

# Impact of control representations on efficiency of local nonholonomic motion planning

I. DULEBA\*

Institute of Computer Engineering, Control and Robotics, Wrocław University of Technology,  
11/17 Janiszewski St., 50-372 Wrocław, Poland

**Abstract.** In this paper various control representations selected from a family of harmonic controls were examined for the task of locally optimal motion planning of nonholonomic systems. To avoid dependence of results either on a particular system or a current point in a state space, considerations were carried out in a sub-space of a formal Lie algebra associated with a family of controlled systems. Analytical and simulation results are presented for two inputs and three dimensional state space and some hints for higher dimensional state spaces were given. Results of the paper are important for designers of motion planning algorithms not only to preserve controllability of the systems but also to optimize their motion.

**Key words:** motion planning, nonholonomic systems, Lie-algebraic methods, controllability, optimization.

## 1. Introduction

Many systems of contemporary robotics (wheeled mobile robots [1], free floating robots, underwater vehicles, nonholonomic manipulators) are subordinated to constraints in the Pfaff form defining, locally, inadmissible directions of motion. From a control perspective it is more desirable to work with admissible directions of motion, therefore, based on constraints, a driftless nonholonomic system is formulated [2]

$$\dot{\mathbf{q}} = \sum_{i=1}^m \mathbf{g}_i(\mathbf{q})u_i = \mathbf{G}(\mathbf{q})\mathbf{u}, \quad (1)$$

$$\dim \mathbf{u} = m < n = \dim \mathbf{q},$$

where  $\mathbf{q}$  is the configuration vector,  $\mathbf{u}$  denote controls and (smooth) vector fields  $\mathbf{g}_i(\mathbf{q})$  are called generators. One of the most important tasks in robotics is the motion planning problem that relies on determining admissible controls  $\mathbf{u}$  that steer the system (1) from a given configuration  $\mathbf{q}_0$  to the goal one  $\mathbf{q}_f$ . In robotic literature many methods and techniques were developed to solve the problem. Their spectrum varies from applying standard tools of the optimal control theory [3] and the calculus of variation [4] to numerical and graph-based discrete approaches [5]. Generally, they can be divided into local and global ones. Local methods, step-by-step in local neighborhood of a current configuration generate pieces of a resulting trajectory joining the initial and the goal configuration. Global methods try to solve the motion planning problem at once usually in an iterative process. Both of aforementioned paradigms have their own advantages and disadvantages with respect to optimality, obstacle avoidance and computational complexity. Nevertheless, they can be incorporated into a higher order system to control single or multi-robot systems [6].

Many fruitful local and global methods are rooted in a Lie algebra associated with the system (1) and spanned by its

generators  $\mathbf{g}_i$ ,  $i = 1, \dots, m$  [2]. Most of motion planning methods are based on the Newtonian principle where a mapping is defined (often called kinematics) from the space of controls to the space of configurations (a desired local direction of motion in local Lie-algebraic methods [2] or an end-trajectory position in global Newtonian methods [7]). Then, an error function is defined as a distance between a configuration reached from a current location (local methods) or the location corresponding to current controls (global ones), and a desired location (the goal configuration for global methods). Finally, an algorithm should be proposed to tune controls in order to decrease the error function. The convergence of the iterative process is preserved when the error decreases monotonically and tends to the zero value. In robotics this sort of tasks is called inverse kinematics.

The theory of motion planning methods often considers control signals that belong to a broad class of admissible functions with a weak constraints, if any, imposed on a total energy. In practice, the signals must be somehow restricted (for example any energy supplying device has got an upper limit on transferred frequencies). To avoid a functional space of controls, it is a common practice, to replace it with a parameter space, after fixing a (Fourier, polynomial) basis. In this case the basis coefficients are searched for. Although computations of motion planning algorithms are performed on powerful computers, still the representation of controls has to be finite and a contradiction between the computational complexity and the convergence property appears. For computational reasons, the representation should be as small as possible. On the other hand, the convergence is easier to be guaranteed by more numerous representations. In this paper an impact of control representations on a solvability and the quality of solution offered by a local Lie-algebraic method of motion planning will be considered. Considerations are restricted to two input

\*e-mail: ignacy.duleba@pwr.wroc.pl

systems,  $m = 2$ , because many nonholonomic systems of robotics have got only two controls and the systems are the most complex to control (for  $m = 2$  and a high dimensional state spaces much more difficult-to-generate higher degree vector fields are required than for  $m > 2$  to preserve controllability). For more than two controls, the approach presented can be also generalized.

