma,\omega,G,v)^T \mu_\omega(f,G,v) + \epsilon_0\|\vartheta(f,\sigma,\omega,G,v)\|^2\Big]\mathrm{d}f \geqslant 0$$

involves

$$c_0(\sigma,\omega,G,v)\left\{\int_a^b \mu(f,\sigma,\omega)\mathrm{d}f - \int_a^b \mu(f,G,v)\mathrm{d}f\right\} \geqslant 0,$$

and

$$\int_a^b \Big[\nu(f,\sigma,\omega,G,v)^T h_\sigma(f,G,\dot G,v) + \frac{d}{df}\nu(f,\sigma,\omega,G,v)^T h_{\dot\sigma}(f,G,\dot G,v)$$
$$+\,\xi(f,\sigma,\omega,G,v)^T h_\omega(f,G,\dot G,v) + \epsilon_1\|\vartheta(f,\sigma,\omega,G,v)\|^2\Big]\mathrm{d}f \geqslant 0$$

implies

$$-c_1(\sigma,\omega,G,v)\int_a^b h(f,G,\dot G,v)\mathrm{d}f \geqslant 0.$$

Definition 6. *The pair* (μ,h) *is said to be pseudo - quasi -* $(\epsilon_0,\epsilon_1) - (c_0,c_1) -$ *type* $-\ I$ *at* $(G,v) \in Q \subset C^1(I,\mathbb{R}^n) \times C(I,\mathbb{R}^l)$ *with respect to* ν,ξ, *and* ϑ, *if there exist* $c_0,c_1 : \left(C^1(I,\mathbb{R}^n) \times C(I,\mathbb{R}^l)\right)^2 \to \mathbb{R}_+$, $\nu,\vartheta : I \times \Big(C^1(I,\mathbb{R}^n) \times C(I,\mathbb{R}^l)\Big)^2 \to \mathbb{R}^n$, $\xi : I \times \Big(C^1(I,\mathbb{R}^n) \times C(I,\mathbb{R}^l)\Big)^2 \to \mathbb{R}^l$, *with* $\nu(f,\sigma,\omega,\sigma,\omega) = \nu(f,\sigma,\omega,G,v)|_{f=a,b} = \xi(f,\sigma,\omega,\sigma,\omega) = 0$, $\epsilon_0,\epsilon_1 \in \mathbb{R}$, *such that, for all* $(\sigma,\omega) \in Q$, *are verified the conditions: if*

$$c_0(\sigma,\omega,G,v)\left\{\int_a^b \mu(f,\sigma,\omega)\mathrm{d}f - \int_a^b \mu(f,G,v)\mathrm{d}f\right\} \leqslant 0$$

we have

$$\int_a^b \Big[\nu(f,\sigma,\omega,G,v)^T \mu_\sigma(f,G,v) + \frac{d}{df}\nu(f,\sigma,\omega,G,v)^T \mu_{\dot\sigma}(f,G,v)$$
$$+\,\xi(f,\sigma,\omega,G,v)^T \mu_\omega(f,G,v) + \epsilon_0\|\vartheta(f,\sigma,\omega,G,v)\|^2\Big]\mathrm{d}f \leqslant 0,$$

and from

$$\int_a^b \Big[\nu(f,\sigma,\omega,G,v)^T h_\sigma(f,G,\dot G,v) + \frac{d}{df}\nu(f,\sigma,\omega,G,v)^T h_{\dot\sigma}(f,G,\dot G,v)$$
$$+\,\xi(f,\sigma,\omega,G,v)^T h_\omega(f,G,\dot G,v) + \epsilon_1\|\vartheta(f,\sigma,\omega,G,v)\|^2\Big]\mathrm{d}f \geqslant 0$$

there result that

$$-c_1(\sigma,\omega,G,v)\int_a^b h(f,G,\dot G,v)\,\mathrm{d}f \geqslant 0.$$

If in the previous definition, we have

$$\int_a^b \Big[\nu(f,\sigma,\omega,G,v)^T h_\sigma(f,G,\dot G,v) + \frac{d}{df}\nu(f,\sigma,\omega,G,v)^T h_{\dot\sigma}(f,G,\dot G,v)$$
$$+\,\xi(f,\sigma,\omega,G,v)^T h_\omega(f,G,\dot G,v) + \epsilon_1\|\vartheta(f,\sigma,\omega,G,v)\|^2\Big]\mathrm{d}f \geqslant 0$$

there result that

$$-c_1(\sigma,\omega,G,v)\int_a^b h(f,G,\dot G,v)\,\mathrm{d}f > 0,$$

then we say that the pair (μ,h) is strictly - pseudo - quasi - $(\epsilon_0,\epsilon_1)-(c_0,c_1)-$ $type-I$ at $(G,v)\in Q\subset C^1(I,\mathbb{R}^n)\times C\left(I,\mathbb{R}^l\right)$ with respect to ν,ξ, and ϑ.

Definition 7. *The pair (μ,h) is said to be strongly - pseudo - quasi - $(\epsilon_0,\epsilon_1)-(c_0,c_1)-type-I$ at $(G,v)\in Q\subset C^1(I,\mathbb{R}^n)\times C\left(I,\mathbb{R}^l\right)$ with respect to ν,ξ, and ϑ, if there exist $c_0,c_1:\left(C^1(I,\mathbb{R}^n)\times C\left(I,\mathbb{R}^l\right)\right)^2\to\mathbb{R}_+$, $\nu,\vartheta:I\times\left(C^1(I,\mathbb{R}^n)\times C\left(I,\mathbb{R}^l\right)\right)^2\to\mathbb{R}^n$, $\xi:I\times\left(C^1(I,\mathbb{R}^n)\times C\left(I,\mathbb{R}^l\right)\right)^2\to\mathbb{R}^l$, with $\nu(f,\sigma,\omega,\sigma,\omega)=\nu(f,\sigma,\omega,G,v)|_{f=a,b}=\xi(f,\sigma,\omega,\sigma,\omega)=0$, $\epsilon_0,\epsilon_1\in\mathbb{R}$,* *such that, for all* $(\sigma, \omega) \in Q$*, are verified the conditions:*

$$\int_a^b \Big[\nu(f,\sigma,\omega,G,v)^T \mu_\sigma(f,G,v) + \frac{d}{df}\nu(f,\sigma,\omega,G,v)^T \mu_{\dot\sigma}(f,G,v) + \xi(f,\sigma,\omega,G,v)^T \mu_\omega(f,G,v) + \epsilon_0 \|\vartheta(f,\sigma,\omega,G,v)\|^2 \Big] \mathrm{d}f \geqslant 0$$

implies

$$c_0(\sigma,\omega,G,v)\left\{ \int_a^b \mu(f,\sigma,\omega)\,\mathrm{d}f - \int_a^b \mu(f,G,v)\,\mathrm{d}f \right\} \geqslant 0,$$

and

$$-c_1(\sigma,\omega,G,v)\int_a^b h(f,G,\dot G,v)\,\mathrm{d}f \leqslant 0$$

gives

$$\int_a^b \Big[\nu(f,\sigma,\omega,G,v)^T h_\sigma(f,G,\dot G,v) + \frac{d}{df}\nu(f,\sigma,\omega,G,v)^T h_{\dot\sigma}(f,G,\dot G,v) + \xi(f,\sigma,\omega,G,v)^T h_\omega(f,G,\dot G,v) + \epsilon_1 \|\vartheta(f,\sigma,\omega,G,v)\|^2 \Big] \mathrm{d}f \leqslant 0.$$

If in the previous definition, if we have

$$\int_a^b \Big[\nu(f,\sigma,\omega,G,v)^T \mu_\sigma(f,G,v) + \frac{d}{df}\nu(f,\sigma,\omega,G,v)^T \mu_{\dot\sigma}(f,G,v) + \xi(f,\sigma,\omega,G,v)^T \mu_\omega(f,G,v) + \epsilon_0 \|\vartheta(f,\sigma,\omega,G,v)\|^2 \Big] \mathrm{d}f \geqslant 0$$

implies

$$c_0(\sigma,\omega,G,v)\left\{ \int_a^b \mu(f,\sigma,\omega)\,\mathrm{d}f - \int_a^b \mu(f,G,v)\,\mathrm{d}f \right\} > 0,$$

then we say that the pair (μ, h) is *strictly – strongly – pseudo – quasi –* $(\epsilon_0, \epsilon_1) - (c_0, c_1) - type - I$ *at* $(G, v) \in Q \subset C^1(I, \mathbb{R}^n) \times C(I, \mathbb{R}^l)$ *with respect to* $\nu, \xi,$ *and* ϑ.

