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Control of a unicycle-like robot with trailers using transverse function approach

D. PAZDERSKI*, K. KOZŁOWSKI, and D.K. WAŚKOWICZ

Poznań University of Technology, Chair of Control and Systems Engineering, 3a Piotrowo St., 60-965 Poznań, Poland

Abstract. The paper presents the application of a smooth kinematic algorithm to control multi-body vehicle which consists of the unicycle-like tractor with three trailers. The controller takes advantage of the transverse functions and properties of the IV-order two input chained system. The derivation of the algorithm is presented in details. In order to improve the performance of the controller in the real application a selected tuning techniques are discussed. The properties of the closed-loop control system are examined based on results of numerical simulations concerning the point stabilization and trajectory tracking tasks.

Key words: transverse functions, vehicle with trailers, nonholonomic system, control motion task, Lie group.

1. Introduction

Nonholonomic systems are of great importance in many applications taking into account that most means of transport are subject to nonintegrable velocity constraints. Well-known examples of these systems include the wheeled vehicles with kinematic structure designed for planar motion without slip between the wheels and the surface. Referring to [1] only a few fundamental kinematics satisfying pure rolling and non slipping assumptions can be distinguished. The most popular vehicles are based on two-wheeled differential (unicycle-like) or car-like platforms. More complex systems employ an active vehicle (tractor) pulling a set of coupled passive segments (trailers). Their mechanical configuration consists of the position and orientation of a selected segment as well as the internal variables describing kinematics of the chain. Each segment can be mounted to the previous one centrically (on-axle mount) or eccentrically (off-axle mount) with respect to the wheels axle [2].

Vehicles with trailers are known to be complicated and highly dimensional systems with significant deficiency of control inputs [3]. Basically, control of these systems is a challenging problem as a result of Brockett's obstruction [4] and highly nonlinear kinematics. Moreover, these systems are usually not regular, namely their nonholonomy degree is not constant [5], that makes the control properties strictly dependent on the desired configuration. They cannot be defined on a Lie group that complicates the controller design even more [6]. One method to overcome this difficulty, considered by Sørdalen [7] with respect to a vehicle with on-axle trailers, is based on coordinate and input transformations to other equivalent control systems with a simpler structure (typically two-input chained system). Another approach proposed by Laumond and Venditelli *et. al.* [5, 8] relies on nilpotent approximations [9] and also can be used with respect to trailers with off-axle hitching [3]. In some applications one can simplify the control solution considering the stabilization or convergence problem with respect to a subset of the system coordinates (for example by defining control error only with respect to the position and orientation of the selected trailer) [10].

This paper is focused on an alternative idea of stabilization of nonholonomic systems formulated by Morin and Samson that is based on so-called transverse functions [11]. It can be stated that these functions define some kind of trajectory in the configuration space such that its derivative along with vector fields of the control system span the tangent space. The time-evolution of the transverse functions are governed by an augmented dynamics and it is strictly dependent on the tracking error. This approach has been effectively used to control invariant (defined on Lie group) system for which global stabilization result can be guaranteed. It is a universal approach that ensures practical (in some cases asymptotic) stabilization with respect to the regulation and trajectory tracking problems. Recently, the controller based on the transverse functions has been adapted by Morin and Samson to some class of non invariant systems including a car-like kinematics and a unicycle with N-on-axle trailers [6, 12].

In this paper we consider the application of the controller using the transverse functions to three on-axle trailers pulled by a unicycle-like mobile robot. In comparison to [6] more classic solution is considered, namely the derivation of the controller is based on the transformation of the considered kinematics to the 6D (IV-order) two input chained system which is defined on a Lie group. According to the authors' best knowledge no results which illustrate performance of the controller based on the transverse functions applied for a such complex vehicle have been reported so far. Accordingly, although in [6] theory has been formulated with respect to general N-trailers system presentation of the simulation results has been limited to a two-trailers system only.

*e-mail: dariusz.pazderski@put.poznan.pl

In order to make the presentation clearer the derivation of the controller and specific calculations are given in details. The main contribution of the paper is related to application of optimal control and some tuning methods in order to improve controller performance during transient states and to increase accuracy in the steady-state. The results of numerical simulation illustrate advantage of the considered control scheme assuming presence of input saturation.

The paper is organized as follows. In Sec. 2 a brief overview of the control algorithm is outlined. Next section is focused on formal description of the kinematic chain of the unicycle-like vehicle with on-axis trailers taking into account Lie group structure. Section 4 is dedicated to the design of the controller. The transformation of the considered kinematics to the chained form is discussed taking into account local nature of the coordinate map. Next, derivation of the transverse function is shown and different control schemes are developed including optimal or suboptimal approach. Finally, possibility of the controller tuning is discussed. In Sec. 5 results of numerical simulations are given in order to illustrate advantages and disadvantages of the controller. Section 6 concludes the paper.

2. Overview of the control approach

The algorithm presented in this paper solves a fundamental motion control task of making the particular multi-segment wheeled vehicle track a reference trajectory defined in the configuration space. The control solution is based on formal methods and has its roots in the differential geometry and Lie group theory. It is defined at kinematic level assuming that velocity signals are the real inputs of the vehicle. Such a simplification can be justified because the control at dynamic level for the particular system is not challenging from a theoretical point of view.

In can be noted that the derivation and the controller structure are quite complex. Hereafter, we give a brief overview of different steps of the design procedure.

The first step is devoted to the analysis of the vehicle kinematics. In the considered case the kinematics can be described by a two-input highly nonlinear control system with nonintegrable velocity constraints. The properties of its control Lie algebra indicate that it is locally small-time controllable in spite of the existence of some singular points at which the structure of the algebra is changed. However, it can be shown that the singular configurations are not an issue for typical motion tasks defined for such a vehicle. As a result the motion of the system can be considered in the restricted configuration set assuming that the magnitudes of angles determined between each adjoint segments are less that $\pi/2$. Hence, the control solution may be defined in the domain where the particular structure of the control Lie algebra is preserved.

Although the control paradigm considered in the paper can be applied to any smooth kinematic system satisfying the Lie Algebra Rank Condition the most efficient and global results can be obtained for an invariant system. The existence of a Lie group is essential since it gives a possibility to translate the solution defined at the particular point to any point in the configuration space and to define conveniently a control error. Since the considered system does not satisfy invariant property the design of the controller is more complicated. In order to facilitate the design, one can notice that the description of motion of the last trailer can take advantage on Special Euclidean Lie group SE(2). Consequently a tracking error, describing the position and orientation of the trailer with respect to the reference target, based on intrinsic symmetry on a plane can be considered.

Next, we define the error kinematics defined on SE(2) group and transform it to a system with a drift which is similar to the IV-order chained system, which can be fully defined on a Lie group. Since the error kinematics does not describe the time-evolution of the internal configuration error (namely defined with respect to the angles between the segments of the vehicle) the reference internal variables are transformed additionally. Then using the group operation a new transformed tracking error is defined that reflects original tracking error defined for the whole vehicle.

As a result, the controller can be defined directly in the auxiliary space using Lie group theory. In order to do that the transverse function should be defined. It is calculated considering the control Lie algebra of the IV-order (6D) chained system and consists of four fundamental functions with derivatives responsible for generating the directions determined by the higher order Lie brackets of the algebra. These derivatives, in a sense, substitute the missing fundamental vector fields of the controlled system (this property is related to the transversality condition). It is worth to emphasize that the selection of a transverse function is not unique. In particular, one has to chose a proper parameters of the function to guarantee the transversality condition. In order to scale the transverse function, namely to adjust its norm, one unique parameter can be chosen.

The transverse function can be considered as some kind of trajectory evolving in the neighborhood of neutral element of the group. Its basic form is characterized by the non zero norm, namely the function never degenerates to the neutral element. Additionally, we take into account an extended transverse function that may satisfy transversality condition even if its norm is zero at some time instants.

Having calculated a transverse function one can define an auxiliary signal which describes the error calculated between it and the transformed tracking error. This signal is obtained taking advantage of the group operation. The aim of the controller is to make the auxiliary error converge to neutral element of the group and to ensure stabilization at this point. Then the transformed tracking error tends to the neighborhood of the neutral element. The similar statement can be made with respect to the tracking error defined for the original system. In the case of the extended transverse function it is even possible to ensure the asymptotic convergence (however, only for some particular reference trajectories).

The controller calculates an extended input which consists of the real and virtual signals. The latter define frequency-like signals which govern the time-evolution of the transverse function. The advantage of this approach is the possibility to fully decouple the auxiliary closed-loop dynamics. On the other hand, it can be seen as an approximate decoupling of the original system with an accuracy determined by the norm of the transverse function. These properties give many possibilities of the control loop design, similarly as it can be done with respect to linear systems. Apart from the most classical solutions using constant gains one can consider optimal control schemes. Then one can improve the performance of the controller during transient states by limiting instantaneous control effort.

It is worth to emphasize that the controller considered in this paper is designed for highly dimensional nonholonomic system. Therefore its tuning is a difficult task. The tuning procedures can take into account a proper selection of parameters of the transverse function (it is quite complicated and is generally supported by numerical methods) and the development of methods for a suitable scaling of the function in order to limit oscillatory behavior. Moreover, in practice careful design with respect to input saturation as well as numerical stability of the algorithm should be taken into account.

At the end of this section we make a comment concerning the notation used in the paper. Element of a Lie group are represented simultaneously in two ways. The first one refers to their representation at abstract generic level and the second one to n-dimensional vectors in $\mathbb{R}^n$. Therefore operations and measures defined in $\mathbb{R}^n$ space can be used with respect to Lie group elements as well. In particular we use Euclidean norm denoted by $\|\cdot\|$ which acts on considered Lie groups elements.

3. Model description

3.1. Kinematics of N on-axle trailers. Let us consider planar motion of N on-axle trailers driven by a unicycle-like tractor. The mechanical coordinates of the vehicle can be defined by

$$q := \begin{bmatrix} x & y & \theta & \varphi_1 & \dots & \varphi_N \end{bmatrix}^\top \in \mathcal{Q} := \mathbb{R}^2 \times \mathbb{T}^{N+1}, \quad (1)$$

where x, y and θ denote the position of the last trailer and its orientation determined with respect to the inertial frame, respectively, while $\varphi_1, \dots \varphi_N \in \mathbb{S}^1$ refer to the internal configuration describing orientation of each segment of the vehicle with respect to the previous one (cf. geometrical interpretation of the coordinates given in Fig. 1 for $N = 3$).

The distances between the origins of the adjacent local frames fixed to each segment (trailer or tractor) are denoted by l_i, $i = 0, 1, \dots, N$. It is assumed that the control input of the system corresponds to the linear, v_1, and angular, v_2, velocity of the unicycle-like tractor and it is defined by $v := [v_1\ v_2]^\top \in \mathbb{R}^2$. Then the kinematics of the vehicle can be described by the following affine driftless control system (cf. [5, 7])

$$\Sigma \coloneqq \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\varphi}_1 \\ \vdots \\ \dot{\varphi}_i \\ \vdots \\ \dot{\varphi}_N \end{bmatrix} = \begin{bmatrix} C\theta \prod_{k=1}^N C\varphi_k \\ S\theta \prod_{k=1}^N C\varphi_k \\ \frac{1}{l_0} S\varphi_1 \prod_{k=2}^N C\varphi_k \\ \frac{l_0 S\varphi_2 - l_1 S\varphi_1 C\varphi_2}{l_0 l_1} \prod_{k=3}^N C\varphi_k \\ \vdots \\ \frac{l_{i-1} S\varphi_{i+1} - l_i S\varphi_i C\varphi_{i+1}}{l_{i-1} l_i} \prod_{k=i+2}^N C\varphi_k \\ \vdots \\ -\dfrac{S\varphi_N}{l_{N-1}} \end{bmatrix} v_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \end{bmatrix} v_2, \quad (2)$$

where abbreviations $C\varphi$ and $S\varphi$ denote sine and cosine of variable symbol$\varphi \in \Re$, respectively, while symbol $\prod$ describes a product operator defined as follows

$$\prod_{k=\underline{n}}^{\overline{n}} a_k = \begin{cases} \text{for } \underline{n} \le \overline{n} & a_{\underline{n}} \cdot a_{\underline{n}+1} \cdot \ldots \cdot a_{\overline{n}-1} \cdot a_{\overline{n}} \\ \text{otherwise} & 1 \end{cases}, \quad (3)$$

with $\underline{n}$ and $\overline{n}$ being some positive integers, while a_k denoting some scalar function (or coefficient).