This paper is organized as follows. In Sec. 2 some preliminary Lie-algebraic facts will be recalled. Then, the optimal control problem will be formulated. In Sec. 3 solutions of the task will be given. Simulations for two input system with three dimensional configuration space are collected in Sec. 4. Section 5 concludes the paper.

## 2. Problem formulation

In order to make considerations almost independent either on a particular system or a particular point in the space, the generators of the system (1) are replaced with their formal counterparts  $g_1 \rightarrow X$ ,  $g_2 \rightarrow Y$  being also Lie monomials. This mapping will be referred to as generalization. Then the bi-linear Lie bracket operation is defined that for two Lie polynomials  $A$ ,  $B$  assigns another one denoted as  $[A, B]$  [8]. Starting with formal generators and using the Lie bracket other Lie monomials can be recursively produced. With each Lie monomial its degree is assigned counting how many generators appear in this particular Lie monomial. All Lie monomials with the same degree form a layer. Not all Lie monomials are independent of each other as the antisymmetry and the Jacobi identity holds [8]. Taking linear combinations (over  $\mathbb{R}$ ) of the Lie monomials a formal Lie algebra can be defined. Independent Lie monomials form a basis of the algebra. Probably the most popular is the Ph. Hall basis with its very first elements equal to  $X, Y, [X, Y], [X, [X, Y]], [Y, [X, Y]], [X, [X, [X, Y]]], [Y, [X, [X, Y]]], [Y, [Y, [X, Y]]]$ .

The formal Lie algebra becomes a controllability algebra of the system (1) when the Lie bracket is defined in coordinates as

$$[A, B] = \frac{\partial B}{\partial q} A - \frac{\partial A}{\partial q} B$$

and the reverse generalization transformation (the specification)  $X \rightarrow g_1$ ,  $Y \rightarrow g_2$  is applied. When the controllability algebra at each configuration has got a full rank, equal to  $n$ , so according to the Chow theorem [9], the system (1) is a small time locally controllable. In literature the controllability requirement is called the Lie algebra rank condition (LARC). Elements of the controllability Lie algebra can be viewed as velocities that allows to change a current state of the system (1) and push it towards the goal state. It is easy to generate  $X, Y$  directions just by switching on only one control  $u_1$  or  $u_2$  and switching off the other. However, to generate compound Lie monomials (vector fields after specification) more complex control scenarios are required. Fortunately, there is the Campbell-Hausdorff-Dynkin formula [10] which allows to locally predict behavior of the system (1) actuated with a given  $u$  (in fact this formula is more powerful as it is also satisfied by non-autonomous systems of differential

equations). For  $m = 2$ , a local (small time motion) shift  $z(t)$  can be expressed as a combination of Lie monomials from the Lie algebra space

$$z(t) = k_1(t)X + k_2(t)Y + k_3(t)[X, Y] + k_4(t)[X, [X, Y]] + k_5(t)[Y, [X, Y]] + \dots \quad (2)$$

where subscripts in  $k = (k_1, k_2, \dots)^T$  enumerate consecutive elements of the Ph. Hall basis elements. Control dependent coefficients (2) are the following [11]

$$\begin{aligned} k_1(t) &= \int_{s=0}^t u_1(s) ds, \\ k_2(t) &= \int_{s=0}^t u_2(s) ds, \\ k_3(t) &= \frac{1}{2} \iint_{t^{(2)}} (u_{12} - u_{21}) ds^2, \\ k_4(t) &= \frac{1}{6} \iiint_{t^{(3)}} (u_{112} - 2u_{121} + u_{211}) ds^3, \\ k_5 &= \frac{1}{6} \iiint_{t^{(3)}} (-u_{122} + 2u_{212} - u_{221}) ds^3, \end{aligned} \quad (3)$$

where  $ds^i = ds_1 \dots ds_i$ , the integration area is the simplex  $t^{(i)} = \{s_1 \in [0, s_2], s_2 \in [0, s_3], \dots, s_i \in [0, t]\}$  and an abbreviated notation of controls is used  $u_{ij} = u_i(s_1)u_j(s_2)$ ,  $u_{ijk} = u_i(s_1)u_j(s_2)u_k(s_3)$ . Note that originally [10] the shift  $z(t)$  is expressed in terms of vector fields from the controllability algebra and evaluated at a current state. In this paper, the form (2) will be preferred as it makes motion independent on a particular state in the configuration space. To get the original form, it is enough to apply specification mapping and evaluate resulting vector fields at a current state. Also locality of the formula (2) is clearly visible, because it is expressed as infinite series of Lie monomials. As the time  $t$  increases the higher degree Lie monomials influences  $z(t)$  more than those with low degrees. To reduce the impact of higher degree Lie monomials on  $z(t)$ , the time  $t$  should be kept as small as possible to get a predictable shift. For nilpotent systems (all Lie monomials with degrees higher than order of nilpotency vanish) the restriction on  $t$  is not valid and the local Lie-algebraic method becomes also the global one.