Lemma 2 ((Nahak and Behera [29])). *Every* $(\epsilon_0, \epsilon_1) - (c_0, c_1) - type - I$ *pair of functions is strongly - pseudo -* $(\epsilon_0, \epsilon_1) - (c_0, c_1) - type - I$*. The converse is false.*

3. Efficiency criteria associated with (VCP)

Theorem 2 (Necessary efficiency conditions)**.** *If* $(\sigma^*, \omega^*) \in C^1(I, \mathbb{R}^n) \times C(I, \mathbb{R}^l)$ *is a properly efficient pair of* (VCP)*, which is assumed to be a normal optimal solution for* $\left(P_q^*\right)$*,* $q = 1, 2, \ldots, t$*, then there exist* $\phi^* \in \mathbb{R}^t$ *and* $\zeta^* : I \to \mathbb{R}^n$ *a piecewise smooth function such that*

$$\phi^{*T} f_x(f, \sigma^*, \omega^*) + \zeta^*(f)^T h_\sigma(f, \sigma^*, \dot{\sigma}^*, \omega^*)$$
$$= \frac{d}{df}\Big(\phi^{*T} \mu_{\dot{\sigma}}(f, \sigma^*, \omega^*) + \zeta^*(f)^T h_{\dot{\sigma}}(f, \sigma^*, \dot{\sigma}^*, \omega^*)\Big), \quad f \in I, \tag{7}$$

$$\phi^{*T} \mu_\omega(f, \sigma^*, \omega^*) + \zeta^*(f)^T h_\omega(f, \sigma^*, \dot{\sigma}^*, \omega^*) = 0, \quad f \in I, \tag{8}$$

$$\zeta^*(f)^T h(f, \sigma^*, \dot{\sigma}^*, \omega^*) = 0, \quad f \in I, \tag{9}$$

$$\zeta^*(f) \geqq 0, \quad \phi^* \geqslant 0, \quad f \in I. \tag{10}$$

Proof. Taking into account Lemma 1 and Theorem 1, the proof is obvious. □

Theorem 3 (Sufficient efficiency conditions)**.** *Let* $(\sigma^*, \omega^*) \in Q$ *be a feasible pair of* (VCP)*. Suppose that there exist* $\phi^* \in \mathbb{R}^t, \phi^* > 0$ *and* $\zeta^* : I \to \mathbb{R}^n$ *a piecewise smooth function such that for all* $f \in I$ *the relations (7)–(10) are satisfied. If* $(\phi^{*T}\mu, \zeta^*(f)^T h)$ *is* $(\epsilon_0, \epsilon_1) - (c_0, c_1) - type - I$ *at* $(\sigma^*, \omega^*) \in Q \subset C^1(I, \mathbb{R}^n) \times C(I, \mathbb{R}^l)$*, with respect to* ν, ξ*, and* ϑ*, and* $\epsilon_0, \epsilon_1 \in \mathbb{R}$*,* $\epsilon_0 + \epsilon_1 \geqslant 0$*, with* $b_0(\sigma, \omega, \sigma^*, \omega^*) > 0$*, for all* $(\sigma, \omega) \in Q$*, then* (σ^*, ω^*) *is a properly efficient pair of* (VCP)*.*

Proof. Due to $(\phi^{*T}\mu, \zeta^*(f)^T h)$ is $(\epsilon_0, \epsilon_1) - (c_0, c_1) - type - I$ at $(\sigma^*, \omega^*) \in Q$, with respect to ν, ξ, and ϑ, and $\epsilon_0, \epsilon_1 \in \mathbb{R}$, $\epsilon_0 + \epsilon_1 \geqslant 0$, with $b_0(\sigma, \omega, \sigma^*, \omega^*) > 0$, for all $(\sigma, \omega) \in Q$, it follows

$$c_0(\sigma, \omega, \sigma^*, \omega^*)\left\{\int_a^b (\phi^*)^T \mu(f, \sigma, \omega)\,\mathrm{d}f - \int_a^b (\phi^*)^T \mu(f, \sigma^*, \omega^*)\,\mathrm{d}f\right\}$$
$$\geqslant \int_a^b \Big[\nu(f, \sigma, \omega, \sigma^*, \omega^*)^T (\phi^*)^T \mu_\sigma(f, \sigma^*, \omega^*)$$
$$+ \frac{d}{df}\nu(f, \sigma, \omega, \sigma^*, \omega^*)^T (\phi^*)^T \mu_{\dot{\sigma}}(f, \sigma^*, \omega^*)$$
$$+ \xi(f, \sigma, \omega, \sigma^*, \omega^*)^T (\phi^*)^T \mu_\omega(f, \sigma^*, \omega^*) + \epsilon_0 \|\vartheta(f, \sigma, \omega, \sigma^*, \omega^*)\|^2\Big]\,\mathrm{d}f$$

and

$$
\begin{aligned}
&-c_1(\sigma,\omega,\sigma^*,\omega^*)\int_a^b(\zeta^*)^T h(f,\sigma^*,\dot{\sigma}^*,\omega^*)\,\mathrm{d}f\\
&\quad\geqslant\int_a^b\Big[\nu(f,\sigma,\omega,\sigma^*,\omega^*)^T(\zeta^*)^T h_\sigma(f,\sigma^*,\dot{\sigma}^*,\omega^*)\\
&\quad+\frac{d}{df}\nu(f,\sigma,\omega,\sigma^*,\omega^*)^T(\zeta^*)^T h_{\dot{\sigma}}(f,\sigma^*,\dot{\sigma}^*,\omega^*)\\
&\quad+\xi(f,\sigma,\omega,\sigma^*,\omega^*)^T(\zeta^*)^T h_\omega(f,\sigma^*,\dot{\sigma}^*,\omega^*)+\epsilon_1\|\vartheta(f,\sigma,\omega,\sigma^*,\omega^*)\|^2\Big]\mathrm{d}f.
\end{aligned}
$$

Since $(\zeta^*)^T h(f,\sigma^*,\dot{\sigma}^*,\omega^*)=0$, we obtain

$$
\begin{aligned}
0&\geqslant\int_a^b\Big[\nu(f,\sigma,\omega,\sigma^*,\omega^*)^T(\zeta^*)^T h_\sigma(f,\sigma^*,\dot{\sigma}^*,\omega^*)\\
&+\frac{d}{df}\nu(f,\sigma,\omega,\sigma^*,\omega^*)^T(\zeta^*)^T h_{\dot{\sigma}}(f,\sigma^*,\dot{\sigma}^*,\omega^*)\\
&+\xi(f,\sigma,\omega,\sigma^*,\omega^*)^T(\zeta^*)^T h_\omega(f,\sigma^*,\dot{\sigma}^*,\omega^*)+\epsilon_1\|\vartheta(f,\sigma,\omega,\sigma^*,\omega^*)\|^2\Big]\mathrm{d}f.
\end{aligned}
$$

By addition the above relations, we obtain

$$
\begin{aligned}
&c_0(\sigma,\omega,\sigma^*,\omega^*)\left\{\int_a^b(\phi^*)^T\mu(f,\sigma,\omega)\,\mathrm{d}f-\int_a^b(\phi^*)^T\mu(f,\sigma^*,\omega^*)\,\mathrm{d}f\right\}\\
&\quad\geqslant\int_a^b\Big[\nu(f,\sigma,\omega,\sigma^*,\omega^*)^T\left[(\phi^*)^T\mu_\sigma(f,\sigma^*,\omega^*)+(\zeta^*)^T h_\sigma(f,\sigma^*,\dot{\sigma}^*,\omega^*)\right]\\
&\quad+\frac{d}{df}\nu(f,\sigma,\omega,\sigma^*,\omega^*)^T\left[(\phi^*)^T\mu_{\dot{\sigma}}(f,\sigma^*,\omega^*)+(\zeta^*)^T h_{\dot{\sigma}}(f,\sigma^*,\dot{\sigma}^*,\omega^*)\right]\\
&\quad+\xi(f,\sigma,\omega,\sigma^*,\omega^*)^T\left[(\phi^*)^T\mu_\omega(f,\sigma^*,\omega^*)+(\zeta^*)^T h_\omega(f,\sigma^*,\dot{\sigma}^*,\omega^*)\right]\\
&\quad+(\rho_0+\rho_1)\|\vartheta(f,\sigma,\omega,\sigma^*,\omega^*)\|^2\Big]\mathrm{d}f. \qquad (11)
\end{aligned}
$$