It can be proved that system Σ satisfies the Lie Algebra Rank Condition (LARC), hence it is locally small-time controllable for any $q \in \mathcal{Q}$. However, for $N = 2, 3, \dots$ nonholonomy degree (dNH) of the system, namely the minimum number of layers in the Lie algebra filtration which is necessary to satisfy LARC, is not constant for any $q \in \mathcal{Q}$ (one can say that the system is not regular). To be more precise, referring to [5, 8] one can introduce a set of singular configurations $\mathcal{Q}^s := \{q \in \mathcal{Q} : \exists j = 2, \dots, N\ \varphi_j = \pm (2k+1)\pi/2\}$. Then it can be shown that

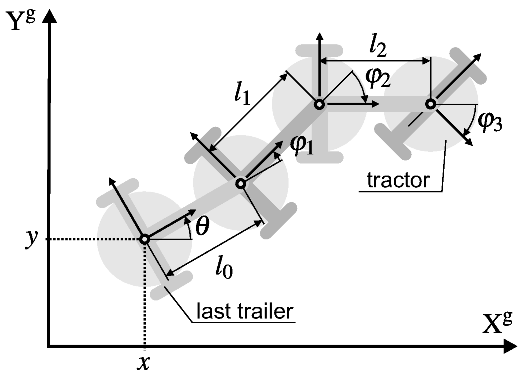


Fig. 1. Illustration of three on-axle trailers driven by one unicycle-like tractor

$$\mathrm{dNH} = \begin{cases} \text{for } N = 1 \text{ and } \forall q \in \mathcal{Q} & 3 \\ \text{for } N = 2, 3, \ldots \text{ and } \forall q \in \mathcal{Q} \setminus \mathcal{Q}^s & N + 2 \\ \text{for } N = 2, 3, \ldots \text{ and } \forall q \in \mathcal{Q}^s & M > N + 2 \end{cases}. \tag{4}$$

The increase of nonholonomy at points $q \in \mathcal{Q}^s$ indicates that control of the system at some neighborhood of these points becomes more challenging. However, in practice, it is very rare to meet a singular configuration for the considered kinematics as a result of mechanical internal structure limitation. Typically in order to prevent a collision between the vehicle's segments it is uncommonly to get $|\varphi_k| > \frac{\pi}{2}$ $(k = 1, 2, \ldots)$. Hence, it is reasonable to assume that system Σ evolves only in the restricted configuration set defined by $\overline{\mathcal{Q}} := \mathbb{R}^2 \times \mathbb{S}^1 \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)^N$.

Following this assumption, one can consider a simpler control system which is equivalent to Σ in the domain $q \in \overline{\mathcal{Q}}$. Referring to [6] and defining the following local coordinate and input transformations

$$\psi = \Psi(\varphi), \tag{5}$$

$$\eta = T(\varphi)\, v, \tag{6}$$

where $\psi \in \mathbb{R}^N$ and

$$\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)^N, \qquad \Psi : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)^N \to \mathbb{R}^N$$

and

$$T : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)^N \to \mathbb{R}^{2\times 2}$$

are smooth maps, one can get

$$\dot{\overline{q}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\psi}_1 \\ \vdots \\ \dot{\psi}_i \\ \vdots \\ \dot{\psi}_N \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ \psi_1 \\ \psi_2 \\ \vdots \\ \psi_{i+1} \\ \vdots \\ 0 \end{bmatrix} \eta_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \end{bmatrix} \eta_2, \tag{7}$$

with $\overline{q} := [x\ y\ \theta\ \psi_1\ \ldots \psi_N]^\top \in \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}^N$ being transformed configuration. It is guaranteed that this system is regular ($\mathrm{dNH} = N + 2 = \mathrm{const}$) in the whole configuration domain.

Although theoretical background and methods used in this paper can be applied for system Σ with $N = 1, 2, \ldots$ and $q \in \overline{\mathcal{Q}}$ the later detailed considerations are focused on the vehicle with three trailers, namely it is assumed that $N = 3$. In order to facilitate the description system (7) is rewritten as follows

$$\dot{\overline{q}} = \begin{bmatrix} \cos\overline{q}_3 \\ \sin\overline{q}_3 \\ \overline{q}_4 \\ \overline{q}_5 \\ \overline{q}_6 \\ 0 \end{bmatrix} \eta_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \eta_2. \tag{8}$$

Then the transformations (5) and (6) are given by

$$\Psi(\varphi) = \begin{bmatrix} \dfrac{\tan\varphi_1}{l_0} \\ \dfrac{l_0 \tan\varphi_2 - l_1 \sin\varphi_1}{l_0^2 l_1 \cos^3\varphi_1} \\ \dfrac{(l_0 \tan\varphi_2 - l_1 \sin\varphi_1)(3\tan\varphi_1 (l_0 \tan\varphi_2 - l_1 \sin\varphi_1) - l_1 \cos\varphi_1)}{l_0^3 l_1^2 \cos^4\varphi_1} + \dfrac{l_1 \tan\varphi_3 - l_2 \sin\varphi_2}{l_0 l_1^2 l_2 \cos^4\varphi_1 \cos^3\varphi_2} \end{bmatrix} \tag{9}$$

and

$$T(\varphi) = \begin{bmatrix} \cos\varphi_1 \cos\varphi_2 \cos\varphi_3 & 0 \\ \left(\dfrac{\partial\psi_3}{\partial\varphi_1}\left(\dfrac{\tan\varphi_2}{l_1} - \dfrac{\sin\varphi_1}{l_0}\right)\cos\varphi_2 + \dfrac{\partial\psi_3}{\partial\varphi_2}\left(\dfrac{\tan\varphi_3}{l_2} - \dfrac{\sin\varphi_2}{l_1}\right) - \dfrac{\partial\psi_3}{\partial\varphi_3}\dfrac{\tan\varphi_3}{l_2}\right)\cos\varphi_3 & \dfrac{\partial\psi_3}{\partial\varphi_3} \end{bmatrix}. \tag{10}$$

In addition, referring to (9) one can obtain the following inverse coordinate map

$$\varphi = \Psi^{-1}(\psi) = \begin{bmatrix} \arctan(l_0\psi_1) \\ \arctan\left(l_1 \kappa(l_0\psi_1)\psi_1 + l_0^2 l_1 \kappa^3(l_0\psi_1)\psi_2\right) \\ \arctan\left(\left(\psi_3 - \dfrac{(l_0\gamma_2 - l_1\kappa(\gamma_1)\gamma_1)(3\gamma_1(l_0\gamma_2 - l_1\kappa(\gamma_1)\gamma_1) - l_1\kappa(\gamma_1))}{l_0^3 l_1^2 \kappa^4(\gamma_1)}\right) l_0 l_1 l_2 \kappa^4(\gamma_1)\kappa^3(\gamma_2) + \dfrac{l_2}{l_1}\kappa(\gamma_2)\gamma_2\right) \end{bmatrix} \tag{11}$$

with $\gamma_1(\psi_1) = l_0\psi_1$, $\gamma_2(\psi_1,\psi_2) = l_1\kappa(\gamma_1)\psi_1 + l_0^2 l_1\kappa^3(\gamma_1)\psi_2$ and $\kappa(\gamma) := \cos(\arctan\gamma) = \dfrac{1}{\sqrt{1+\gamma^2}}$, $(\mathbb{R}\to(0,1])$. It can be proved that the coordinate and input maps are well defined in the assumed domain. Hence, for any bounded $\psi_i \in \mathbb{R}$ it is guaranteed that variables $\varphi_j \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ for $i,j = 1,2,3$.

3.2. Kinematics description based on Lie group theory.

Recalls of Lie group and Lie algebra. Consider a Lie group G with the operation $\circ$ and neutral element e. Assuming that σ_1 and σ_2 are elements of the Lie group ($\sigma_1,\sigma_2 \in G$) we define left and right translations denoted by $l_{\sigma_1}(\sigma_2) := \sigma_1\circ\sigma_2$ and $r_{\sigma_1}(\sigma_2) := \sigma_2\circ\sigma_1$, respectively. For any $\sigma\in G$ an inverse element σ^{-1} satisfies: $\sigma\circ\sigma^{-1} = \sigma^{-1}\circ\sigma = e$.

In order to specify a tangent map the following differential operators related to the left and right translations can be taken into account: $dl_{\sigma_1}(\sigma_2) := \dfrac{\partial}{\partial\sigma_2} l_{\sigma_1}(\sigma_2)$ and $dr_{\sigma_1}(\sigma_2) := \dfrac{\partial}{\partial\sigma_2} r_{\sigma_1}(\sigma_2)$.

For any Lie group G associated Lie algebra $\mathfrak{g}$ can be considered. In this case the algebra is the set of left-invariant vector fields X_i which satisfy

$$dl_{\sigma_1}(\sigma_2)\, X_i(\sigma_2) = X_i(\sigma_1\circ\sigma_2)\,. \tag{12}$$

From relationship (12) it follows that any left-invariant vector field can be transformed via push operator in the same way as evaluating it at the given point using left translation. The Lie algebra $\mathfrak{g}$ basis consists of independent vector fields $X_1, X_2, \ldots, X_n$ and in the given coordinates it can be described using matrix notation by $X = [X_1\ X_2\ \ldots\ X_n]$, where $n = \dim G$ is dimension of the manifold G.

The other important differential operator is the adjoint operator $Ad:\ G\times\mathfrak{g}\to\mathfrak{g}$ which is given by

$$Ad(\sigma)\,\zeta := dl_\sigma\left(\sigma^{-1}\right) dr_{\sigma^{-1}}(e)\,\zeta = dr_{\sigma^{-1}}(\sigma)\, dl_\sigma(e)\,\zeta, \tag{13}$$

where $\zeta\in\mathfrak{g}$. Additionally, to shorten the notation the adjoint operator expressed in the algebra basis X is introduced

$$Ad^X(\sigma) := X(e)^{-1} Ad(\sigma)\, X(e)\,. \tag{14}$$

In this paper we also take advantage of the following relationships (cf. [12]):

$$\left(dl_{\sigma_1}(\sigma_2)\right)^{-1} = dl_{\sigma_1^{-1}}(\sigma_1\circ\sigma_2)\,, \tag{15}$$

$$\left(dr_{\sigma_1}(\sigma_2)\right)^{-1} = dr_{\sigma_1^{-1}}(\sigma_2\circ\sigma_1)\,, \tag{16}$$

$$dr_{\sigma_1\circ\sigma_2}(\sigma_3) = dr_{\sigma_2}(\sigma_3\circ\sigma_1)\, dr_{\sigma_1}(\sigma_3)\,. \tag{17}$$

Next, consider a general m-input control affine driftless system

$$\dot{h} = \sum_{i=1}^{m} X_i(h)\, u_i, \tag{18}$$

where h is the state, X_i denotes i^{th} vector field ($i = 1,2,\ldots,m$) and u_i is the control input. In some cases the system can be described on a Lie group. This possibility is in part related to the following lemma.

Lemma 1. Consider a control Lie algebra generated by basic vector fields of the driftless system given by (18). If this algebra is infinite dimensional system (18) cannot be defined on a Lie group.

Proof 1. Assume that the dimension of the control Lie algebra of system (18) is infinite (the dimension is calculated over $\mathbb{R}$). Next, by contradiction assume that there exists some finite-dimensional Lie group G which is associated with the given Lie algebra. It implies that $h\in G$ and there exists some group operation $\circ$ such that the vector fields X_i satisfy the left-invariance property, namely $dl_{h_1}(h_2)\, X_i(h_2) = X_i(h_1\circ h_2)$.