Let us fix the time horizon  $t = T$  and consider a family of harmonic controls

$$u(s) = a_0 + \sum_{i=1}^N (a_{2i-1} \sin(i\omega s) + a_{2i} \cos(i\omega s)) \quad (4)$$

$$\text{where } \omega = 2\pi/T \text{ and } s \in [0, T],$$

with undetermined vectors-coefficients  $a$  and  $N$  setting the upper limit on admissible frequencies.

For harmonic controls (4) it is easy to check whether they can be generated with a real supplying device or not. It is not the case for polynomial controls that generate large number of high frequencies (to check it, it is enough to expand

polynomials in the Fourier series). The energy of signal (4) is equal to

$$\begin{aligned} \text{energy}(\mathbf{u}(\cdot)) &= \int_0^T \mathbf{u}^T(s)\mathbf{u}(s)ds = \\ &= \frac{T}{2} \left( 2\|\mathbf{a}_0\|^2 + \sum_{i=1}^{2N} \|\mathbf{a}_i\|^2 \right). \end{aligned} \quad (5)$$

In subsets of the family (4) (some elements in the sum are not present) controls  $u_1, u_2$  will be searched for. Those elements of the series (4) which are taken for controls  $u_1, u_2$  will be called their representations.

Now the following task can be formulated: for a fixed representation of controls and a given motion expressed as a linear combination  $k_1^c \mathbf{X} + k_2^c \mathbf{Y} + \dots$  with known coordinates  $k^c$ , find coefficients of the representations to minimize the total energy of motion.

Some comments on the task formulated follow:

- When a representation of controls is fixed, coefficients of controls become independent variables  $\mathbf{x} = (\mathbf{a}_0^T, \mathbf{a}_1^T, \dots, \mathbf{a}_{2N}^T)^T$ . Substituting the representation into Eq. (3) a kinematic mapping  $\mathbf{x} \rightarrow k(\mathbf{x})$  is obtained and the total energy of controls (4) is a function of  $\mathbf{x}$ .
- The kinematics  $k(\mathbf{x})$  display some differences with kinematics of manipulators like PUMA, SCARA, or the Stanford arm [12]. First, each component is a uniform polynomial with the same degree as the Lie monomial the coordinate of kinematics correspond to. Manipulators' kinematics are composed mainly of trigonometric functions. Second, components  $k_1, k_2$  (cf. Eq. (3)) corresponding to Lie monomials  $\mathbf{X}, \mathbf{Y}$  are equal to constant terms  $\mathbf{a}_0$  of harmonic series (4) multiplied by  $T$ . Consequently, the constant terms are determined by given values of  $k_1^c, k_2^c$  and can be excluded from the vector of variables  $\mathbf{x}$  reducing the dimensionality of  $\mathbf{x}$  and also complexity of the inverse kinematic task. Moreover, after the reduction and neglecting a positive constant multiplier, the total energy of controls equals

$$\text{energy}(\mathbf{u}(\cdot)) = \sum_i x_i^2. \quad (6)$$

- Important questions concern with coefficients  $k^c$  should be posed. How many of them should be supplied and how to determine their values in practice? The first answer is that all  $k^c$  coefficients should be provided corresponding to vector fields with the same degrees as those vector fields that were used to satisfy the LARC. Their number will be denoted as  $r \geq n$ . It may look strange that it may happen  $r > n$ . But all vector fields (Lie monomials) occupying the same layer have got a comparable impact on motion (cf. coefficients  $k_4, k_5$  for the third layer in Eq. (3)) so any of them should not be missed. To answer the second question the following construction is carried out. At a particular point  $\mathbf{q}_c$  in the configuration space and having given motion shift towards the goal described by a vector  $\mathbf{v}_c$  of the size  $(n \times 1)$ , all  $r$  vector fields are evaluated

and collected in columns of a constant, full rank matrix  $A(\mathbf{q}_c)$  of the size  $(n \times r)$  as the LARC is satisfied. Finally  $k^c$  are determined by solving the equation  $\mathbf{v}_c = A(\mathbf{q}_c)\mathbf{k}^c$  (pseudo-inverse when  $r > n$ ).