The relation in (11) becomes

$$c_0(\sigma,\omega,\sigma^*,\omega^*)\left\{\int_a^b(\phi^*)^T\mu(f,\sigma,\omega)\,\mathrm{d}f-\int_a^b(\phi^*)^T\mu(f,\sigma^*,\omega^*)\,\mathrm{d}f\right\}$$
$$\geqslant\int_a^b\Big[\nu(f,\sigma,\omega,\sigma^*,\omega^*)^T((\phi^*)^T\mu_\sigma(f,\sigma^*,\omega^*))+(\zeta^*)^T h_\sigma(f,\sigma^*,\dot\sigma^*,\omega^*)$$
$$-\frac{d}{df}((\phi^*)^T\mu_{\dot\sigma}(f,\sigma^*,\omega^*)+(\zeta^*)^T h_{\dot\sigma}(f,\sigma^*,\dot\sigma^*,\omega^*))$$
$$+\xi(f,\sigma,\omega,\sigma^*,\omega^*)^T\left((\phi^*)^T\mu_\omega(f,\sigma^*,\omega^*)+(\zeta^*)^T h_\omega(f,\sigma^*,\dot\sigma^*,\omega^*)\right)$$
$$+(\rho_0+\rho_1)\|\vartheta(f,\sigma,\omega,\sigma^*,\omega^*)\|^2\Big]\,\mathrm{d}f.$$

Considering the relations in (7), (8) and $\rho_0+\rho_1\geqslant 0$, we get

$$c_0(\sigma,\omega,\sigma^*,\omega^*)\left\{\int_a^b(\phi^*)^T\mu(f,\sigma,\omega)\,\mathrm{d}f-\int_a^b(\phi^*)^T\mu(f,\sigma^*,\omega^*)\,\mathrm{d}f\right\}\geqslant 0.$$

We know, by hypothesis, that $c_0(\sigma,\omega,\sigma^*,\omega^*)>0$, so, it follows

$$\int_a^b(\phi^*)^T\mu(f,\sigma,\omega)\,\mathrm{d}f-\int_a^b(\phi^*)^T\mu(f,\sigma^*,\omega^*)\,\mathrm{d}f\geqslant 0,$$

or, equivalent,

$$\int_a^b(\phi^*)^T\mu(f,\sigma,\omega)\,\mathrm{d}f\geqslant\int_a^b(\phi^*)^T\mu(f,\sigma^*,\omega^*)\,\mathrm{d}f,$$

which means that (σ^*,ω^*) is a minimizer for $\int_a^b(\phi^*)^T\mu(f,\sigma,\omega)\,\mathrm{d}f$ over Q, with $\phi^*>0$. Therefore, (σ^*,ω^*) becomes properly efficient pair of (VCP) (see Theorem 1 of Bector and Husain [5]). □

Theorem 4. *Consider that $(\sigma^*,\omega^*)\in Q$ is a feasible pair of (VCP) and there exist $\phi^*\in\mathbb{R}^t$, $\phi^*>0$ and $\zeta^*: I\to\mathbb{R}^n$ such that, for every $f\in I$, the relations (7)–(10) are satisfied. If $({\phi^*}^T\mu,\zeta^*(f)^T h)$ is $semi-strictly-(\epsilon_0,\epsilon_1)-(c_0,c_1)-type-I$ at (σ^*,ω^*), with respect to ν, ξ, and ϑ, and ϵ_0, $\epsilon_1\in\mathbb{R}$, $\epsilon_0+\epsilon_1\geqslant 0$,* *with* $b_0(\sigma,\omega,\sigma^*,\omega^*) > 0$*, for all* $(\sigma,\omega) \in Q$*, then* (σ^*,ω^*) *is an efficient pair of* (VCP).

Proof. Let us suppose that (σ^*,ω^*) is not an efficient pair of (VCP). Thus, there exist $(\sigma,\omega) \in Q$ and, for at least an index $s = 1, 2, ..., t$, we have

$$\int_a^b f_j(f,\sigma,\omega)\,\mathrm{d}f \leqslant \int_a^b f_j(f,\sigma^*,\omega^*)\,\mathrm{d}f$$

and

$$\int_a^b f_s(f,\sigma,\omega)\,\mathrm{d}f < \int_a^b f_s(f,\sigma^*,\omega^*)\,\mathrm{d}f.$$

But, since $\phi^* > 0$ and $c_0(\sigma,\omega,\sigma^*,\omega^*) > 0$, considering (26), we get

$$c_0(\sigma,\omega,\sigma^*,\omega^*)\left\{\int_a^b (\phi^*)^T\mu(f,\sigma,\omega)\,\mathrm{d}f - \int_a^b (\phi^*)^T\mu(f,\sigma^*,\omega^*)\,\mathrm{d}f\right\} \leqslant 0.$$

We take in consideration that $\zeta^*(f)^T h\,(f,\sigma^*,\dot{\sigma}^*,\omega^*) = 0$, $\zeta^*(f) \geqslant 0$, $f \in I$, and obtain

$$-c_1(\sigma,\omega,\sigma^*,\omega^*)\int_a^b (\zeta^*)^T h(f,\sigma^*,\dot{\sigma}^*,\omega^*)\,\mathrm{d}f = 0.$$

As $(\phi^{*T}\mu, \zeta^*(f)^T h)$ is semi-strictly - $(\epsilon_0,\epsilon_1) - (c_0,c_1) - type - I$ at (σ^*,ω^*), with respect to ν, ξ, and ϑ, we have

$$\begin{aligned}
0 \geqslant\ & c_0(\sigma,\omega,\sigma^*,\omega^*)\left\{\int_a^b (\phi^*)^T\mu(f,\sigma,\omega)\,\mathrm{d}f - \int_a^b (\phi^*)^T\mu(f,\sigma^*,\omega^*)\,\mathrm{d}f\right\} \\
> & \int_a^b \Big[\nu(f,\sigma,\omega,\sigma^*,\omega^*)^T((\phi^*)^T\mu_\sigma(f,\sigma^*,\omega^*)) \\
& + \frac{d}{df}\nu(f,\sigma,\omega,\sigma^*,\omega^*)^T(\phi^*)^T\mu_{\dot{\sigma}}(f,\sigma^*,\omega^*) \\
& + \xi(f,\sigma,\omega,\sigma^*,\omega^*)^T(\phi^*)^T\mu_\omega(f,\sigma^*,\omega^*) + \rho_0\|\vartheta(f,\sigma,\omega,\sigma^*,\omega^*)\|^2\Big]\,\mathrm{d}f
\end{aligned}$$

and

$$
\begin{aligned}
0 &= -c_1(\sigma, \omega, \sigma^*, \omega^*) \int_a^b (\zeta^*)^T h(f, \sigma^*, \dot{\sigma}^*, \omega^*) \mathrm{d} f \\
&> \int_a^b \Big[\nu(f, \sigma, \omega, \sigma^*, \omega^*)^T (\zeta^*)^T h_x(f, \sigma^*, \dot{\sigma}^*, \omega^*) \\
&\quad + \frac{d}{df} \nu(f, \sigma, \omega, \sigma^*, \omega^*)^T (\zeta^*)^T h_{\dot{\sigma}}(f, \sigma^*, \dot{\sigma}^*, \omega^*) \\
&\quad + \xi(f, \sigma, \omega, \sigma^*, \omega^*)^T (\zeta^*)^T h_\omega(f, \sigma^*, \dot{\sigma}^*, \omega^*) + \rho_1 \|\vartheta(f, \sigma, \omega, \sigma^*, \omega^*)\|^2 \Big] \mathrm{d} f.
\end{aligned}
$$