Taking into account that any Lie bracket of left-invariant vector fields is also left-invariant it follows that any vector field which belongs to the control Lie algebra is also left-invariant. Since it is assumed that the control Lie algebra is infinite dimensional over $\mathbb{R}$, there is infinite number of higher order vector fields which cannot be expressed as linear combination of lower dimensional vector fields. It follows that there is infinite number of left-invariant vector fields. However, for any finite dimensional Lie group from definition one can conclude that number of left-invariant vector fields is finite. It gives contradiction and shows that any Lie algebra of finite dimensional Lie group cannot be infinite. As a result system (18) with infinite dimensional Lie algebra is not a system on a Lie group.

Description of the control system using Lie group theory. Now we investigate the structure of kinematics (7) in more detail. Firstly, we consider that the system cannot be defined on any Lie group. Taking into account the control Lie algebra associated with system (7) (cf. [5]) it can be shown that the dimension of the Lie algebra is infinite over $\mathbb{R}$ for any $N = 1,2,\ldots$. Hence, in view of Lemma 1 one can conclude that the system is not invariant. In spite of this difficulty it was shown in [6] that the system has specific structure which gives possibility to take advantage of Lie group theory with respect to some part of it. Namely, the first three variables of $\overline{q}$ can be seen as coordinate variables specifying position and orientation on the plane. Consequently, defining $g := [x\ y\ \theta]^\top$, the following decomposition of vector $\overline{q}$ can be introduced

$$\overline{q} := \begin{bmatrix} g \\ \psi \end{bmatrix}. \tag{19}$$

Next, we consider g as an element of Lie group $G^E \sim \mathrm{SE}(2)$ (to be more precise we assume vector representation of the Special Euclidean Group) with the operation given by

$$g_1 \bullet g_2 := \begin{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + R(\theta_1)\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \\ \theta_1+\theta_2 \end{bmatrix}, \tag{20}$$

where $g_i := [x_i\ y_i\ \theta_i]^\top \in G^E$ for $i = 1,2$ and $R\in\mathrm{SO}(2)$ is an element of the Special Orthogonal Group determining a rotation on the plane. The neutral element of the group G^E satisfies $e_g := [0\ 0\ 0]^\top$.

Next, we chose canonical basis of the Lie algebra associated to the given group $G^E \sim \mathrm{SE}(2)$ which consists of the following vector fields evaluated at g

$$X_1^E(g) := \begin{bmatrix}\cos\theta\\ \sin\theta\\ 0\end{bmatrix}, \qquad X_2^E(g) := \begin{bmatrix}-\sin\theta\\ \cos\theta\\ 0\end{bmatrix}, \qquad X_3^E := \begin{bmatrix}0\\0\\1\end{bmatrix}. \tag{21}$$

It can be verified that X_i^E, with $i = 1, 2, 3$ are left invariant vector fields, namely the following relationship is met: $dl_{g_1}(g_2)\, X_i^E(g_2) = X_i^E(l_{g_1}(g_2))$. Further, the basis of Lie algebra of Lie group G^E is described by matrix $X^E(g) := \begin{bmatrix}X_1^E(g) & X_2^E(g) & X_3^E\end{bmatrix}$. Referring to (7) and using basis X^E the following kinematics with respect to g can be derived

$$\dot{g} = X^E(g)\, C_g(\psi)\, \eta, \tag{22}$$

where

$$C_g(\psi) := \begin{bmatrix}1 & 0\\ 0 & 0\\ \psi_1 & 0\end{bmatrix}.$$

It is worth to emphasize that in spite of the fact that the overall system (7) is not defined on a Lie group (namely it is non-invariant) one can still take advantage of Lie group theory with respect to some part of it. Then control system (8) can be seen as the composition of two subsystems given by

$$\dot{g} = X^E(g)\, C_g(\psi)\, \eta, \tag{23}$$

$$\dot{\psi} = C_\psi(\psi)\, \eta, \tag{24}$$

where

$$C_\psi := \begin{bmatrix}\psi_2 & 0\\ \psi_3 & 0\\ 0 & 1\end{bmatrix}.$$

4. Control design

4.1. General description of the control task. In order to define the reference motion of the vehicle we consider a smooth reference trajectory $q_r := \begin{bmatrix}g_r^\top & \varphi_r^\top\end{bmatrix} = [x_r\ y_r\ \theta_r\ \varphi_{r1}\ \varphi_{r2}\ \varphi_{r3}]^\top \in \overline{\mathcal{Q}}$ specified for the last trailer and the internal configuration of system Σ with $N = 3$. Referring to Subsec. 3.2 it is assumed that the reference motion of the last trailer is governed by

$$\dot{g}_r = X^E(g_r)\, u_r, \tag{25}$$

where $u_r = [v_{rx}\ v_{ry}\ v_{r\theta}]^\top$ is the reference signal with v_{rx}, v_{ry} being the longitudinal and lateral velocities and $v_{r\theta}$ describing the angular velocity of the target expressed with respect to its local frame. The chosen reference trajectory can be admissible or even non-admissible, namely it can be a solution of the reference kinematics based on Eq. (2) with $N = 3$ or not. For example, defining the non-admissible trajectory one can assume that the reference target describing last trailer moves sideways (namely $v_{ry} \neq 0$) and nonholonomic constraints are violated.

To quantify the control objective the following tracking error is defined $q_e = [x_e\ y_e\ \theta_e\ \varphi_{e1}\ \varphi_{e2}\ \varphi_{e3}]^\top := q - q_r$. Then, the control problem can be stated as follows:

Problem 1. (Main control problem). Find bounded smooth kinematic control inputs, v_1 and v_2, of the tractor such that for any smooth reference trajectory q_r and $q_e(0) \in B_{0,\epsilon_0}$, where B_{0,ϵ_0} denotes the hyperball with center at the origin and radius $\epsilon_0 > 0$, configuration error is bounded and asymptotically converges to the neighborhood with arbitrarily chosen radius $\epsilon > 0$, while

$$\sup_{t\geq 0} \|q_e(t)\| < \infty, \qquad \lim_{t\to\infty} q_e(t) \in B_{0,\epsilon} \tag{26}$$

and configuration q stays in the restricted configuration set, namely

$$q(0) \in \overline{\mathcal{Q}} \Rightarrow \forall t > 0, \quad q(t) \in \overline{\mathcal{Q}}. \tag{27}$$

Defining the control problem it is assumed that the configuration error does not necessary tend to zero. It means that the requirement of asymptotic convergence is relaxed and only the convergence to the neighborhood of zero is taken into account (on the other hand asymptotic convergence is not completely excluded at least for some class of admissible reference trajectories). It is important to recall that for the non-admissible reference trajectory only the approximate tracking can be considered.

Following the assumption that $q_r \in \overline{\mathcal{Q}}$ and motivated by the description presented in Subsec. 3.1 we introduce the auxiliary reference trajectory

$$\overline{q}_r := \begin{bmatrix}g_r\\ \psi_r\end{bmatrix} \in G^E \times \mathbb{R}^3, \tag{28}$$

where $\psi_r = [\psi_{r1}\ \psi_{r2}\ \psi_{r3}]^\top \overset{(5)}{:=} \Psi(\varphi_r)$ with $\Psi(\cdot)$ being defined by (9).

Since the current and reference configurations of the last trailer denoted by g and g_r, respectively, can be seen as the elements of the group G^E one can define tracking error using group operation as follows

$$\tilde{g} := g_r^{-1} \bullet g = l_{g_r^{-1}}(g). \tag{29}$$

Taking the time derivative of (29) and utilizing the left-invariance property of system (22) one can obtain (the details are given in the Appendix)

$$\dot{\tilde{g}} = X^E(\tilde{g}) \left(C_g(\psi)\, \eta - Ad^{X^E}(\tilde{g}^{-1})\, u_r \right), \tag{30}$$

where $Ad^{X^E}(g)$ is the adjoint operator evaluated at point g and expressed in the base X^E of the Lie algebra. Then, the kinematics related to the internal variables ψ governed by Eq. (24) remains unchanged.

4.2. Control solution

Coordinate and input transformations with open-loop error dynamics derivation. The control solution considered in this paper takes advantage of the coordinate and input transformations which give possibility to transform non-invariant control system to invariant one. Here, derivation of the controller is based on two input IV-order chained system defined as follows

$$\dot{\xi} = X_1^C(\xi)\, w_1 + X_2^C w_2, \tag{31}$$

where $\xi = [\xi_1 \dots \xi_6]^\top$ is the coordinate vector, w_1 and w_2 denote the control inputs, while X_1^C and X_2^C are vector fields given by

$$\begin{aligned} X_1^C(\xi) &:= [1\ 0\ \xi_2\ \xi_3\ \xi_4\ \xi_5]^\top, \\ X_2^C &:= [0\ 1\ 0\ \dots 0]^\top. \end{aligned} \tag{32}$$

In fact, systems (31) and (8) can be seen as equivalent control systems since they can be transformed (at least locally) to each other using the coordinate and input maps.

Referring to results given in the Appendix the coordinate transformation (114) is local and it is well defined for $|\theta| \neq \frac{\pi}{2}$. This limitation is too restrictive since it imposes that orientation of the last trailer could not be arbitrary and some configurations would become no feasible. In order to overcome this difficulty, the coordinate transformation based on (114) is applied with respect to variable $\tilde{g}$ instead of g. Then, the given restriction affects orientation error $\left(|\tilde{\theta}| \neq \frac{\pi}{2}\right)$ that is clearly more acceptable.

Taking into account definitions (30), (24) the coordinate transformation (114) is redefined as follows

$$\xi := \Phi(\tilde{g}, \psi) = \begin{bmatrix} \tilde{x} \\ \dfrac{3\psi_1^3 + 12\psi_1^3 \sin^2\tilde{\theta} + 5\psi_1\psi_2 \sin\left(2\tilde{\theta}\right) + \psi_3\cos^2\tilde{\theta}}{\cos^7\tilde{\theta}} \\ \dfrac{3\psi_1^2 \tan\tilde{\theta} + \psi_2}{\cos^4\tilde{\theta}} \\ \dfrac{\psi_1}{\cos^3\tilde{\theta}} \\ \tan\tilde{\theta} \\ \tilde{y} \end{bmatrix} \tag{33}$$

and the transformed system becomes

$$\begin{aligned} \dot{\xi} &= \frac{\partial\Phi(\tilde{g},\psi)}{\partial\tilde{g}}\dot{\tilde{g}} + \frac{\partial\Phi(\tilde{g},\psi)}{\partial\psi}\dot{\psi} \\ &\overset{(30),(24)}{=} \left(\frac{\partial\Phi(\tilde{g},\psi)}{\partial\tilde{g}} X^E(\tilde{g})\, C_g(\psi) + \frac{\partial\Phi(\tilde{g},\psi)}{\partial\psi} C_\psi(\psi)\right)\eta \\ &\quad - \frac{\partial\Phi(\tilde{g},\psi)}{\partial\tilde{g}} X^E(\tilde{g})\, Ad^{X^E}\left(\tilde{g}^{-1}\right) u_r \\ &= X_1^C(\xi)\, w_1 + X_2^C w_2 + p(\tilde{g},\psi,u_r), \end{aligned} \tag{34}$$

with

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} := U\left(\tilde{\theta},\psi\right)\eta \overset{(115)}{=} \begin{bmatrix} \cos\tilde{\theta} & 0 \\ \frac{\partial\xi_2(\tilde{\theta},\psi)}{\partial\tilde{\theta}}\psi_1 + \frac{\partial\xi_2(\tilde{\theta},\psi)}{\partial\psi_1}\psi_2 + \frac{\partial\xi_2(\tilde{\theta},\psi)}{\partial\psi_2}\psi_3 & \frac{\partial\xi_2(\tilde{\theta},\psi)}{\partial\psi_3} \end{bmatrix}\eta \tag{35}$$

being the control input, while

$$p(\tilde{g},\psi,u_r) = -\frac{\partial\Phi(\tilde{g},\psi)}{\partial\tilde{g}} X^E(\tilde{g})\, Ad^{X^E}\left(\tilde{g}^{-1}\right) u_r = \begin{bmatrix} -v_{rx} + \tilde{y}v_{r\theta} \\ -\dfrac{15\psi_1^3\tan\tilde{\theta}\left(3+4\sin^2\tilde{\theta}\right) + 10\psi_1\psi_2\left(5\sin^2\tilde{\theta}+1\right) + \frac{5}{2}\psi_3\sin\left(2\tilde{\theta}\right)}{\cos^7\tilde{\theta}} v_{r\theta} \\ -\dfrac{3\psi_1^2\left(4\sin^2\tilde{\theta}+1\right) + 2\psi_2\sin\left(2\tilde{\theta}\right)}{\cos^6\tilde{\theta}} v_{r\theta} \\ -\dfrac{3\psi_1\sin\left(\tilde{\theta}\right)}{\cos^4\tilde{\theta}} v_{r\theta} \\ -\dfrac{1}{\cos^2\tilde{\theta}} v_{r\theta} \\ -v_{ry} - \tilde{x}v_{r\theta} \end{bmatrix} \tag{36}$$

denotes the drift. It is worth to note that system (34) correspond to the IV-order chained system only for $u_r \equiv 0$, namely in the case when a constant reference point q_r is assumed. In the more general case system (34) is not a driftless system anymore, however its vector fields associated with inputs are the same as for system (31).