- The task defined is very similar to a task of determining nonholonomic spheres in the sub-Riemannian geometry [13]. However, there are two significant differences. Nonholonomic spheres assume that the energy of motion is fixed and controls are free to choose, and the question is how far the system can move in any directions. In the task presented, a motion is fixed while an energy is minimized in a subclass of controls admissible for the nonholonomic spheres. The formulated task is more practical because usually a motion is known and an energy is to be minimized.
- It should be noticed that in the formulated task an implicit assumption on a collision free configuration space has been made.

### 3. Solution

The formulated problem is a standard optimization task with equality constraints (in robotic terms – the inverse kinematics). To solve it, the Lagrange's multiplier technique was applied [14]. At first the Lagrange's function is defined

$$L(\boldsymbol{\lambda}, \mathbf{x}) = \sum_{i=1}^D x_i^2 + \sum_{j=3}^r \lambda_j (k_j(\mathbf{x}) - k_j^c), \quad (7)$$

where  $\dim \boldsymbol{\lambda} = r - 2$ ,  $k(\mathbf{x}) = (k_1, k_2, \dots, k_r)^T$ , and  $D = \dim \mathbf{x}$  is the total number of variables appearing in a representation of controls. Then, among those  $\mathbf{x}$  that respect the necessary condition of optimality

$$\frac{\partial L(\boldsymbol{\lambda}, \mathbf{x})}{\partial x_i} = 0, \quad i = 1, \dots, D \quad (8)$$

a solution is searched for. The Lagrange's multiplier technique based on Eq. (7) works quite well and provides analytical results when the dimensionality of vector  $\boldsymbol{\lambda}$  is small. Otherwise, due to a specific and particularly regular minimized function (6) the Newton method is preferred [12, 15]. The method is initialized with any vector  $\mathbf{x}_0$  and modifies it iteratively according to the formula

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \xi J^\#(\mathbf{x}_i)(\mathbf{k}^c - k(\mathbf{x}_i)), \quad (9)$$

where  $i$  is the iteration counter,  $\xi$  is a small positive real coefficient,  $J$  denotes  $((r - 2) \times D)$  the Jacobi matrix of the kinematics  $k(\mathbf{x})$ ,  $J = \partial k / \partial \mathbf{x}$  and  $J^\#$  is the generalized Moore-Penrose matrix inversion,  $J^\# = J^T (J J^T)^{-1}$  where  $T$  denotes transposition (if  $D = r - 2$  then  $J$  is a square matrix and  $J^\# = J^{-1}$ ). When no singular values of  $\mathbf{x}$  (the locations where the matrix  $J J^T$  loses its full rank, so invertability) or their small neighborhood were generated, the limit value of  $\mathbf{x}$  satisfies constraints and minimize also the quality function (6). The only drawback of the Newton method is its locality because being based on linearization of the kinematics around a current  $\mathbf{x}_i$  it implements the steepest descent algorithm to minimize the quality function. However, to increase the chance of getting the global optimum, a multi-start approach (many initial values of  $\mathbf{x}_0$ ) can be utilized.

A clear advantage of the Newton method is that it is faster than any other numerical approach solving the general optimization problem Eq. (8).

#### 4. Simulations

In this section various representations will be examined for two input control systems and with three dimensional state space spanned by the Lie monomials  $\mathbf{X}, \mathbf{Y}, [\mathbf{X}, \mathbf{Y}]$ . To short-hand notations an abbreviated form of exposing controls was utilized. The constant terms of the first and the second control are denoted as  $p_1$  and  $p_2$ , respectively (the terms correspond to the  $\mathbf{a}_0$  term in Eq. (4)). Then, consecutive harmonics of the first and the second control are enumerated and multiplied with consecutive coordinates of the vector of variables  $\mathbf{x}$  ( $\mathbf{a}_i, i \geq 1$  of the series (4)). In Table 1 the third components ( $k_3$ ) of considered kinematics were collected as the remaining two coordinates are equal to  $k_1 = Tp_1, k_2 = Tp_2$  for all the cases. To clarify the notations, in the sixth row of Table 1 the code 0, 1, 2 of control  $u_1$  denotes  $u_1(s) = p_1 + x_1s_1 + x_2c_1$  while the code 0, 1, 2, 3, 4 codes control  $u_2(s) = p_2 + x_3s_1 + x_4c_1 + x_5s_2 + x_6c_2$  where  $s_i = \sin(i\omega s), c_i = \cos(i\omega s), \omega = 2\pi/T$ .