By joining the relations given above, we obtain

$$
\begin{aligned}
&\int_a^b \Big[\nu(f, \sigma, \omega, \sigma^*, \omega^*)^T ((\phi^*)^T \mu_\sigma(f, \sigma^*, \omega^*)) + (\zeta^*)^T h_x(f, \sigma^*, \dot{\sigma}^*, \omega^*) \\
&\quad + \frac{d}{df} \nu(f, \sigma, \omega, \sigma^*, \omega^*)^T ((\phi^*)^T \mu_{\dot{\sigma}}(f, \sigma^*, \omega^*) + (\zeta^*)^T h_{\dot{\sigma}}(f, \sigma^*, \dot{\sigma}^*, \omega^*)) \\
&\quad + \xi(f, \sigma, \omega, \sigma^*, \omega^*)^T ((\phi^*)^T \mu_\omega(f, \sigma^*, \omega^*) + (\zeta^*)^T h_\omega(f, \sigma^*, \dot{\sigma}^*, \omega^*)) \\
&\quad (\rho_0 + \rho_1) \|\vartheta(f, \sigma, \omega, \sigma^*, \omega^*)\|^2 \Big] \mathrm{d} f < 0,
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
&\int_a^b \Big[\nu(f, \sigma, \omega, \sigma^*, \omega^*)^T ((\phi^*)^T \mu_\sigma(f, \sigma^*, \omega^*) + (\zeta^*)^T h_\sigma(f, \sigma^*, \dot{\sigma}^*, \omega^*) \\
&\quad - \frac{d}{df} ((\phi^*)^T \mu_{\dot{\sigma}}(f, \sigma^*, \omega^*) + (\zeta^*)^T h_{\dot{\sigma}}(f, \sigma^*, \dot{\sigma}^*, \omega^*))) \\
&\quad + \xi(f, \sigma, \omega, \sigma^*, \omega^*)^T ((\phi^*)^T \mu_\omega(f, \sigma^*, \omega^*) + (\zeta^*)^T h_\omega(f, \sigma^*, \dot{\sigma}^*, \omega^*)) \\
&\quad (\rho_0 + \rho_1) \|\vartheta(f, \sigma, \omega, \sigma^*, \omega^*)\|^2 \Big] \mathrm{d} f < 0,
\end{aligned}
$$

which is contrary to Theorem 2 and $\rho_0 + \rho_1 \geqslant 0$. This means that our supposition that (σ^*, ω^*) is not an efficient pair of (VCP) is false. So, we obtain that (σ^*, ω^*) is an efficient pair of (VCP). □

Theorem 5. *Consider $(\sigma^*, \omega^*) \in Q$ is a feasible pair of (VCP), and there exist $\phi^* \in \mathbb{R}^t, \phi^* > 0$ and $\zeta^* : I \to \mathbb{R}^n$ such that, for every $f \in I$, the relations in (7) and (8) are satisfied. If $(\phi^{*T} \mu, \zeta^*(f)^T h)$ is strongly - pseudo - quasi -*

$(\epsilon_0,\epsilon_1)-(c_0,c_1)-type-I$ at (σ^*,ω^*), with respect to functions ν, ξ, and ϑ, and $\epsilon_0,\epsilon_1\in\mathbb{R}$, $\epsilon_0+\epsilon_1\geqslant 0$, with $b_0(\sigma,\omega,\sigma^*,\omega^*)>0$, for all $(\sigma,\omega)\in Q$, then (σ^*,ω^*) is a properly efficient pair of (VCP).

Proof. Using relation given in (9), we obtain

$$-c_1(\sigma,\omega,\sigma^*,\omega^*)\int_a^b(\zeta^*)^T h(f,\sigma^*,\dot{\sigma}^*,\omega^*)\,\mathrm{d}f=0.$$

From hypothesis, we know that $({\phi^*}^T\mu,\zeta^*(f)^T h)$ is strongly - pseudo - quasi - $(\epsilon_0,\epsilon_1)-(c_0,c_1)-type-I$ at (σ^*,ω^*), with respect to functions ν,ξ, and ϑ. Thus, we get

$$\begin{aligned}&\int_a^b\Big[\nu(f,\sigma,\omega,\sigma^*,\omega^*)^T(\zeta^*)^T h_\sigma(f,\sigma^*,\dot{\sigma}^*,\omega^*)\\&\quad+\frac{d}{df}\nu(f,\sigma,\omega,\sigma^*,\omega^*)^T((\zeta^*)^T h_{\dot{\sigma}}(f,\sigma^*,\dot{\sigma}^*,\omega^*))\\&\quad+\xi(f,\sigma,\omega,\sigma^*,\omega^*)^T(\zeta^*)^T h_\omega(f,\sigma^*,\dot{\sigma}^*,\omega^*)+\rho_1\|\vartheta(f,\sigma,\omega,\sigma^*,\omega^*)\|^2\Big]\mathrm{d}f\\&\leqslant 0,\end{aligned}$$

or, equivalently,

$$\begin{aligned}&\int_a^b\Big[\nu(f,\sigma,\omega,\sigma^*,\omega^*)^T((\zeta^*)^T h_\sigma(f,\sigma^*,\dot{\sigma}^*,\omega^*)-\frac{d}{df}(\zeta^*)^T h_{\dot{\sigma}}(f,\sigma^*,\dot{\sigma}^*,\omega^*))\\&\quad+\xi(f,\sigma,\omega,\sigma^*,\omega^*)^T(\zeta^*)^T h_\omega(f,\sigma^*,\dot{\sigma}^*,\omega^*)+\rho_1\|\vartheta(f,\sigma,\omega,\sigma^*,\omega^*)\|^2\Big]\mathrm{d}f\\&\leqslant 0.\end{aligned}$$

Using relations (7) and (8), it follows

$$\begin{aligned}&\int_a^b\Big[\nu(f,\sigma,\omega,\sigma^*,\omega^*)^T((\phi^*)^T\mu_\sigma(f,\sigma^*,\omega^*)-\frac{d}{df}(\phi^*)^T\mu_{\dot{\sigma}}(f,\sigma^*,\omega^*))\\&\quad+\xi(f,\sigma,\omega,\sigma^*,\omega^*)^T(\phi^*)^T\mu_\omega(f,\sigma^*,\omega^*)-\rho_1\|\vartheta(f,\sigma,\omega,\sigma^*,\omega^*)\|^2\Big]\mathrm{d}f\\&\geqslant 0,\end{aligned}$$

or, equivalently,

$$\int_a^b \Big[\nu(f,\sigma,\omega,\sigma^*,\omega^*)^T(\phi^*)^T\mu_\sigma(f,\sigma^*,\omega^*) + \frac{d}{df}\nu(f,\sigma,\omega,\sigma^*,\omega^*)^T(\phi^*)^T\mu_{\dot\sigma}(f,\sigma^*,\omega^*) + \xi(f,\sigma,\omega,\sigma^*,\omega^*)^T(\phi^*)^T\mu_\omega(f,\sigma^*,\omega^*) - \rho_1\|\vartheta(f,\sigma,\omega,\sigma^*,\omega^*)\|^2\Big]\mathrm{d}f \geqslant 0,$$

for all $(\sigma,\omega)\in Q$. But, since $\rho_0+\rho_1\geqslant 0$, we get

$$\int_a^b \Big[\nu(f,\sigma,\omega,\sigma^*,\omega^*)^T(\phi^*)^T\mu_\sigma(f,\sigma^*,\omega^*) + \frac{d}{df}\nu(f,\sigma,\omega,\sigma^*,\omega^*)^T(\phi^*)^T\mu_{\dot\sigma}(f,\sigma^*,\omega^*) + \xi(f,\sigma,\omega,\sigma^*,\omega^*)^T(\phi^*)^T\mu_\omega(f,\sigma^*,\omega^*) + \rho_0\|\vartheta(f,\sigma,\omega,\sigma^*,\omega^*)\|^2\Big]\mathrm{d}f \geqslant 0.$$

Now, we corroborate the last equation with the fact that $({\phi^*}^T\mu,\zeta^*(f)^Th)$ is strongly - pseudo - quasi - $(\epsilon_0,\epsilon_1)-(c_0,c_1)-type-I$ at (σ^*,ω^*), with respect to functions ν, ξ, and ϑ, and $\epsilon_0,\epsilon_1\in\mathbb{R}$, $\epsilon_0+\epsilon_1\geqslant 0$, with $b_0(\sigma,\omega,\sigma^*,\omega^*)>0$, for all $(\sigma,\omega)\in Q$, and obtain

$$c_0(\sigma,\omega,\sigma^*,\omega^*)\left\{\int_a^b(\phi^*)^T\mu(f,\sigma,\omega)\,\mathrm{d}f - \int_a^b(\phi^*)^T\text{‘}\mu(f,\sigma^*,\omega^*)\,\mathrm{d}f\right\}\geqslant 0.$$