From (33) it is clear that auxiliary configuration ξ reflects configuration error $\tilde{g} \in G$ and the current internal configuration of the vehicle. Hence, in order to satisfy the control objective and to include also the error defined with respect to the internal variables it is assumed that ξ converges to some neighborhood of the reference trajectory given by

$$\xi_r := \Phi(e_g, \psi_r) = \left[0\ \ 3\psi_{r1}^3 + \psi_{r3}\ \ \psi_{r2}\ \ \psi_{r1}\ \ 0\ \ 0\right]^\top. \tag{37}$$

To be more specific, the following implication can be considered

$$(\xi \to \xi_r) \Rightarrow ((\tilde{g} \to 0) \text{ and } (\psi \to \psi_r)). \tag{38}$$

Consequently, in order to define the transformed tracking error we take advantage of the fact that system (31) is defined on some Lie group G^C with the group operation given by Eq. (117) in the Appendix. Then, we have

$$\tilde{\xi} = \xi_r^{-1} \star \xi, \tag{39}$$

with ξ_r^{-1} in view of (37) being $\xi_r^{-1} = -\xi_r$. Taking the time derivative of (39) the following open-loop dynamics is obtained

$$\begin{aligned} \dot{\tilde{\xi}} &= dl_{\xi_r^{-1}}(\xi)\,\dot{\xi} + dr_\xi\left(\xi_r^{-1}\right)\frac{d}{dt}\xi_r^{-1} = X_1^C\left(\tilde{\xi}\right) w_1 \\ &\quad + X_2^C w_2 + dl_{\xi_r^{-1}}(\xi)\, p + dr_\xi\left(\xi_r^{-1}\right)\frac{d}{dt}\xi_r^{-1}, \end{aligned} \tag{40}$$

where $dl_{\xi_a}(\xi_b)$ and $dr_{\xi_a}(\xi_b)$ describe differential operators defined by Eqs. (118) and (119) in the Appendix while the time derivative of the term associated to the reference trajectory can be written as

$$\frac{d}{dt}\xi_r^{-1} = -\dot{\xi}_r = -\frac{\partial\Phi\left(e,\psi_r\right)}{\partial\psi_r}\dot{\psi}_r = -\begin{bmatrix} 0 \\ 9\psi_{r1}^2\dot{\psi}_{r1} + \dot{\psi}_{r3} \\ \dot{\psi}_{r2} \\ \dot{\psi}_{r1} \\ 0 \\ 0 \end{bmatrix}. \tag{41}$$

In order to facilitate design of the controller, error dynamics (40) is rewritten as follows

$$\dot{\tilde{\xi}} = X^C\left(\tilde{\xi}\right)\left(C_\xi w + \tilde{p}\right) \tag{42}$$

where

$$C_\xi := \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{bmatrix}^\top \in \mathbb{R}^{6\times 2} \tag{43}$$

and

$$\tilde{p} = \left(X^C\left(\tilde{\xi}\right)\right)^{-1}\left(dl_{\xi_r^{-1}}\left(\xi\right)p + dr_\xi\left(\xi_r^{-1}\right)\frac{d}{dt}\xi_r^{-1}\right), \tag{44}$$

with X^C being a basis of Lie algebra associated to Lie group G^C and defined in the Appendix.

Transverse function design. The control approach considered here is based on the transverse functions which are designed for system (31). Now we recall a definition of the transverse function.

Definition 1. Let $f^\xi : \mathbb{T}^k \times (0,\varepsilon_{\max}) \to G^C$, with $k = 4,5,\dots$ and $\varepsilon_{\max} > 0$, be a smooth function which satisfies

$$\begin{gathered} \forall\alpha\in\mathbb{T}^k,\ \varepsilon\in(0,\varepsilon_{\max}) \\ \operatorname{rank}\left[X_1^C\left(f^\xi\left(\alpha,\varepsilon\right)\right) \quad X_2^C\left(f^\xi\left(\alpha,\varepsilon\right)\right) \quad -\frac{\partial f^\xi\left(\alpha,\varepsilon\right)}{\partial\alpha}\right] \\ = \dim G^C = 6 \end{gathered} \tag{45}$$

and

$$\forall\alpha\in\mathbb{T}^k \lim_{\varepsilon\to 0^+}\left\|f^\xi\left(\alpha,\varepsilon\right)\right\| = 0. \tag{46}$$

Then function f^ξ is transverse with respect to vector fields of control system (31) and is centered at the neutral element e_ξ of group G^C.

According to the result given in [11] a transverse function exists for any driftless systems satisfying the LARC condition. Taking into account the general formula introduced in [11] one can formally calculate transverse function by finding a flow of differential equation evaluated at time $t = 1$. Moreover, due to the existence of the Lie group for the particular control Lie algebra the derivation of the transverse function is simplified, [13]. For system (31) one can consider the following definition of the transverse function

$$f^\xi\left(\alpha\right) := {}^4f^\xi\left(\alpha_4\right)\star{}^3f^\xi\left(\alpha_3\right)\star{}^2f^\xi\left(\alpha_2\right)\star{}^1f^\xi\left(\alpha_1\right) \tag{47}$$

where

$$\begin{aligned} {}^1f^\xi\left(\alpha_1\right) &= \exp\left(X_1^C\varepsilon\beta_{1,1}\sin\alpha_1 + X_2^C\varepsilon\beta_{1,2}\cos\alpha_1\right), \\ {}^2f^\xi\left(\alpha_2\right) &= \exp\left(X_1^C\varepsilon\beta_{2,1}\sin\alpha_2 + X_3^C\varepsilon^2\beta_{2,2}\cos\alpha_2\right), \end{aligned} \tag{48}$$

$$\begin{aligned} {}^3f^\xi\left(\alpha_3\right) &= \exp\left(X_1^C\varepsilon\beta_{3,1}\sin\alpha_3 + X_4^C\varepsilon^3\beta_{3,2}\cos\alpha_3\right), \\ {}^4f^\xi\left(\alpha_4\right) &= \exp\left(X_1^C\varepsilon\beta_{4,1}\sin\alpha_4 + X_5^C\varepsilon^4\beta_{4,2}\cos\alpha_4\right) \end{aligned} \tag{49}$$

are basic components of the transverse function, such that ${}^if^\xi$ is responsible for generating of the direction associated to vector field X_{i+2}^C (namely the Lie bracket which is necessary to ensure the controllability of the chained system – refer to (116)). It is assumed that functions ${}^if^\xi$ are parametrized by the set of eight coefficients $\beta_{i,j}\in\mathbb{R}$, where $i = 1,2,3,4$ and $j = 1,2$, and $\varepsilon > 0$. Calculating each component function ${}^if^\xi$ one has (cf. the result given in [13, 6])

$$\begin{gathered} {}^1f^\xi = \begin{bmatrix} \varepsilon\beta_{1,1}\sin\alpha_1 \\ \varepsilon\beta_{1,2}\cos\alpha_1 \\ \frac{\varepsilon^2\beta_{1,1}\beta_{1,2}}{4}\sin\left(2\alpha_1\right) \\ \frac{\varepsilon^3\beta_{1,1}^2\beta_{1,2}}{6}\sin^2\alpha_1\cos\alpha_1 \\ \frac{\varepsilon^4\beta_{1,1}^3\beta_{1,2}}{24}\sin^3\alpha_1\cos\alpha_1 \\ \frac{\varepsilon^5\beta_{1,1}^4\beta_{1,2}}{120}\sin^4\alpha_1\cos\alpha_1 \end{bmatrix}, \\ {}^2f^\xi = \begin{bmatrix} \varepsilon\beta_{2,1}\sin\alpha_2 \\ 0 \\ \varepsilon^2\beta_{2,2}\cos\alpha_2 \\ \frac{\varepsilon^3\beta_{2,1}\beta_{2,2}}{4}\sin\left(2\alpha_2\right) \\ \frac{\varepsilon^4\beta_{2,1}^2\beta_{2,2}}{6}\sin^2\alpha_2\cos\alpha_2 \\ \frac{\varepsilon^5\beta_{2,1}^3\beta_{2,2}}{24}\sin^3\alpha_2\cos\alpha_2 \end{bmatrix}, \end{gathered} \tag{50}$$

and

$$\begin{gathered} {}^3f^\xi = \begin{bmatrix} \varepsilon\beta_{3,1}\sin\alpha_3 \\ 0 \\ 0 \\ \varepsilon^3\beta_{3,2}\cos\alpha_3 \\ \frac{\varepsilon^4\beta_{3,1}\beta_{3,2}}{4}\sin\left(2\alpha_3\right) \\ \frac{\varepsilon^5\beta_{3,1}^2\beta_{3,2}}{6}\sin^2\alpha_3\cos\alpha_3 \end{bmatrix}, \\ {}^4f^\xi = \begin{bmatrix} \varepsilon\beta_{4,1}\sin\alpha_4 \\ 0 \\ 0 \\ 0 \\ \varepsilon^4\beta_{4,2}\cos\alpha_4 \\ \frac{\varepsilon^5\beta_{4,1}\beta_{4,2}}{4}\sin\left(2\alpha_4\right). \end{bmatrix} \end{gathered} \tag{51}$$

The example of calculations are presented in the Appendix.

Parameter ε can be seen as a scaling coefficient. According to the result given in [13] for the chained system the transverse function f^ξ is well defined for any $\varepsilon > 0$ (it means that upper bound $\varepsilon_{\max}$ introduced in definition can be made arbitrary large). To be more specific, it can be shown that function f^ξ satisfies

$$f^\xi(\alpha, \varepsilon) = \Delta_\varepsilon^r\left(f^\xi(\alpha, 1)\right) = D(\varepsilon) f^\xi(\alpha, 1), \tag{52}$$

where Δ_ε^r stands for the dilation operator with $r = [1, 1, 2, 3, 4, 5]$ being the weights (for the definition of homogeneity and dilation the reader may refer to [9]) while

$$D(\varepsilon) := \begin{bmatrix} \varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon^5 \end{bmatrix} \in \mathbb{R}^{6\times 6} \tag{53}$$

is a scaling matrix which invertible for any $\varepsilon > 0$. Usually parameter ε is assumed to be a constant (or piecewise constant) coefficient. However, in this paper we assume that in the general case ε can be time-varying. This gives a possibility to scale the transverse function during the control process in order to improve the transient states. Then the time derivative of f^ξ can be expressed as follows

$$\dot{f}^\xi = \frac{\partial f^\xi}{\partial \alpha}\dot{\alpha} + \frac{\partial f^\xi}{\partial \varepsilon}\dot{\varepsilon}, \tag{54}$$

where derivatives $\frac{\partial f^\xi}{\partial \alpha}$ and $\frac{\partial f^\xi}{\partial \varepsilon}$ are given by Eqs. (124) and (125) in the Appendix. Alternatively, taking advantage of Eq. (52) derivative $\frac{\partial f^\xi}{\partial \varepsilon}$ can be defined in a simpler way, namely

$$\frac{\partial f^\xi}{\partial \varepsilon} = \frac{\partial D(\varepsilon)}{\partial \varepsilon} f^\xi(\alpha, 1). \tag{55}$$