Table 1

Codes of controls and the third coordinate of kinematics corresponding to the controls

case	$u_1$	$u_2$	$k_3$ multiplied by $T^2/(8\pi)$
a	0, 1	0	$4p_2x_1$
b	0, 1	0, 1	$4(p_2x_1 - p_1x_2)$
c	0, 1	0, 2	$2(2p_2x_1 - x_1x_2)$
d	0, 1, 2	0, 1	$2(2p_2x_1 - 2p_1x_3 + x_2x_3)$
e	0, 1, 2	0, 1, 2	$2(2p_2x_1 - 2p_1x_3 + x_2x_3 - x_1x_4)$
e	0, 1, 2	0, 1, 2, 3, 4	$2(2p_2x_1 - 2p_1x_3 + x_2x_3 - x_1x_4 - p_1x_5)$
f	0, 3, 4	0, 1	$2(p_2x_1 - 2p_1x_3)$
f	0, 3, 4	0, 1, 2	$2(p_2x_1 - 2p_1x_3)$
g	0, 3, 4	0, 3, 4	$2p_2x_1 - 2p_1x_3 + x_2x_3 - x_1x_4$
g	0, 3, 4	0, 1, 2, 3, 4	$2p_2x_1 - 4p_1x_3 - 2p_1x_5 + x_2x_5 - x_1x_6$

For the kinematics

$$k(\mathbf{x}) = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} Tp_1 \\ Tp_2 \\ k_3 \end{bmatrix}$$

goal points  $\mathbf{k}^c$  were selected on the sphere with radius  $R$  parametrized with spherical angles  $\alpha, \beta$

$$k_1^c = R \cos \beta \cos \alpha, \quad k_2^c = R \cos \beta \sin \alpha, \quad k_3^c = R \sin \beta$$

where  $\alpha \in [-180^\circ, 180^\circ]$  was changed with the step of  $20^\circ$  while  $\beta \in [-90^\circ, 90^\circ]$  with the step of  $10^\circ$ . Immediately constant terms of controls were determined

$$p_1 = k_1^c/T, \quad p_2 = k_2^c/T.$$

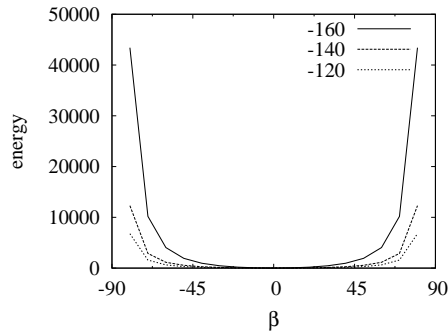
Consequently, the Lagrange's multiplier technique with only one multiplier was used to minimize the energy (6). In all simulations  $T$  was set to 0.5 and  $R = 1$ . Results are presented in Fig. 1 with the energy (6) supplemented by the contribution of constant terms  $p_1, p_2$  and equal to  $2(p_1^2 + p_2^2)$ .

It appears, Fig. 1a, that the three dimensional non-redundant,  $r = n$ , representation 01-0 was not enough to preserve motion ability in any direction, so controllability. The uncontrollable plane is spanned by the versors  $\mathbf{X}, [\mathbf{X}, \mathbf{Y}]$  and it is characterized by the angle  $\alpha = \pm 180^\circ$ . This value forces  $p_2 = 0$ , cf. Table 1, leaving the coordinate  $k_3$  of kinematics uncontrollable. One can complain that the plane is not massive object in  $\mathbb{R}^3$  (but still it separates points placed on its opposite sides). However, outside the plane in a small (but open, so massive) neighborhood of the direction  $[\mathbf{X}, \mathbf{Y}]$ ,  $|\beta| \rightarrow 90^\circ$ , the energy consumption grows rapidly because  $p_2$  is small enough to set a value of coordinate  $\mathbf{Y}$  while to get desired  $k_3$ , the value of  $x_1$  has to be huge, so is the energy, Fig. 1a. Nevertheless, the non-redundant representation can be quite useful for directions with small values of  $[\mathbf{X}, \mathbf{Y}]$  and placed far from the plane  $\mathbf{X}, [\mathbf{X}, \mathbf{Y}]$ .