But we know that $c_0(\sigma,\omega,\sigma^*,\omega^*>0$. Therefore, it results

$$\int_a^b(\phi^*)^T\mu(f,\sigma,\omega)\,\mathrm{d}f \geqslant \int_a^b(\phi^*)^T\mu(f,\sigma^*,\omega^*)\,\mathrm{d}f,$$

for all $(\sigma,\omega)\in Q$, that is, the pair (σ^*,ω^*) minimizes $\int_a^b(\phi^*)^T\mu(f,\sigma,\omega)\,\mathrm{d}f$ over Q, with $\phi^*>0$, that is, (σ^*,ω^*) is a properly efficient pair of (VCP). □

Theorem 6. *Consider $(\sigma^*,\omega^*)\in Q$ is a feasible pair of (VCP), and there exist $\phi^*\in\mathbb{R}^t,\phi^*>0$ and $\zeta^*:I\to\mathbb{R}^n$ such that, for every $f\in I$, the relations in (7)–(10) are verified. If $({\phi^*}^T\mu,\zeta^*(f)^Th)$ is strictly - pseudo - quasi - $(\epsilon_0,\epsilon_1)-$*

$(c_0, c_1)-type-I$ at (σ^*, ω^*), with respect to functions ν, ξ, and ϑ and $\epsilon_0, \epsilon_1 \in \mathbb{R}$, $\epsilon_0 + \epsilon_1 \geqslant 0$, with $c_1(\sigma, \omega, \sigma^*, \omega^*) > 0$, for all $(\sigma, \omega) \in Q$, then (σ^*, ω^*) is an efficient pair of (VCP).

Proof. We suppose, contrary to the conclusion, that (σ^*, ω^*) is not an efficient pair of (VCP). This means there exist $(\sigma, \omega) \in Q$ and, for at least an index $s = 1, ..., t$, we have

$$\int_a^b f_j(f, \sigma, \omega) \mathrm{d}f \leqslant \int_a^b f_j(f, \sigma^*, \omega^*) \mathrm{d}f$$

and

$$\int_a^b f_s(f, \sigma, \omega) \mathrm{d}f < \int_a^b f_s(f, \sigma^*, \omega^*) \mathrm{d}f.$$

Since $\phi^* > 0$ and $c_0(\sigma, \omega, \sigma^*, \omega^*) > 0$, we obtain

$$c_0(\sigma, \omega, \sigma^*, \omega^*) \left\{ \int_a^b (\phi^*)^T \mu(f, \sigma, \omega) \mathrm{d}f - \int_a^b (\phi^*)^T \mu(f, \sigma^*, \omega^*) \mathrm{d}f \right\} \leqslant 0.$$

But, by hypothesis, $(\phi^{*T}\mu, \zeta^*(f)^T h)$ is strictly - pseudo - quasi - $(\epsilon_0, \epsilon_1) - (c_0, c_1) - type - I$ at (σ^*, ω^*), with respect to functions ν, ξ, and ϑ. Thus, we obtain:

$$\begin{aligned}
&\int_a^b \Big[\nu(f, \sigma, \omega, \sigma^*, \omega^*)^T (\phi^*)^T \mu_\sigma(f, \sigma^*, \omega^*) \\
&\quad + \frac{d}{df} \nu(f, \sigma, \omega, \sigma^*, \omega^*)^T (\phi^*)^T \mu_{\dot{\sigma}}(f, \sigma^*, \omega^*) \\
&\quad + \xi(f, \sigma, \omega, \sigma^*, \omega^*)^T (\phi^*)^T \mu_\omega(f, \sigma^*, \omega^*) + \rho_0 \|\vartheta(f, \sigma, \omega, \sigma^*, \omega^*)\|^2 \Big] \mathrm{d}f \leqslant 0,
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
&\int_a^b \Big[\nu(f, \sigma, \omega, \sigma^*, \omega^*)^T ((\phi^*)^T \mu_\sigma(f, \sigma^*, \omega^*) - \frac{d}{df} (\phi^*)^T \mu_{\dot{\sigma}}(f, \sigma^*, \omega^*)) \\
&\quad + \xi(f, \sigma, \omega, \sigma^*, \omega^*)^T (\phi^*)^T \mu_\omega(f, \sigma^*, \omega^*) + \rho_0 \|\vartheta(f, \sigma, \omega, \sigma^*, \omega^*)\|^2 \Big] \mathrm{d}f \leqslant 0.
\end{aligned}$$

But we have

$$(\phi^*)^T \mu_\sigma(f,\sigma^*,\omega^*) + (\zeta^*(f))^T h_x(f,\sigma^*,\dot\sigma^*,\omega^*)$$
$$= \frac{d}{df}\Big((\phi^*)^T \mu_{\dot\sigma}(f,\sigma^*,\omega^*)\Big) + (\zeta^*(f))^T h_{\dot\sigma}(f,\sigma^*,\dot\sigma^*,\omega^*),$$
$$(\phi^*)^T \mu_\omega(f,\sigma^*,\omega^*) + (\zeta^*(f))^T h_\omega(f,\sigma^*,\dot\sigma^*,\omega^*) = 0.$$

In consequence, we get

$$\int_a^b \Big[\nu(f,\sigma,\omega,\sigma^*,\omega^*)^T((\zeta^*(f))^T h_\sigma(f,\sigma^*,\dot\sigma^*,\omega^*)$$
$$- \frac{d}{df}\Big((\zeta^*(f))^T h_{\dot\sigma}(f,\sigma^*,\dot\sigma^*,\omega^*)\Big)$$
$$- \xi(f,\sigma,\omega,\sigma^*,\omega^*)^T(\phi^*)^T\mu_\omega(f,\sigma^*,\omega^*) - \rho_0\|\vartheta(f,\sigma,\omega,\sigma^*,\omega^*)\|^2\Big]\,\mathrm{d}f \geqslant 0,$$

equivalent with

$$\int_a^b \Big[\nu(f,\sigma,\omega,\sigma^*,\omega^*)^T((\zeta^*(f))^T h_\sigma(f,\sigma^*,\dot\sigma^*,\omega^*)$$
$$- \frac{d}{df}\Big((\zeta^*(f))^T h_{\dot\sigma}(f,\sigma^*,\dot\sigma^*,\omega^*)\Big)$$
$$+ \xi(f,\sigma,\omega,\sigma^*,\omega^*)^T(\zeta^*(f))^T h_\omega(f,\sigma^*,\dot\sigma^*,\omega^*)\Big]\,\mathrm{d}f$$
$$\geqslant \rho_0\|\vartheta(f,\sigma,\omega,\sigma^*,\omega^*)\|^2.$$

We use the condition $\rho_0+\rho_1 \geqslant 0$, and obtain

$$\int_a^b \Big[\nu(f,\sigma,\omega,\sigma^*,\omega^*)^T((\zeta^*(f))^T h_\sigma(f,\sigma^*,\dot\sigma^*,\omega^*)$$
$$- \frac{d}{df}\Big((\zeta^*(f))^T h_{\dot\sigma}(f,\sigma^*,\dot\sigma^*,\omega^*)\Big)$$
$$+ \xi(f,\sigma,\omega,\sigma^*,\omega^*)^T(\zeta^*(f))^T h_\omega(f,\sigma^*,\dot\sigma^*,\omega^*)$$
$$+ \rho_1\|\vartheta(f,\sigma,\omega,\sigma^*,\omega^*)\|^2\Big]\,\mathrm{d}f \geqslant 0,$$

that is,

$$\int_a^b \Big[\nu(f,\sigma,\omega,\sigma^*,\omega^*)^T(\zeta^*(f)^T h_\sigma(f,\sigma^*,\dot{\sigma}^*,\omega^*)$$
$$+ \frac{d}{df}\nu(f,\sigma,\omega,\sigma^*,\omega^*)^T((\zeta^*(f))^T h_{\dot{\sigma}}(f,\sigma^*,\dot{\sigma}^*,\omega^*)$$
$$+ \xi(f,\sigma,\omega,\sigma^*,\omega^*)^T(\zeta^*(f))^T h_\omega(f,\sigma^*,\dot{\sigma}^*,\omega^*)$$
$$+ \rho_1\|\vartheta(f,\sigma,\omega,\sigma^*,\omega^*)\|^2\Big]\mathrm{d}f \geqslant 0,$$

which, due to our hypotheses, implies

$$-c_1(\sigma,\omega,\sigma^*,\omega^*)\int_a^b(\zeta^*(f))^T h(f,\sigma^*,\dot{\sigma}^*,\omega^*)\mathrm{d}f > 0.$$