To facilitate the control design and analysis it is convenient to express the derivatives of function f^ξ in basis of Lie algebra X^C such that

$$\dot{f}^\xi = X^C\left(f^\xi\right)\left(A_\alpha \dot{\alpha} + A_\varepsilon \dot{\varepsilon}\right), \tag{56}$$

where

$$\begin{aligned} A_\alpha &:= X^C\left(f^\xi\right)^{-1} \frac{\partial f^\xi}{\partial \alpha} \in \mathbb{R}^{6\times 4}, \\ A_\varepsilon &:= X^C\left(f^\xi\right)^{-1} \frac{\partial f^\xi}{\partial \varepsilon} \in \mathbb{R}^{6}. \end{aligned} \tag{57}$$

In particular, the matrix notation introduced in (56) can be convenient for checking the transversality condition. Consequently, transversality condition (45) can be rewritten as follows

$$\begin{bmatrix} X_1^C\left(f^\xi\right) & X_2^C\left(f^\xi\right) & -X^C\left(f^\xi\right) A_\alpha \end{bmatrix} = X^C\left(f^\xi\right) \overline{C}_\xi, \tag{58}$$

with

$$\overline{C}_\xi := \begin{bmatrix} C_\xi & \begin{matrix} -A_1 \\ -A_2 \end{matrix} \end{bmatrix} \in \mathbb{R}^{6\times 6}, \tag{59}$$

where $A_1 \in \mathbb{R}^{2\times 4}$ and $A_2 \in \mathbb{R}^{4\times 4}$ satisfies $\begin{bmatrix} A_1^\top & A_2^\top \end{bmatrix} = A_\alpha^\top$. Assuming that f^ξ is the transverse function, matrix $\overline{C}_\xi$ is invertible $\forall \alpha \in \mathbb{T}^4$. Moreover, taking into account the particular structure of matrix C_ξ the following implication can be concluded: $(\det A_2 \neq 0) \Rightarrow \left(\det \overline{C}_\xi \neq 0\right) \Rightarrow$ transversality condition is met.

It is important to emphasize that function (47) with (50) and (51) satisfies transversality condition (45) only with a proper selection of parameters $\beta_{i,j}$. Namely, a given set of $\beta_{i,j}$ can be verified by checking of the determinant of matrix A_2 instead of verifying the rank condition for a higher dimensional matrix defined by Eq. (45). Taking into account that parameter ε does not have any impact on the transversality condition, the determinant of A_2 can be verified for any arbitrary chosen positive value of ε (typically it is assumed that $\varepsilon = 1$).

The selection of parameters of function f^ξ and utilizing scaling factor ε is not a unique way to shape the transverse function. In [12] an extension of the transverse function is proposed that gives a possibility to change norm of the function to zero without violating the transversality condition (namely for $\varepsilon > 0$). Referring to the Lie structure once again the modified transverse function can be defined as follows

$$\overline{f}^\xi(\alpha, \alpha_r, \varepsilon) = f^\xi(\alpha_r, \varepsilon)^{-1} \star f^\xi(\alpha, \varepsilon), \tag{60}$$

where $\alpha_r \in \mathbb{T}^4$ is an auxiliary variable. In particular based on (60) the following relationship can be taken into account

$$\lim_{\alpha \to \alpha_r} \overline{f}^\xi(\alpha, \alpha_r, \varepsilon) = 0. \tag{61}$$

It can be proved (see [12]) that $\overline{f}^\xi$ defined by Eq. (60) is a transverse function if f^ξ satisfies transversality condition given by (45). Moreover, function $\overline{f}^\xi$ still satisfies homogeneity property defined by Eq. (52).

For the modified function (60) calculating the time derivative yields in

$$\dot{\overline{f}}^\xi = \frac{\partial \overline{f}^\xi}{\partial \alpha}\dot{\alpha} + \frac{\partial \overline{f}^\xi}{\partial \alpha_r}\dot{\alpha}_r + \frac{\partial \overline{f}^\xi}{\partial \varepsilon}\dot{\varepsilon}. \tag{62}$$

Taking into account definition (60) the time derivative given by (62) can be rewritten as follows

$$\begin{aligned} \dot{\overline{f}}^\xi &= dl_{f^\xi(\alpha_r,\varepsilon)^{-1}}\left(f^\xi(\alpha, \varepsilon)\right) \dot{f}^\xi \\ &+ dr_{f^\xi(\alpha,\varepsilon)}\left(f^\xi(\alpha_r, \varepsilon)^{-1}\right) \frac{d}{dt}\left(f^\xi(\alpha_r, \varepsilon)^{-1}\right) \\ &= dl_{f^\xi(\alpha_r,\varepsilon)^{-1}}\left(f^\xi(\alpha, \varepsilon)\right) \dot{f}^\xi(\alpha, \varepsilon) \\ &- dr_{f^\xi(\alpha,\varepsilon)}\left(f^\xi(\alpha_r, \varepsilon)^{-1}\right) dr_{f^\xi(\alpha_r,\varepsilon)^{-1}}(e_\xi) \\ &\cdot dl_{f^\xi(\alpha_r,\varepsilon)^{-1}}\left(f^\xi(\alpha_r, \varepsilon)\right) \dot{f}^\xi(\alpha_r, \varepsilon). \end{aligned} \tag{63}$$

Comparing (62) and (63) yields in

$$\frac{\partial \overline{f}^\xi}{\partial \alpha} = dl_{f^\xi(\alpha_r,\varepsilon)^{-1}}\left(f^\xi(\alpha, \varepsilon)\right) \frac{\partial}{\partial \alpha} f^\xi(\alpha, \varepsilon), \tag{64}$$

$$\begin{aligned}\frac{\partial \overline{f}^{\xi}}{\partial \varepsilon} &= dl_{f^{\xi}(\alpha_r,\varepsilon)^{-1}}\left(f^{\xi}(\alpha,\varepsilon)\right)\frac{\partial}{\partial \varepsilon}f^{\xi}(\alpha,\varepsilon)\\ &-dr_{f^{\xi}(\alpha,\varepsilon)}\left(f^{\xi}(\alpha_r,\varepsilon)^{-1}\right)dr_{f^{\xi^{-1}}(\alpha_r,\varepsilon)}(e_{\xi})\\ &\cdot dl_{f^{\xi^{-1}}(\alpha_r,\varepsilon)}\left(f^{\xi}(\alpha_r,\varepsilon)\right)\frac{\partial}{\partial \varepsilon}f^{\xi}(\alpha_r,\varepsilon)\end{aligned} \tag{65}$$

$$\begin{aligned}\frac{\partial \overline{f}^{\xi}}{\partial \alpha_r} &= -dr_{f^{\xi}(\alpha,\varepsilon)}\left(f^{\xi}(\alpha_r,\varepsilon)^{-1}\right)dr_{f^{\xi^{-1}}(\alpha_r,\varepsilon)}(e_{\xi})\\ &\cdot dl_{f^{\xi^{-1}}(\alpha_r,\varepsilon)}\left(f^{\xi}(\alpha_r,\varepsilon)\right)\frac{\partial}{\partial \varepsilon}f^{\xi}(\alpha_r,\varepsilon).\end{aligned} \tag{66}$$

It is worth to mention that in order to calculate $\frac{\partial \overline{f}^{\xi}}{\partial \varepsilon}$ instead of using Eq. (65) one can also refer to the dilation operator. Following Eq. (55) one obtains

$$\frac{\partial \overline{f}^{\xi}}{\partial \varepsilon} = \frac{\partial D(\varepsilon)}{\partial \varepsilon}\overline{f}^{\xi}(\alpha,\alpha_r,1). \tag{67}$$

Taking advantage of Lie algebra basis Eq. (62) can be rewritten as follows

$$\dot{\overline{f}}^{\xi} = X^C\left(\overline{f}^{\xi}\right)\left(A_{\alpha}\dot{\alpha} + A_{\alpha_r}\dot{\alpha}_r + A_{\varepsilon}\dot{\varepsilon}\right), \tag{68}$$

where A_{α} and A_{ε} are defined similar as in Eq. (57), while

$$A_{\alpha_r} := X^C\left(\overline{f}^{\xi}\right)^{-1}\frac{\partial \overline{f}^{\xi}}{\partial \alpha_r} \in \mathbb{R}^{6\times 4}. \tag{69}$$

Control law design. The idea of control considered in this paper is based on tracking an auxiliary trajectory which evolves on group G^C and is contained in some neighborhood of the origin (namely, the neutral element of the group). This trajectory is directly defined by the tranverse function. The auxiliary tracking error can be defined using the Lie group structure as follows

$$z_{\xi} := \tilde{\xi} \star \overline{f}^{\xi^{-1}}. \tag{70}$$

In order to simplify the derivation of error dynamics Eq. (70) can be rewritten as

$$\tilde{\xi} = z_{\xi} \star \overline{f}^{\xi}. \tag{71}$$

Taking the time derivative of (71) and using Lie group and Lie algebra operators one obtains

$$\dot{\tilde{\xi}} = dl_{z_{\xi}}\left(\overline{f}^{\xi}\right)\dot{\overline{f}}^{\xi} + dr_{\overline{f}^{\xi}}(z_{\xi})\,\dot{z}_{\xi}. \tag{72}$$

Calculating $\dot{z}_{\xi}$ gives (for details see the Appendix)

$$\dot{z}_{\xi} = X^C(z_{\xi})\,Ad^{X^C}\left(\overline{f}^{\xi}\right)\left(\overline{C}_{\xi}\overline{w} + \tilde{p} - A_{\varepsilon}\dot{\varepsilon} - A_{\alpha_r}\dot{\alpha}_r\right), \tag{73}$$

where

$$\overline{w} := \left[w^{\top}\ \dot{\alpha}^{\top}\right]^{\top} \in \mathbb{R}^6 \tag{74}$$

is the extended control input with $\dot{\alpha}$ being the virtual input determining the evolution of the transverse functions.

Based on the open-loop dynamics (73) and considering that matrices $X^C(z_{\xi})$, $Ad^{X^C}\left(\overline{f}^{\xi}\right)$, and $\overline{C}_{\xi}$ are invertible one can formulate the following control algorithm.

Proposition 1 (Classic control feedback). The feedback given as follows

$$\overline{w} = \overline{C}_{\xi}^{-1}\left(Ad^{X^C}\left(\overline{f}^{\xi^{-1}}\right)X^C(z_{\xi})^{-1}Kz_{\xi} - \tilde{p} + A_{\varepsilon}\dot{\varepsilon} + A_{\alpha_r}\dot{\alpha}_r\right), \tag{75}$$

where $K \in \mathbb{R}^2$ is a Hurwitz-stable matrix, applied to system (73) ensures its exponential stabilization, namely

$$\forall t \geq 0,\ \|z_{\xi}(t)\| \leq \|z_{\xi}(0)\|\exp(-ct), \tag{76}$$

where $c > 0$ is dependent on the eigenvalues of matrix K, and practical stabilization such that

$$\lim_{t\to\infty}\|\tilde{\xi}(t)\| \leq \epsilon_{\xi}, \tag{77}$$

with $\epsilon_{\xi} \geq \|\overline{f}^{\xi}\|$ being the radius of the neighborhood of zero in the steady state, which is dependent on parameters of the transverse functions and can be made arbitrary small.

In order to apply the given control law one has to calculate the inverse of matrix $\overline{C}_{\xi}$ given by Eq. (59). The calculation can be simplified taking into account the particular structure of C_{ξ} – cf. (43). Considering the inverse of the block square matrix the following relationship can be derived

$$\overline{C}_{\xi}^{-1} = \begin{bmatrix} I_{2\times 2} & -A_1A_2^{-1} \\ 0_{4\times 2} & -A_2^{-1}\end{bmatrix}. \tag{78}$$

It is worth recalling that in view of Eq. (74) the control signal $\overline{w}$ consists of control input w and virtual input $\dot{\alpha}$. Referring to the control task defined with respect to the original system the real input v that governs the tractor motion has the most important meaning. Moreover $\dot{\alpha}$ can be seen as only an auxiliary signal that could be subordinated to the other control objectives. Taking into account this property one can consider the design of the control loop based on the optimization technique.