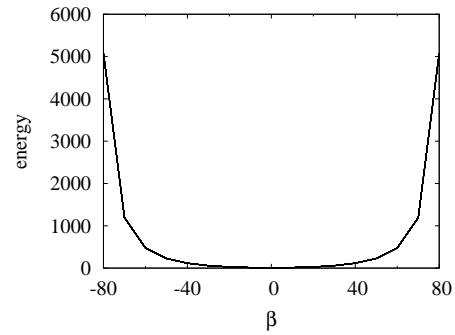
Similar effects to those presented can be observed for another non-redundant representation 02-0 (not presented here). For this representation ill-conditioned plane is spanned by versors  $\mathbf{Y}, [\mathbf{X}, \mathbf{Y}]$ ,  $\alpha = \pm 90^\circ$ .

All remaining analyzed representations were redundant. Although redundant, the representation 01-01 still is not controllable along the direction  $\pm[\mathbf{X}, \mathbf{Y}]$  because for this direction  $p_1 = p_2 = 0$ , so  $k_3 = 0 \neq k_3^c$ . The representation generates axially symmetric results with the axis of rotation  $[\mathbf{X}, \mathbf{Y}]$ , therefore Fig. 1b presents the characteristics for any value of  $\alpha$ . All goal directions surrounding  $\pm[\mathbf{X}, \mathbf{Y}]$  are energy expensive, as they require small values of  $p_1, p_2$  forcing large amplitudes of  $x_1, x_2$ . Quite different situation is observed for representation 01-02 visualized in Fig. 1c. From the 3D plot one can deduce that the system is fully controllable and a relatively small amount of energy is required to move along difficult directions  $\pm[\mathbf{X}, \mathbf{Y}]$ . When the absolute value of  $k_3$  tends to zero,  $|\beta| \rightarrow 0^\circ$ , the energy decreases taking its minimal value equal to 8 for  $\beta = 0$ . Those energy-cheap motions assume only a slight shifts into the  $[\mathbf{X}, \mathbf{Y}]$  direction. Although the results do not display symmetry around the versor  $[\mathbf{X}, \mathbf{Y}]$  the symmetry is broken only slightly. A big difference between four dimensional representations 01-01 and 01-02 is that the first one is energy expensive in some directions while the second is energy mild for any direction.

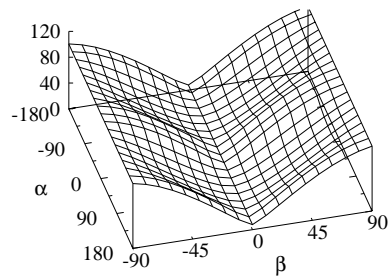
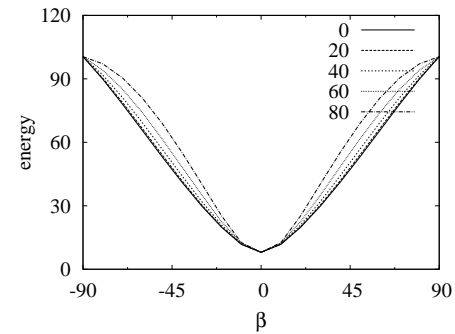
Let us add one more component to the control  $u_1$ , Fig. 1d. It appears that results are the same as in Fig. 1c, slightly asymmetric and motions with a mild energy consumption, more energy demanding for directions close to the  $\pm[\mathbf{X}, \mathbf{Y}]$ . Consequently, one extra component has not improved the energy efficiency of motion. If one more component was added, this time to the control  $u_2$ , Fig. 1e, full axial symmetry with respect to the axis  $[\mathbf{X}, \mathbf{Y}]$  was retrieved but results were improved slightly (Figs. 1de). Adding two more components, the representation 012-01234, has not improved anything. It does not mean that the components are completely useless as they impact coefficients of higher degree Lie monomials  $[\mathbf{X}, [\mathbf{X}, \mathbf{Y}]], [\mathbf{Y}, [\mathbf{X}, \mathbf{Y}]], \dots$  and the coefficients may be useful for higher dimensional state spaces. However, to control nilpotent systems of order two there is no need to apply the extra components.