Since $c_1(\sigma,\omega,\sigma^*,\omega^*) > 0$, we get that $-\int_a^b(\zeta^*(f))^T h(f,\sigma^*,\dot{\sigma}^*,\omega^*)\mathrm{d}f > 0$, which is contradiction to our hypothesis, namely, $\zeta^*(f))^T h(f,\sigma^*,\dot{\sigma}^*,\omega^*) = 0$. We find that our initial supposition is false, so, the pair (σ^*,ω^*) is an efficient pair of (VCP). □

4. Reciprocal results associated with (VCP)

Consider the following multi-objective variational control problem:

$$(VCD)\quad \max_{(z,\pi)}\left\{\int_a^b \mu(f,z,\pi)\mathrm{d}f := \left(\int_a^b \mu_1(f,z,\pi)\mathrm{d}f,\ldots,\int_a^b \mu_t(f,z,\pi)\mathrm{d}f\right)\right\}$$

subject to

$$z(a)=\alpha,\quad z(b)=\beta,$$

$$\phi^T\mu_\sigma(f,z,\pi)+\zeta(f)^T h_\sigma(f,z,\dot{z},\pi)$$
$$= \frac{d}{df}\Big(\phi^T\mu_{\dot{\sigma}}(f,z,\pi)+\zeta(f)^T h_{\dot{\sigma}}(f,z,\dot{z},\pi)\Big),\qquad f\in I, \tag{12}$$

$$\phi^T\mu_\omega(f,z,\pi)+\zeta(f)^T h_\omega(f,z,\dot{z},\pi)=0,\qquad f\in I, \tag{13}$$

$$\zeta(f)^T h(f,z,\dot{z},\pi)\geqslant 0,\qquad f\in I,$$
$$\zeta(f)\geqq 0,\qquad f\in I,$$

$$\phi \in \mathbb{R}^t, \quad \phi \geqslant 0, \quad \phi^T e = 1, \quad e = (1, ..., 1) \in \mathbb{R}^t.$$

In the following, we will establish various duality results between (VCP) and (VCD) under the generalized $(\epsilon_0, \epsilon_1) - (c_0, c_1) - type - I$ hypotheses of the functionals involved.

Theorem 7 (Weak duality). *Let $(\sigma, \omega) \in Q$ be a feasible pair of (VCP) and let (z, π, ϕ, ζ) be a feasible pair of (VCD). We suppose that one of the following conditions is verified:*

(i) *$(\phi^T \mu,\ \zeta(f)^T h)$ is $semi-strictly-(\epsilon_0,\ \epsilon_1)-(c_0, c_1)-type-I$ at (z, π) with respect to ν, ξ and ϑ, and $\epsilon_0, \epsilon_1 \in \mathbb{R}$, $\epsilon_0 + \epsilon_1 \geqslant 0$;*

(ii) *$\phi > 0$ and $(\phi^T \mu, \zeta(f)^T h)$ is $strongly - pseudo - quasi - (\epsilon_0,\ \epsilon_1) - (c_0, c_1) - type - I$ at (z, π) with respect to ν, ξ and ϑ, and ϵ_0, $\epsilon_1 \in \mathbb{R}$, $\epsilon_0 + \epsilon_1 \geqslant 0$, with $c_1(\sigma, \omega, z, \pi) > 0$, for all $(\sigma, \omega) \in Q$;*

(iii) *$(\phi^T \mu, \zeta(f)^T h)$ is $strictly-pseudo-quasi-(\epsilon_0, \epsilon_1)-(c_0, c_1)-type-I$ at (z, π) with respect to ν, ξ and ϑ, and ϵ_0, $\epsilon_1 \in \mathbb{R}$, $\epsilon_0 + \epsilon_1 \geqslant 0$, with $c_0(\sigma, \omega, z, \pi) > 0$, for all $(\sigma, \omega) \in Q$.*

Then, we have $\displaystyle\int_a^b \mu(f, \sigma, \omega)\,\mathrm{d}f \nleq \int_a^b \mu(f, z, \pi)\,\mathrm{d}f.$

Proof. (i) We assume on the contrary, that we have

$$\int_a^b \mu(f, \sigma, \omega)\,\mathrm{d}f \leqslant \int_a^b \mu(f, z, \pi)\,\mathrm{d}f,$$

that is, there exist an index r, $1 \leqslant r \leqslant t$, such that, for every $j \neq r, j = 1, ..., t$, we have

$$\int_a^b f_j(f, \sigma, \omega)\,\mathrm{d}f \leqslant \int_a^b \mu_j(f, z, \pi)\,\mathrm{d}f$$

and

$$\int_a^b f_r(f, \sigma, \omega)\,\mathrm{d}f < \int_a^b \mu_r(f, z, \pi)\,\mathrm{d}f.$$

Due to $\phi > 0$ and $b_0(\sigma, \omega, z, \pi) > 0$, the previous relation performs to

$$b_0(\sigma, \omega, z, \pi) \int_a^b \phi^T \mu(f, \sigma, \omega)\,\mathrm{d}f \leqslant b_0(\sigma, \omega, z, \pi) \int_a^b \phi^T \mu(f, z, \pi)\,\mathrm{d}f,$$

equivalent to

$$b_0(\sigma,\omega,z,\pi)\left(\int_a^b \phi^T\mu(f,\sigma,\omega)\,\mathrm{d}f-\int_a^b \phi^T\mu(f,z,\pi)\,\mathrm{d}f\right)\leqslant 0.$$

We proceed to a parameter change as $(z,\pi)\hookrightarrow(\sigma^*,\omega^*)$ and $\phi\hookrightarrow\phi^*$, and the proof follows similar as in Theorem 4.

(ii) The inequality $\zeta(f)^T h(f,z,\dot{z},\pi)\geqslant 0,\ f\in I$, involves

$$-\int_a^b \zeta(f)^T h(f,z,\dot{z},\pi)\,\mathrm{d}f\leqslant 0.$$

Multiplying this relation with $b_1(\sigma,\omega,z,\pi)>0$, we obtain

$$-b_1(\sigma,\omega,z,\pi)\int_a^b \zeta(f)^T h(f,z,\dot{z},\pi)\,\mathrm{d}f\leqslant 0,$$

involving, by hypothesis,

$$\begin{aligned}\int_a^b \Big[&\nu(f,\sigma,\omega,z,\pi)^T\zeta(f)^T h_\sigma(f,z,\dot{z},\pi)\\ &+\frac{d}{df}\nu(f,\sigma,\omega,z,\pi)^T\zeta(f)^T h_{\dot{\sigma}}(f,z,\dot{z},\pi)\\ &+\xi(f,\sigma,\omega,z,\pi)^T\zeta(f)^T h_\omega(f,z,\dot{z},\pi)+\epsilon_1\|\vartheta(f,\sigma,\omega,z,\pi)\|^2\Big]\,\mathrm{d}f\leqslant 0.\end{aligned}$$

Now, we make a parameter change as $(z,\pi)\hookrightarrow(\sigma^*,\omega^*)$ and $\zeta\hookrightarrow\zeta^*$, and the proof follows similar to Theorem 5.