Following [14] we introduce the performance index

$$J(\overline{v}) = \frac{1}{2}\overline{v}^{\top}W_1\overline{v}, \tag{79}$$

where $\overline{v} := \left[v^{\top}\ \dot{\alpha}^{\top}\right]^{\top} \in \mathbb{R}^6$ is the modified extended control input (cf. definition of $\overline{w}$) and $W_1 \in \mathbb{R}^{6\times 6}$ denotes symmetric positive definite matrix. The optimization problem is solved assuming that $\|z_{\xi}\|$ converges to zero exponentially, namely the following constraint is satisfied

$$z_{\xi}^{\top}\dot{z}_{\xi} + z_{\xi}^{\top}W_2z_{\xi} = 0, \tag{80}$$

where $W_2 \in \mathbb{R}^{6\times 6}$ is the positive definite square matrix.

Proposition 2 (Optimal control feedback). The control law defined as follows

$$\overline{v} = -\frac{z_{\xi}^{\top}W_2z_{\xi} + z_{\xi}^{\top}H\left(\tilde{p} - A_{\varepsilon}\dot{\varepsilon} - A_{\alpha_r}\dot{\alpha}_r\right)}{z_{\xi}^{\top}QW_1{}^{-1}Q^{\top}z_{\xi}}W_1{}^{-1}Q^{\top}z_{\xi} \tag{81}$$

with Q being the matrix defined by Eq. (135), and applied to system (2) with $N = 3$ ensures exponential convergence of

z_ξ and optimal control effort at each time instant in the sense of minimization of the performance index (79).

The derivation of optimal control is given in the Appendix.

It can be noted that the optimal control law given by (81) is quite different from that proposed in [12]. Firstly, the original (not transformed) control input is taken into account (cf. [6]). Secondly, apart from the components related to the static case (namely when $\dot{\varepsilon} = 0$, $\dot{\alpha}_r = 0$ and $\tilde{p} = 0$), the drift terms are considered in the optimization problem (a similar approach can be found in [14]).

The significant drawback of the optimal control law is related to the fact that Eq. (81) is not well-defined at the origin. Namely, considering that $\lim_{z_\xi \to 0} \overline{v} = 0/0$ one can conclude that the solution may be not determined properly. In particular, this becomes a critical issue for the numerical implementation of the algorithm. Namely, for $\|z_\xi\| \leq \sigma$, where $\sigma > 0$ is some positive constant, a numerical division of two small numbers may not give accurate and robust results.

In order to overcome this difficulty one can use a suboptimal solution (cf. Artus *et. al.* [14]) that is based on combination of the classic controller with the optimal one. In order to facilitate the description referring to (75), (81) and (135) we introduce

$$\nu_s^c = Q^{-1} K z_\xi,\ \nu_d^c = -\overline{T}^{-1}\overline{C}_\xi^{-1}\left(\tilde{p} - A_\varepsilon \dot{\varepsilon} - A_{\alpha_r}\dot{\alpha}_r\right), \quad (82)$$

where $\overline{T}$ is given by Eq. (132) and

$$\begin{aligned} \nu_s^o &= -z_\xi^\top W_2 z_\xi {W_1}^{-1} Q^\top z_\xi, \\ \nu_d^o &= -z_\xi^\top H\left(\tilde{p} - A_\varepsilon \dot{\varepsilon} - A_{\alpha_r}\dot{\alpha}_r\right) {W_1}^{-1} Q^\top z_\xi, \end{aligned} \quad (83)$$

where superscript “s” and “d” are used in order to describe the static and dynamic terms of the controller.

Proposition 3 (Suboptimal control law). The control law defined as follows

$$\overline{v} = \frac{\lambda_s}{m+\lambda_s}\nu_s^c + \frac{1}{m+\lambda_s}\nu_s^o + \frac{\lambda_d}{m+\lambda_d}\nu_d^c + \frac{1}{m+\lambda_d}\nu_d^o \quad (84)$$

with λ_s and $\lambda_d > 0$ being positive coefficients, while

$$m := z_\xi^\top Q {W_1}^{-1} Q^\top z_\xi, \quad (85)$$

applied to system (2) with $N = 3$ ensures exponential stabilization in the sense given by Eq. (77).

Proof 2. Consider open-loop error dynamics described by (133). Then applying the control law given by Eq. (84) yields in

$$\begin{aligned} \dot{z}_\xi = &\frac{\lambda_s}{m+\lambda_s} K z_\xi - \frac{Q W_1^{-1} Q^\top z_\xi}{m+\lambda_s} z_\xi^\top W_2 z_\xi \\ &- \frac{\lambda_d}{m+\lambda_d} H\left(\tilde{p} - A_\varepsilon \dot{\varepsilon} - A_{\alpha_r}\dot{\alpha}_r\right) \\ &- \frac{Q W_1^{-1} Q^\top z_\xi}{m+\lambda_d} z_\xi^\top H\left(\tilde{p} - A_\varepsilon \dot{\varepsilon} - A_{\alpha_r}\dot{\alpha}_r\right) \\ &+ H\left(\tilde{p} - A_\varepsilon \dot{\varepsilon} - A_{\alpha_r}\dot{\alpha}_r\right). \end{aligned} \quad (86)$$

Next, calculating the term $z_\xi^\top \dot{z}_\xi$ and assuming that $K = -W_2$ gives

$$\begin{aligned} z_\xi^\top \dot{z}_\xi = &-\frac{z_\xi^\top Q W_1^{-1} Q^\top z_\xi + \lambda_s}{m+\lambda_s} z_\xi^\top W_2 z_\xi \\ &- \frac{z_\xi^\top Q W_1^{-1} Q^\top z_\xi + \lambda_d}{m+\lambda_d} z_\xi^\top H\left(\tilde{p} - A_\varepsilon \dot{\varepsilon} - A_{\alpha_r}\dot{\alpha}_r\right) \\ &+ z_\xi^\top H\left(\tilde{p} - A_\varepsilon \dot{\varepsilon} - A_{\alpha_r}\dot{\alpha}_r\right) \overset{(85)}{=} -z_\xi^\top W_2 z_\xi. \end{aligned} \quad (87)$$

From (87) it follows that exponential stabilization is ensured with the convergent rate related to selection of matrix W_2.

The idea of suboptimal control scheme is based on weighting the classical and optimal algorithms. Namely based on Eq. (84) it can be concluded that for the significant tracking error z_ξ, such that $m >> \lambda_s$ and $m >> \lambda_d$, the optimal part of the controller is predominant. On the other hand, when the magnitude of z_ξ decreases and m becomes much less that λ_d and λ_s the classical solution based on Proposition 1 is responsible for generation of control input. Hence, at some neighborhood of equilibrium point $z_\xi = e_\xi$ the feedback is not optimal in the sense of minimization of the performance index (79). However, for a small norm of error z_ξ the requirement of the control optimization is not critical. As a result, the suboptimal controller can be seen as a compromise between the robust and optimal control solution.

Now, we return to the main control problem formulated in Subsec. 4.1. Each variant of the proposed controller guarantees that for $\|z_\xi(0)\| < \infty$ the auxiliary control task is satisfied, namely $z_\xi = e_\xi$ is the equilibrium point and z_ξ converges exponentially to zero (neutral element of group G^C). In view of definition (71) it implies that $\tilde{\xi} \to \overline{f}^\xi$. Recalling that norm of $\overline{f}^\xi$ is dependent on parameter ε it follows that transformed error $\tilde{\xi}$ tends to the neighborhood of zero with the radius which can be arbitrarily adjusted. Further, taking into account definitions (39), (37) and inverse map of (33) one can write that

$$\lim_{t\to\infty}\left(\begin{bmatrix}\tilde{g}\\ \psi\end{bmatrix}(t) - \Phi^{-1}\left(\Phi\left(e_g, \psi_r\right) \star \overline{f}^\xi\right)(t)\right) = 0. \quad (88)$$

Finally, it can be shown that

$$\lim_{t\to\infty} q_e(t) \in B_{0,\epsilon}, \quad (89)$$

with

$$\epsilon = \delta\left(\varphi_r\right)\|\overline{f}^\xi\| \quad (90)$$

being some positive constant, while $\delta(\varphi_r)$ is a nonlinear positive function that is dependent on the selected reference angles φ_r. Hence, it can be stated that the control problem defined in the configuration space is solved assuming that the magnitude of the initial orientation error does not exceeded $\pi/2$ and that the reference trajectory $q_r \in \overline{\mathcal{Q}}$ is smooth enough. The control signals are bounded and q stays in the set $\overline{\mathcal{Q}}$ as a result of the regularity of coordinate transformations (33) and (9).

Controller tuning. Tuning the controller based on the transverse functions becomes a very important and non trivial task, especially when practical aspects are taken into account such as input saturation, limitation of oscillatory behavior during the transient stage, limitation of the frequency bandwidth, etc. – cf. [15].

The properties of the controller investigated in this paper are strongly dependent on the parameters of the transverse functions. The parametrization of the transverse function becomes an important issue. The selection of coefficients $\beta_{i,j}$ $(i = 1, 2,\ j = 1, 2, 3, 4)$ should take into account the transversality condition and in the considered case is relatively complicated. Consequently, it is difficult to change these coefficients in order to shape the transverse functions in an arbitrary way. Instead, one can scale the transverse function by changing parameter ε for the selected set of $\beta_{i,j}$.

From relationship (89) it follows that the accuracy in the configuration space can be easily increased by making norm $\|\overline{f}^{\xi}\|$ small enough. This norm can be efficiently adjusted by parameter ε according to Eq. (52). We consider this problem in more detail. In order to do that matrix A_α defined by (57) is rewritten as follows: $A_\alpha := X^C\left(D(\varepsilon)\overline{f}^{\xi}(\alpha,\alpha_r,1)\right)^{-1} D(\varepsilon)\dfrac{\partial \overline{f}^{\xi}(\alpha,\alpha_r,1)}{\partial\alpha}$ (notice that in this case function f^{ξ} is replaced by the more general form $\overline{f}^{\xi}$). Then referring to Eq. (59) one has

$$\overline{C}_\xi := \left[C_\xi \quad -X^C\left(D(\varepsilon)\overline{f}^{\xi}(\alpha,\alpha_r,1)\right)^{-1} D(\varepsilon)\frac{\partial\overline{f}^{\xi}(\alpha,\alpha_r,1)}{\partial\alpha}\right]. \tag{91}$$

Considering the determinant of matrix $\overline{C}_\xi$ more thoroughly it can be shown that

$$\det\overline{C}_\xi = \varepsilon^{14}\det\left[C_\xi \quad -X^C\left(\overline{f}^{\xi}(\alpha,\alpha_r,1)\right)^{-1}\frac{\partial\overline{f}^{\xi}(\alpha,\alpha_r,1)}{\partial\alpha}\right]. \tag{92}$$

Hence, evoking Eq. (75) one can interpret that the selection of small value of parameter ε increases the resultant gain in the control loop. Consequently, it implies that for $\|z_\xi(0)\| << \epsilon$ the extremely high magnitude and high frequency control input, that deteriorates the performance of the controller, may appear. This problem can be solved partly using the suboptimal control law formulated by Proposition 3; however this method may still not attenuate the oscillations to a satisfactory level.

Another possibility of improving the controller performance is based on a proper scaling of parameter ε during the transient states. Namely, the initial value of this parameter should correspond to the initial auxiliary error z_ξ in order to prevent high control input magnitude. Then, the parameter can be smoothly decreased towards the final value which determine the desired accuracy. For example one can consider the following off-line scaling

$$\varepsilon(t) = \zeta_0\exp(-\zeta_1 t) + \zeta_2, \tag{93}$$

where ζ_0, ζ_1 and ζ_2 are some positive coefficients. It can be easily find that $\varepsilon(0) = \zeta_0 + \zeta_2$, while $\lim_{t\to\infty}\varepsilon(t) = \zeta_2$. Parameter ζ_1 determines the convergence rate and it should be selected such that $\zeta_1 < c$, where c describes the convergent rate of z_ξ according to Eq. (76). The explanation of this rule is quite clear – namely ε should not tend to zero faster than z_ξ.