*Impact of control representations on efficiency of local nonholonomic motion planning*
a) 01-0  $\alpha$  parameter

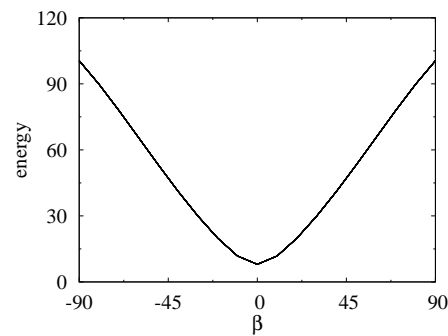
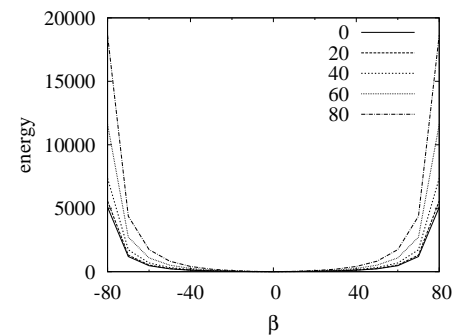
b) 01-01



c) 01-02

d) 012-01  $\alpha$  parameter

e) 012-012 (also 012-01234)

f) 034-01  $\alpha$  parameter (also 034-012)

g) 034-034 (also 012-01234)

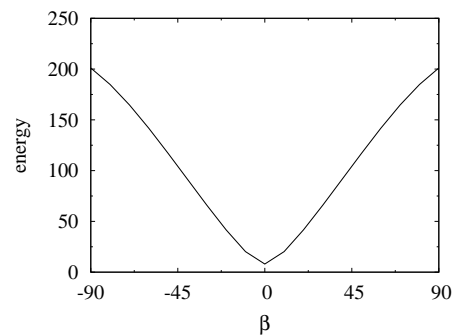


Fig. 1. Energy as a function of  $\beta$  angle (mostly) for various control representations (the notation of cases and description of controls are the same as in Table 1)

Next simulations considered the redundant case with the second harmonics, coded as 034, in control  $u_1$  replacing the first harmonics, 012. Results presented in Fig. 1f were obtained for  $u_2$  represented as 01 or 012. Despite of 5 or 7 dimensional control-parameter spaces it was unable to steer

the system (1) in the three dimensional space  $X, Y, [X, Y]$ , more specifically, the direction  $[X, Y]$  was uncontrollable. To explain this case one can notice, Table 1f, that  $k_3$  component is the same for both representations and for  $p_1 = p_2 = 0$  this component vanish.

When full harmonics (both sinus and co-sinus components) of the same order appear in both control, Fig. 1g, controllability is once more retrieved. Comparing Fig. 1e with Fig. 1g, or Table 1e with Table 1g, it is easy to observe that lower harmonics are more energy efficient than higher order harmonics.

One more aspect, this time algebraic one, will be explained. Let us consider the scenario 012-012 with controls  $u_1(s) = p_1 + x_1 s_1 + x_2 c_1$ ,  $u_2(s) = p_2 + x_3 s_1 + x_4 c_1$  and kinematics given in the fifth row of Table 1. The only constraint is  $w(\mathbf{x}) = 2p_2 x_1 - 2p_1 x_3 + x_2 x_3 - x_1 x_4 - p_3$  with the fixed parameter  $p_3 = k_3^c/2$  as  $k_3^c$  is given. The Lagrange's function  $L(\mathbf{x}, \lambda) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + \lambda \cdot w(\mathbf{x})$  when substituted to the necessary optimality conditions (8) generate expressions for  $\mathbf{x}$  as follows

$$\begin{aligned} x_1 &= 2p_2 h(\lambda), & x_2 &= \lambda p_1 h(\lambda), \\ x_3 &= -2p_1 h(\lambda), & x_4 &= \lambda p_2 h(\lambda), \end{aligned} \quad (10)$$

where  $h(\lambda) = 2\lambda/(\lambda^2 - 4)$ . Substituting (10) to the constraint  $w(\mathbf{x})$  the fourth order polynomial equation on  $\lambda$  is obtained

$$32\lambda(p_1^2 + p_2^2) + p_3(\lambda^2 - 4)^2 = 0. \quad (11)$$

Its real solutions, when substituted to Eq. (10), allows to determine the minimum of the energy function for almost any goal direction given. However, not for all, as  $\pm[\mathbf{X}, \mathbf{Y}]$  direction requires  $p_1 = p_2 = 0$  and the only real solutions of Eq. (11) are  $\lambda = \pm 2$  and Eq. (10) is ill-conditioned. This example warns us that very rare cases should be considered separately. Fortunately, they are easy to determine algebraically. In the aforementioned case it is enough to take  $p_1 = p_2 = 0$  and once more apply the Lagrange's multiplier technique to deal with the direction  $\pm[\mathbf{X}, \mathbf{Y}]$ .