(iii) The proof of this part is omitted (see Theorem 6). □

Theorem 8 (Strong duality)**.** *Consider $(\sigma^*,\omega^*)\in Q$ is a properly efficient pair of (VCP), which is assumed to be a normal solution for $(P_q^*), q=1,\dots,t$. Then there exist $\phi^*\in\mathbb{R}^t,\phi^*\geqslant 0$ and $\zeta^*:I\to\mathbb{R}^n$ a piecewise smooth function such that $(\sigma^*,\omega^*,\phi^*,\zeta^*)$ is a feasible pair for (VCD). In addition, if for each feasible solution (z,π,ϕ,ζ) of (VCD), any of the conditions (i)–(iii) of the previous theorem is verified, then $(\sigma^*,\omega^*,\phi^*,\zeta^*)$ is a properly efficient solution of (VCD).*

Proof. By using Theorem 2 and Theorem 7, the proof follows immediately. □

Theorem 9 (Strict converse duality). *Let $(\sigma^*, \omega^*) \in Q$ be a feasible pair of (VCP) and let $(z^*, \pi^*, \phi^*, \zeta^*)$ be a feasible solution of (VCD) such that*

$$\int_a^b \phi^{*T} \mu(f, \sigma^*, \omega^*) \mathrm{d} f = \int_a^b \phi^{*T} \mu(f, z^*, \pi^*) \mathrm{d} f.$$

Let $(\phi^{*T} \mu, \zeta^*(f)^T h)$ be strictly-pseudo-quasi-$(\epsilon_0, \epsilon_1) - (c_0, c_1) - type - I$ in (z^*, π^*) with respect to ν, ξ and ϑ, and ρ_0, $\rho_1 \in \mathbb{R}$ with $\rho_0 + \rho_1 \geqslant 0$, $b_0(\sigma^*, \omega^* z^*, \pi^*) > 0$, $b_1(\sigma^*, \omega^*, z^*, \pi^*) > 0$, for all $(\sigma^*, \omega^*) \in Q$. Then (σ^*, ω^*) is a properly efficient pair of (VCP).

Proof. We suppose that $(\sigma^*, \omega^*) \neq (z^*, \pi^*)$. Let $(z^*, \pi^*, \phi^*, \zeta^*)$ to be feasible solution for (VCD). Then, we get

$$-\int_a^b \zeta^*(f)^T h(f, z^*, \dot{z}^*, \pi^*) \mathrm{d} f \leqslant 0.vadjust$$

Since, $b_1(\sigma^*, \omega^*, z^*, \pi^*) > 0$, we obtain

$$-b_1(\sigma^*, \omega^*, z^*, \pi^*) \int_a^b \zeta^*(f)^T h(f, z^*, \dot{z}^*, \pi^*) \leqslant 0.vadjust$$

As $(\phi^{*T} f, \zeta^*(f)^T h)$ is strictly-pseudo-quasi-(ϵ_0, ϵ_1)-(c_0, c_1)-$type$-I in (z^*, π^*), the previous relation implies

$$\begin{aligned} &\int_a^b \Big[\nu(f, \sigma^*, \omega^*, z^*, \pi^*)^T \zeta^*(f)^T h_\sigma(f, z^*, \dot{z}^*, \pi^*) \\ &\quad + \frac{d}{df} \nu(f, \sigma^*, \omega^*, z^*, \pi^*)^T \zeta^*(f)^T h_{\dot{\sigma}}(f, z^*, \dot{z}^*, \pi^*) \\ &\quad + \xi(f, \sigma^*, \omega^* \$, z^*, \pi^*)^T h_\omega(f, z^*, \dot{z}^*, \pi^*) + \epsilon_1 \| \vartheta(f, \sigma^*, \omega^*, z^*, \pi^*) \|^2 \Big] \mathrm{d} f \leqslant 0, \end{aligned}$$

or, equivalently,

$$\begin{aligned} &\int_a^b \Big\{ \nu(f, \sigma^*, \omega^*, z^*, \pi^*)^T \Big[\zeta^*(f)^T h_\sigma(f, z^*, \dot{z}^*, \pi^*) - \frac{d}{df} \zeta^*(f)^T h_{\dot{\sigma}}(f, z^*, \dot{z}^*, \pi^*) \Big] \\ &\quad + \xi(f, \sigma^*, \omega^*, z^*, \pi^*)^T \zeta^*(f)^T h_\omega(f, z^*, \dot{z}^*, \pi^*) + \epsilon_1 \| \vartheta(f, \sigma^*, \omega^*, z^*, \pi^*) \|^2 \Big\} \mathrm{d} f \\ &\leqslant 0. \end{aligned}$$

We use (12) and (13) and obtain

$$\int_a^b \Big\{ \nu(f,\sigma^*,\omega^*,z^*,\pi^*)^T \Big[\phi^{*T} \mu_\sigma(f,z^*,\pi^*) - \frac{d}{df} \phi^{*T} \mu_{\dot\sigma}(f,z^*,\pi^*) \Big]$$
$$+ \xi(f,\sigma^*,\omega^*,z^*,\pi^*)^T \phi^{*T} \mu_\omega(f,z^*,\pi^*) - \epsilon_1 \|\vartheta(f,\sigma^*,\omega^*,z^*,\pi^*)\|^2 \Big\} \mathrm{d}f \geqslant 0.$$

As $\rho_0 + \rho_1 \geqslant 0$, we obtain

$$\int_a^b \Big\{ \nu(f,\sigma^*,\omega^*,z^*,\pi^*)^T \Big[\phi^{*T} \mu_\sigma(f,z^*,\pi^*) - \frac{d}{df} \phi^{*T} \mu_{\dot\sigma}(f,z^*,\pi^*) \Big]$$
$$+ \xi(f,\sigma^*,\omega^*,z^*,\pi^*)^T \phi^{*T} \mu_\omega(f,z^*,\pi^*) + \epsilon_0 \|\vartheta(f,\sigma^*,\omega^*,z^*,\pi^*)\|^2 \Big\} \mathrm{d}f \geqslant 0,$$

or, equivalently,

$$\int_a^b \Big[\nu(f,\sigma^*,\omega^*,z^*,\pi^*)^T \phi^{*T} \mu_\sigma(f,z^*,\pi^*)$$
$$+ \frac{d}{df} \nu(f,\sigma^*,\omega^*,z^*,\pi^*)^T \phi^{*T} \mu_{\dot\sigma}(f,z^*,\pi^*)$$
$$+ \xi(f,\sigma^*,\omega^*,z^*,\pi^*)^T \phi^{*T} \mu_\omega(f,z^*,\pi^*) + \epsilon_0 \|\vartheta(f,\sigma^*,\omega^*,z^*,\pi^*)\|^2 \Big] \mathrm{d}f \geqslant 0.$$

Using the hypothesis formulated in theorem, we obtain

$$b_0(\sigma^*,\omega^*,z^*,\pi^*) \left[\int_a^b \phi^{*T} \mu(f,\sigma^*,\omega^*) \mathrm{d}f - \int_a^b \phi^{*T} \mu(f,z^*,\pi^*) \right] \mathrm{d}f > 0.$$

Because $b_0(\sigma^*,\omega^*,z^*,\pi^*) > 0$, we obtain

$$\int_a^b \phi^{*T} \mu(f,\sigma^*,\omega^*) \mathrm{d}f > \int_a^b \phi^{*T} \mu(f,z^*,\pi^*) \mathrm{d}f,$$

which is contradictory with our hypotheses. So, our supposition is false, and the proof is complete. □

5. Conclusions

Our outcomes provided a flexible and powerful theoretical foundation for the analysis and solution of a given class of multi-objective variational control problems. Their application to theoretical development and real-world decision-making offers a rich approach for continued exploration. This study established (necessary and sufficient) efficiency criteria and a duality framework for the considered multi-objective variational control problem (VCP) and its dual (VCD) under new (generalized) $(\epsilon_0, \epsilon_1) - (c_0, c_1) - type - I$ hypotheses. Through weak, strong and strict converse duality theorems, we provided necessary and sufficient conditions for proper efficiency, extending classical results to more general settings. These results are applicable in a variety of multi-objective optimization problems, such as robotics, aerospace logistic, and energy systems, where trade-offs between conflicting objectives like cost, time, and safety, which should be considered. Future research may focus on developing computational algorithms, relaxing smoothness assumptions, and applying the theory to real world problems. Further exploration into the integration of machine learning techniques also holds promise for advancing theoretical and practical aspects of multi-objective control.

References

[1] S. Adly and T.K. Bui: KKT optimality conditions for multiobjective optimal control problems with endpoint and mixed constraints: Application to sustainable energy management. *arXiv*, (2024), DOI: 10.48550/arXiv.2411.09719

[2] B. Aghezzaf and K. Khazafi: Sufficient optimality conditions and duality in multiobjective variational programming problems with generalized B-invexity. *Control and Cybernetics*, **33**(1), (2004), 113–124.