Alternatively, the scaling of ϵ can be realized by introducing an additional dynamic system. For example, one may take advantage of the following low-pass filter

$$T_1T_2\ddot{\varepsilon} + (T_1+T_2)\dot{\varepsilon} + \varepsilon = \sqrt{\mu_1^2 z_\xi^\top z_\xi + \mu_2^2}, \tag{94}$$

with $\dot{\varepsilon}(0) = 0$, $\varepsilon(0) > 0$, T_1, T_2, μ_1 and $\mu_2 > 0$ being the parameters. Taking into account that $z_\xi \to e_\xi$ and the filter (94) is asymptotically stable, one can easily prove that $\varepsilon \to \mu_2$. Coefficients T_1 and T_2 describe the inertia of the filter (94), while μ_2 determines the accuracy in the steady state.

Additional possibility of tuning the controller may follow from definition of transverse function (60) and implication (61). Namely, one can decrease $\|\overline{f}^{\xi}\|$ by a suitable changing of variable α_r. Unfortunately, in the case of regulation problem this possibility, at least for highly dimensional systems, is still not well developed (some initial results related to this concept have been presented in [16]). The main advantage of the form (60) can be found for some class of admissible reference trajectories. In [12] it is proved that for these reference trajectories there exist constant values of α_r such that for a proper selection of signs of coefficients $\beta_{i,j}$ asymptotic convergence of configuration error can be restored (in this particular case $\overline{f}^{\xi}$ is shrink to zero). Namely, the following implication holds

$$\left(\left(\overline{f}^{\xi} \to e_\xi\right) \text{ and } (z_\xi \to e_\xi)\right) \Rightarrow (q_e \to 0). \tag{95}$$

The input control saturation can be relatively easy solved for the regulation case using the magnitude and time scaling. Assume that $\overline{v}$ is saturated as follows

$$|v_1| \le V_{1\max},\ |v_2| \le V_{2\max},\ |\dot{\alpha}_i| \le \Omega_{\max}, \tag{96}$$

where $i = 1, 2, 3, 4$ while $V_{1\max}$, $V_{2\max}$, and $\Omega_{\max}$ denote positive upper bounds. In order to determine how much the nominal control input given by the kinematic controller violates the assumed bounds we introduce the following scaling factor

$$\chi := \max\left\{1, \frac{|v_1|}{V_{1\max}}, \frac{|v_2|}{V_{2\max}}, \frac{|\dot{\alpha}_1|}{\Omega_{\max}}, \ldots, \frac{|\dot{\alpha}_4|}{\Omega_{\max}}\right\}. \tag{97}$$

In order to guarantee that the control input is within the assumed range, the control input is redefined as follows

$$\overline{v} := \chi^{-1}\overline{v}. \tag{98}$$

Similarly, in order to guarantee the stability result, each time-dependent term in the control law should be scaled as well. This gives a motivation to scale the tuning function in order to ensure Lyapunov stability of the closed-loop system, [17]. As a consequence this method may fail for more general tracking problem if the evolution of reference trajectory is not properly slowed down. In this case it is necessary to assume that – at least locally – the reference trajectory can be tracked with the assumed limitation of the control input. Unfortunately, the global stability may not be guaranteed. On the other hand, considering an implementation of the algorithm in the discrete time domain, control input scaling gives possibility to improve numerical stability and to relax the requirement with respect to sampling time.

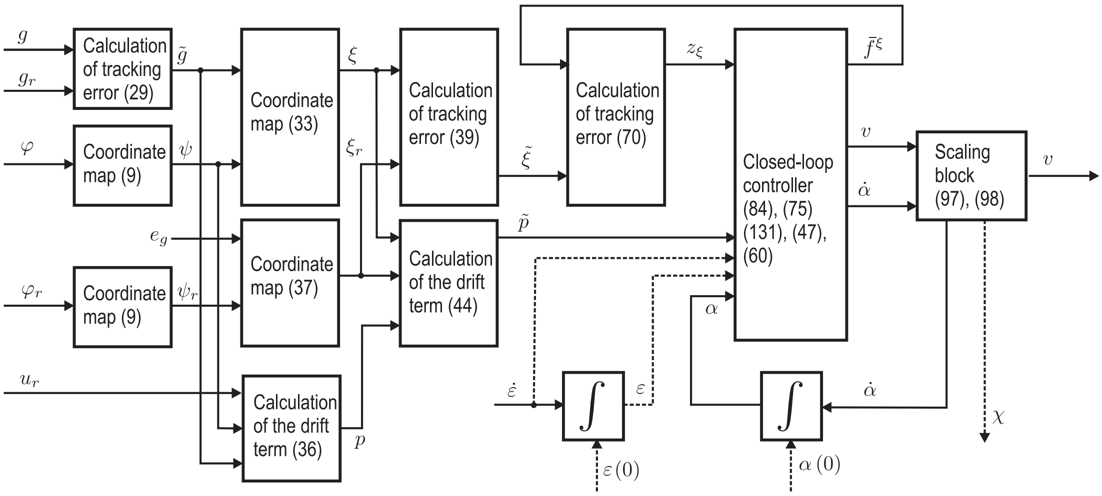


Fig. 2. Diagram of the control loop

The diagram of the controller including tuning and scaling blocks is given in Fig. 2. The numbers put in the brackets refer to the equations which describe the particular blocks.

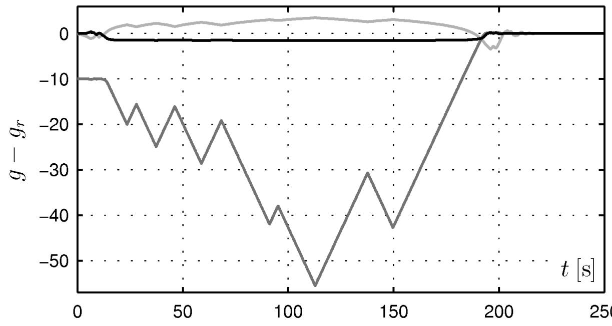
5. Simulation results

General description. In order to illustrate the properties of the described controller, extensive numerical simulations have been conducted in Matlab/Simulink environment. Here, we present the results of simulations obtained for selected scenarios which are outlined below.

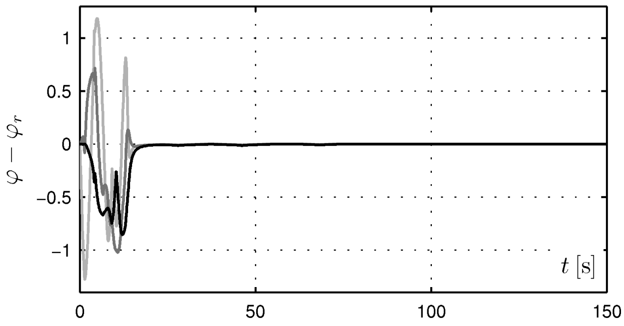
- Sim 1 – regulation case (parallel parking): $q(0) = 0$, $q_r = [0\ 10\ 0\ 0\ 0\ 0]^\top$, classic feedback with $K = -5 \cdot I_{6\times 6}$, $\alpha(0) = 0$, $\zeta_0 = 2$, $\zeta_1 = 0.5$ and $\zeta_2 = 0.02$
- Sim 2 – regulation case – suboptimal controller with $W_1 = \mathrm{diag}\{200, 100, 1, 1, 1, 1\}$, $W_2 = 5 \cdot I_{6\times 6}$; other conditions are the same as in simulation Sim 1
- Sim 3 – regulation case (reconfiguration of the kinematic chain), suboptimal controller, $q_r = \left[0\ 0\ -\frac{\pi}{3}\ \frac{\pi}{6}\ -\frac{\pi}{3}\ 0\right]^\top$, $\zeta_0 = 2.5$, $\zeta_1 = 0.5$ and $\zeta_2 = 0.02$; other conditions are the same as in simulation Sim 2
- Sim 4 – tracking of an eight-like shaped admissible trajectory parametrized as follows: $x_r(t) = 5\sin(0.05t)$ and $y_r(t) = 5\sin(0.025t)$, suboptimal controller, $q(0) = [0\ -3\ 0\ 0\ 0\ 0]^\top$; other conditions are the same as simulation Sim 2
- Sim 5 – tracking of an eight-like shaped admissible trajectory, classic feedback with $K = -5 \cdot I_{6\times 6}$, $\zeta_0 = 2$, $\zeta_1 = 0.5$ and $\zeta_2 = 0.02$; other conditions are the same as simulation Sim 2
- Sim 6 – tracking of a nonadmissible reference trajectory (rotation only) parametrized as follows: $x_r = y_r = 0$, $\theta_r(t) = 0.05t$, $\varphi_r = 0$, suboptimal controller with $W_2 = I_{6\times 6}$, $\zeta_0 = 1$, $\zeta_1 = 0.2$ and $\zeta_2 = 1$, $q_r = [0\ -2\ 0\ 0\ 0\ 0]^\top$; other conditions are the same as in simulation Sim 1
- Sim 7 – tracking of a nonadmissible reference trajectory (linear motion violating the nonholonomic constraint for the last trailer) parametrized as follows: $x_r = 0$, $y_r(t) = 0.05t$, $\theta_r = 0$, $\varphi_r = 0$; other conditions are the same as in simulation Sim 6

The set of parameters $\beta_{i,j}$ ($i = 1, 2, 3, 4$ and $j = 1, 2$) have been verified numerically in order to meet transversality condition and have been selected as: $\beta_{1,1} = 0.2$, $\beta_{1,2} = 1$, $\beta_{2,1} = 0.4$, $\beta_{2,2} = 0.8$, $\beta_{3,1} = 0.8$, $\beta_{3,2} = 0.4$, $\beta_{4,1} = 1.8$, and $\beta_{4,2} = 0.2$. In simulations Sim 1, Sim 2, Sim 3, Sim 6 and Sim 7 typical form of transverse function given by Eq. (47) is used while the extended transverse function defined by Eq. (60) is taken into account in simulations Sim 4 and Sim 5 with $\alpha_r = \left[-\frac{\pi}{2}\ -\frac{\pi}{2}\ -\frac{\pi}{2}\ -\frac{\pi}{2}\right]^\top$. The transverse functions are scaled using an off-line method that is based on relationship (93) with parameters ζ_i for $i = 1, 2, 3$ defined above. The control input is saturated as follows: $V_{1\max} = V_{2\max} = 1$ while $\Omega_{\max} = 30$. The saturation is taken into account in simulations Sim 1, Sim 2, Sim 3, Sim 4 and Sim 5.

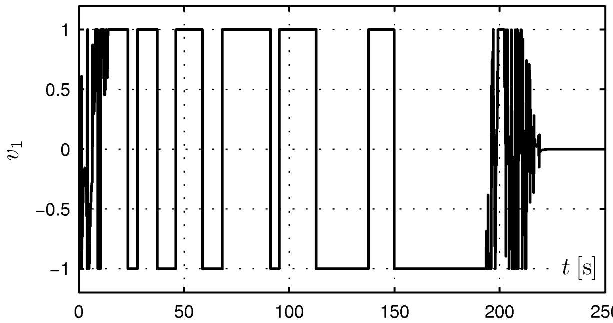
Description of simulation results. The results of simulations Sim 1 and Sim 2 presented in Figs. 3 and 4 illustrate a significant advantage of the suboptimal feedback design. Although for both cases the goal of the control is achieved (namely the configuration of the vehicle with trailers converges to the neighborhood of the desired point and practical stabilization at this point is ensured), the performance of the controller based on Proposition 3 is clearly better. From Figs. 10a and 3a one can conclude that in Sim 1 the position error increases significantly in the y-axle as a result of the vehicle extensive motion. This phenomenon is related to the selection of quite a high initial value of ε in order to suppress oscillatory behavior. Therefore, penalizing the real control inputs v_1 and v_2 more significantly than the virtual signal $\dot{\alpha}$ (notice the weights assumed in matrix W_1) gives a possibility to limit unnecessary motion of the tractor during the transient stage (cf. 10a and 3a). Consequently, the regulation time in the presence of input saturation for the optimal controller is decreased, compared to the classic feedback solution. As a result the transient stage performance can be improved considerably.