## 5. Conclusions

In this paper an impact of representations of controls on controllability and optimality of a local motion was analyzed for two input and three dimensional systems. Two input space was selected to cover systems the most frequently encountered in practice while the low dimensional configuration space was selected to use purely analytical approach based on the Lagrange's multiplier technique. To avoid a shape of a particular system or a particular point of its state space considerations were carried out in the sub-space of formal Lie algebra associated with controllability algebra of a particular system and spanned by the first three elements of the Ph. Hall basis. The simulation results revealed that non-redundant representations can not guarantee controllability and the directions where controllability was missed were different for various non-redundant representation. It may suggest that switching non-redundant representations is a good method to retrieve controllability. Unexpectedly, uncontrollable systems may also appear for redundant representations, even with high redundancy. It was shown that as low as possible harmonics should be used to optimize a motion. When there is restriction on the number of elements in representations it is advised to use

sinus function in one control while co-sinus function for the other.

Difficult tests for controllability are along directions corresponding to pure higher degree Lie monomials. Those directions set restrictive conditions on parameters of controls and exclude some components from the kinematics. Moreover, those directions may require a unique treatment as they generate ill-posed optimization problems easily. Therefore, those directions should be obligatory added to a set of data for testing designed representations for high dimensional systems. To deal with higher dimensional state spaces, more harmonics will be required and the Newton algorithm (9) seems to be promising alternative to the Lagrange's multiplier method for this case.

Results presented in this paper has got also some value for global methods of nonholonomic motion planning as those methods resemble local methods for boundary points placed close one another.

## REFERENCES

- [1] J. Szrek and P. Wójtowicz, "Idea of wheel-legged robot and its control system", *Bull. Pol. Ac.: Tech.* 58 (1), 43–50 (2010).
- [2] I. Duleba, *Algorithms of Motion Planning for Nonholonomic Robots*, Wrocław Univ. Publ. House, Wrocław, 1998.
- [3] D.B. Reister and F.G. Pin, "Time-optimal trajectories for mobile robots with two independently driven wheels", *Int. J. Robotics Research* 13 (1), 38–54 (1991).
- [4] C. Fernandez, L. Gurvits, and Z.X. Li, "A variational approach to optimal nonholonomic motion planning", *Proc. IEEE Conf. Robotics and Automat.* 1, 680–685 (1991).
- [5] S. LaValle, *Planning Algorithms*, Cambridge Univ. Press, Cambridge, 2006.
- [6] C. Zieliński and T. Winiarski, "General specification of multi-robot control system structures", *Bull. Pol. Ac.: Tech.* 58 (1), 15–28 (2010).
- [7] K. Tchon and J. Jakubiak, "Endogenous configuration space approach to mobile manipulators: a derivation and performance assessment of jacobian inverse kinematics algorithms", *Int. J. Control* 26 (14), 1387–1419 (2003).
- [8] J-P. Serre, *Lie Algebras and Lie Groups*, Springer, Berlin, 1964.
- [9] W.L. Chow, "Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung", *Math. Annalen* 117 (1), 98–105 (1939).
- [10] R.S. Strichartz, "The Campbell-Baker-Hausdorff-Dynkin formula and solutions of differential equations", *J. Funct. Analysis* 72 (2), 320–345 (1987).
- [11] I. Duleba, "Pre-control form of the generalized Campbell-Baker-Hausdorff-Dynkin formula for affine nonholonomic systems", *Systems & Control Letters* 55 (2), 146–157 (2006).
- [12] M. Spong and M. Vidyasagar, *Robot Dynamics and Control*, MIT Press, Cambridge, 1989.
- [13] A. Belaiche, F. Jean, and J.J. Risler, "Geometry of nonholonomic systems", *Robot Motion Planning and Control, Lecture Notes in Control and Information Sciences* 229, 55–92 (1993).
- [14] D.P. Bertsekas, *Constrained Optimization and Lagrange Multiplier Methods*, Athena Scientific, Nashua, 1999.
- [15] Y. Nakamura, *Advanced Robotics: Redundancy and Optimization*, Addison Wesley, New York, 1991.