[3] I. Ahmad and T.R. Gulati: Mixed type duality for multiobjective variational problems with generalized (F, ρ)-convexity. *Journal of Mathematical Analysis and Applications*, **306**(2), (2005), 669–683, DOI: 10.1016/j.jmaa.2004.10.019

[4] T. Antczak, M. Arana-Jimenez and S. Treanţă: On efficiency and duality for a class of nonconvex nondifferentiable multiobjective fractional variational control problems. *Opuscula Mathematica*, **43**(3), (2023), 335–391, DOI: 10.7494/OpMath.2023.43.3.335

[5] C.R. Bector and I. Husain: Duality for multiobjective variational problems. *Journal of Mathematical Analysis and Applications*, **166**(1), (1992), 214–229, DOI: 10.1016/0022-247X(92)90337-D

[6] R. Bellman: Quasi-linearization and upper and lower bounds for variational problems. *Quarterly of Applied Mathematics*, **19** (1962), 349–350, DOI: 10.1090/qam/130585

[7] D. Bhatia and P. Kumar: Multiobjective control problem with generalized invexitys. *Journal of Mathematical Analysis and Applications*, **189**(3), (1995), 676–692, DOI: 10.1006/jmaa.1995.1045

[8] D. Bhatia and A. Mehra: Optimality conditions and duality for multiobjective variational problems with generalized B-invexity. *Journal of Mathematical Analysis and Applications*, **234**(2), (1999), 341–360, DOI: 10.1006/jmaa.1998.6256

[9] V. Chankong and Y.Y. Haimes: *Multiobjective Decision Making: Theory and Methodology*, North-Holland, **8** (1983), ISSN: 0885-5110.

[10] B.D. Craven: On continuous programming with generalized convexity. *Asia-Pacific Journal of Operational Research*, **10** (1993), 219--232.

[11] G. Fantuzzi: Duality of convex relaxations for constrained variational problems. *arXiv*, (2019), DOI: 10.48550/arXiv.1906.12136

[12] A.M. Geoffrion: Proper efficiency and the theory of vector maximization. *Journal of Mathematical Analysis and Applications*, **22**(3), (1968), 618–630, DOI: 10.1016/0022-247X(68)90201-1

[13] S. Gramatovici: Optimality conditions in multiobjective control problems with generalized invexity. *Annals of the University Craiova, Mathematics and Computer Science Series*, **32** (2005), 150–157, DOI: 10.52846/ami.v32i0.187

[14] L. Grüne, L. Krügel and M.A. Müller: Multiobjective strict dissipativity via a weighted sum approach. *Systems and Control Letters*, **170** (2022), 105396, DOI: 10.1016/j.sysconle.2022.105396

[15] M.A. Hanson: Bounds for functionally convex optimal control problems. *Journal of Mathematical Analysis and Applications*, **8**(1), (1964), 84–89, DOI: 10.1016/0022-247X(64)90086-1

[16] N.H. Hung and N. Van Tuyen: Optimality conditions and duality relations in nonsmooth fractional interval-valued multiobjective optimization. *Applied Set-Valued Analysis and Optimization*, **5**(1), (2023), 31–47, DOI: 10.23952/asvao.5.2023.1.03

[17] B.C. Joshi: Duality results on mathematical programs with vanishing constraints involving generalized invex functions. *Control and Cybernetics*, **52**(4), (2023), 351–379, DOI: 10.2478/candc-2023-0042

[18] K. Khazafi and N. Rueda: Multiobjective variational programming under generalized type I univexity. *Journal of Mathematical Analysis and Applications*, **142**(2), (2009), 363–376, DOI: 10.1007/s10957-009-9526-3

[19] K. Khazafi, N. Rueda and P. Enflo: Sufficiency and duality for multiobjective control problems under generalized (B,ρ)-type I functions. *Journal of Global Optimization*, **46**(1), (2010), 111–132, DOI: 10.1007/s10898-009-9412-4

[20] D.S. Kim and M.H. Kim: Generalized type I invexity and duality in multiobjective variational problems. *Journal of Mathematical Analysis and Applications*, **307**(2), (2005), 533–554, DOI: 10.1016/j.jmaa.2005.02.018

[21] J.C. Liu: Optimality and duality for generalized fractional variational problems involving generalized (F,ρ)-convex functions. *Optimization*, **37**(4), (1996), 369–383, DOI: 10.1080/02331939608844171

[22] S. K. Mishra, S.Y. Wang andK.K. Lai: *Generalized Convexity and Vector Optimization*, Springer-Verlag, **90** (2009), DOI: 10.1007/978-3-540-85671-9

[23] B. Mond, S. Chandra and I. Husain: Duality for variational problems with invexity. *Journal of Mathematical Analysis and Applications*, **134**(2), (1988), 322–328, DOI: 10.1016/0022-247X(88)90026-1

[24] B. Mond and M.A. Hanson: Duality for variational problems. *Journal of Mathematical Analysis and Applications*, **18**(2), (1967), 355–364, DOI: 10.1016/0022-247X(67)90063-7

[25] B. Mond and I. Smart: Duality and sufficiency in control problems with invexity. *Journal of Mathematical Analysis and Applications*, **136**(1), (1988), 325–333, DOI: 10.1016/0022-247X(88)90026-1

[26] R.N. Mukherjee and S.K. Mishra: Generalized invexity and duality in multiobjective objective variational problems. *Journal of Mathematical Analysis and Applications*, **195** (1995), 307–322, DOI: 10.1006/jmaa.1995.1357

[27] R.N. Mukherjee and S.K. Mishra: Multiobjective control problem with V-invexity. *Journal of Mathematical Analysis and Applications*, **235** (1999), 1–12, DOI: 10.1006/jmaa.1998.6110

[28] C. Nahak and S. Nanda: Duality for multiobjective variational problems with invexity. *Optimization*, **36(3)** (1996), 235–248, DOI: 10.1080/02331939608844111

[29] C. Nahak and N. Behera: Optimality conditions and duality for multiobjective variational problems with generalized ρ-(η, θ)-B-type-I functions. *Journal of Control Science and Engineering*, **2011**(1), (2011). DOI: 10.1155/2011/497376

[30] V. Preda: On duality and sufficiency in control problems with general invexity. *Bulletin Mathematique de la Societe des Sciences Mathematiques de Roumanie*, **35** (1991), 271–280.

[31] L.V. Reddy and R.N. Mukherjee: Efficiency and duality of multiobjective fractional control problems under (F, ρ)-convexity. *Indian Journal of Pure and Applied Mathematics*, **30** (1999), 51–69.

[32] M. Saadati and M. Oveisiha: Optimality conditions for robust nonsmooth multiobjective optimization problems in Asplund spaces. *Bulletin of the Belgian Mathematical Society – Simon Stevin*, **28**(4), (2022), 579–601, DOI: 10.36045/j.bbms.210705

[33] S. Treanţă: Efficiency in generalised $V-KT$-pseudoinvex control problems. *International Journal of Control*, **93**(3), (2018), 611–618, DOI: 10.1080/00207179.2018.1483082

[34] S. Treanţă: Necessary and sufficient optimality conditions for some robust variational problems. *Optimal Control Applications and Methods*, **44**(1), (2023), 81–90, DOI: 10.1002/oca.2931

[35] S. Treanţă and Şt. Mititelu: Duality with (ρ, b)–quasiinvexity for multidimensional vector fractional control problems. *Journal of Information and Optimization Sciences*, **40**(7), (2019), 1429–1445, DOI: 10.1080/02522667.2018.1522798

[36] S. Treanţă, V. Ionică, G. Ye and W. Liu: On symmetric dual models associated with multiple cost control problems. *Archives of Control Sciences*, **35**(2), (2025), 205–228, DOI: 10.24425/acs.2025.155395

[37] Ch. Xiuhong: Duality for a class of multiobjective control problems. *Journal of Mathematical Analysis and Applications*, **267** (2002), 377–394, DOI: 10.1006/jmaa.2001.7873

[38] L. Zhian and Y. Qingkai: Duality for a class of multiobjective control problems with generalized invexity. *Journal of Mathematical Analysis and Applications*, **256**(2), (2001), 446–461, DOI: 10.1006/jmaa.2000.7284