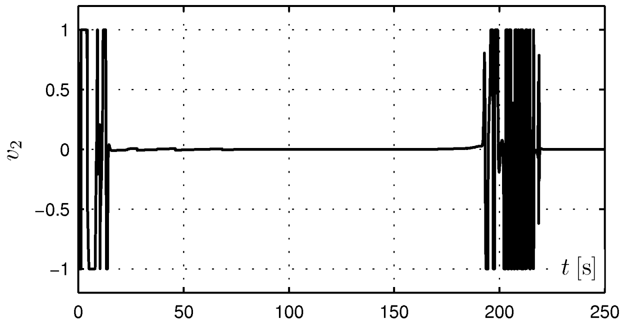
(a) Errors: x_e ■, y_e ■ [m], θ_e ■ [rad]



(b) Errors: φ_{e1} ■, φ_{e2} ■, φ_{e3} ■ [rad]

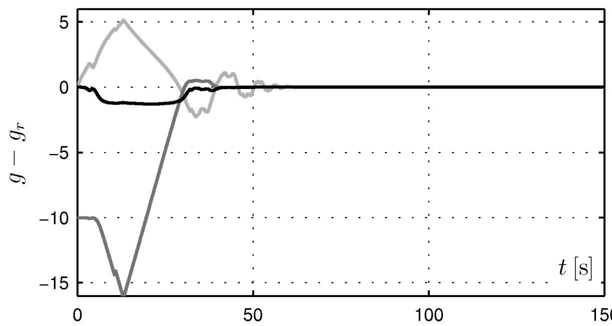


(c) Control input v_1 [m/s]

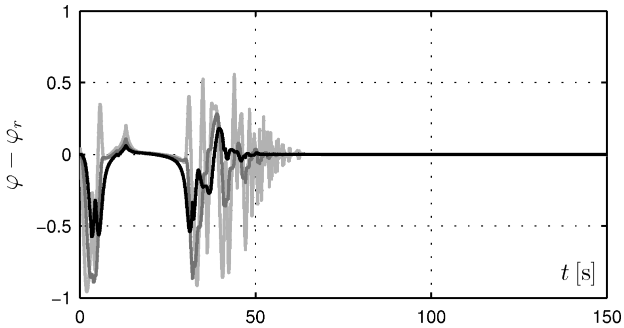


(d) Control input v_2 [rad/s]

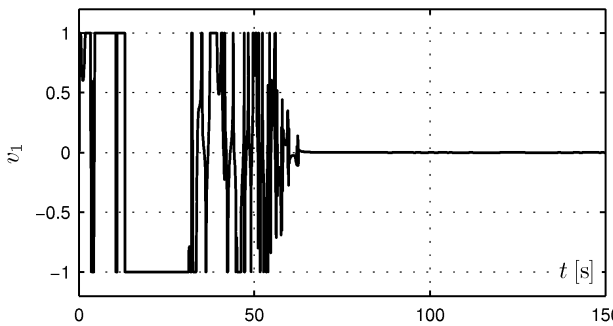
Fig. 3. Results of simulation Sim 1



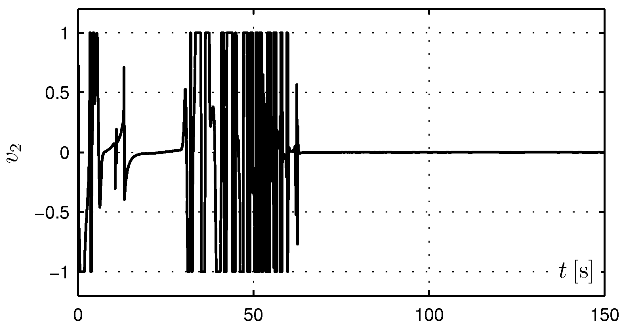
(a) Errors: x_e ■, y_e ■ [m], θ_e ■ [rad]



(b) Errors: φ_{e1} ■, φ_{e2} ■, φ_{e3} ■ [rad]

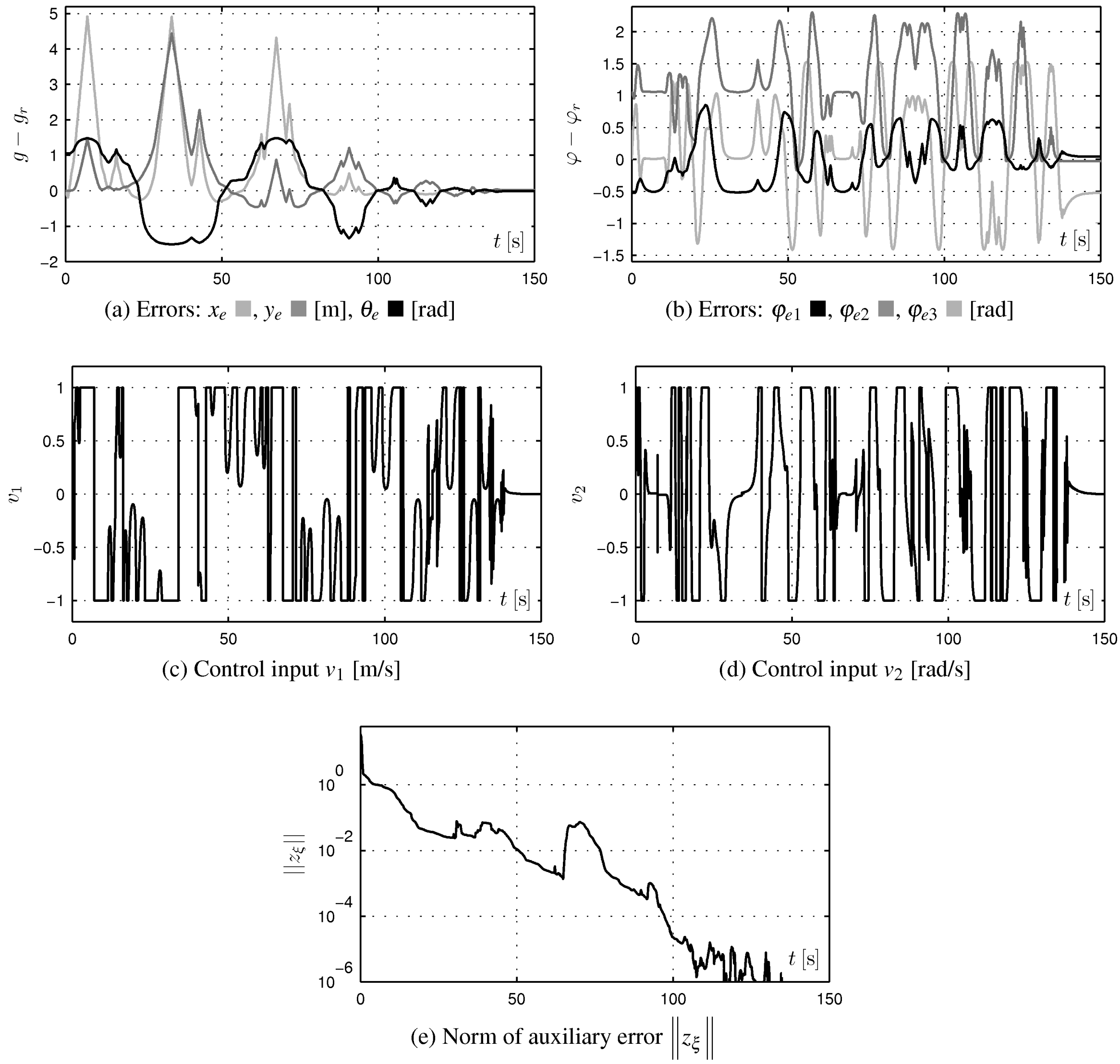


(c) Control input v_1 [m/s]



(d) Control input v_2 [rad/s]

Fig. 4. Results of simulation Sim 2



(a) Errors: x_e ■, y_e ■ [m], θ_e ■ [rad]

(b) Errors: φ_{e1} ■, φ_{e2} ■, φ_{e3} ■ [rad]

(c) Control input v_1 [m/s]

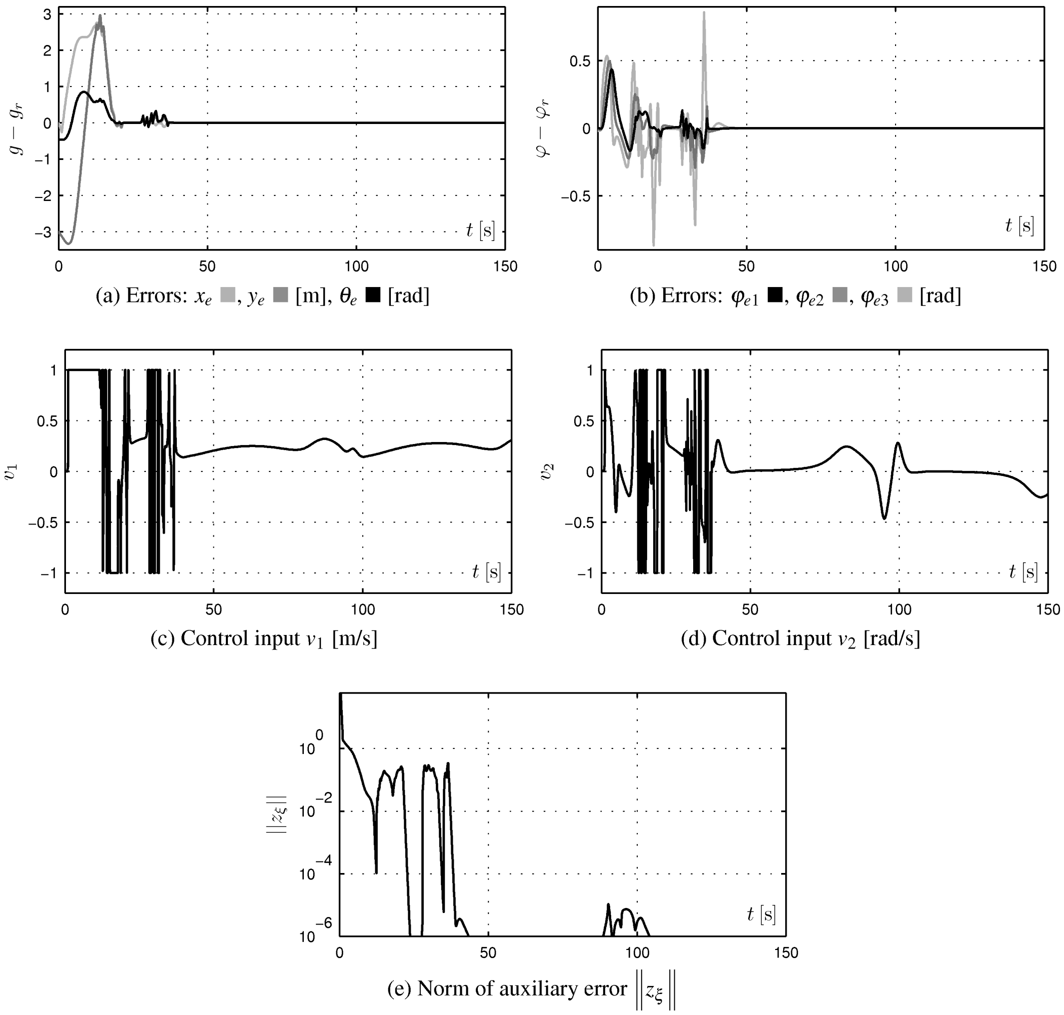
(d) Control input v_2 [rad/s]

(e) Norm of auxiliary error $\left\|z_\xi\right\|$

Fig. 5. Results of simulation Sim 3

In simulation Sim 3 a more demanding control task is considered. The goal is to change the orientation of the last trailer as well as the internal configuration of the chain assuming that the desired position of the trailer remains unchanged. According to the results obtained (see Fig. 5) it follows that the task is realized in an oscillatory way. This issue has been observed in many simulation experiments. Namely if the desired point describes no straight kinematic chain ($\varphi_r \neq 0$) the performance of the controller is negatively affected during the transient stage. Moreover, in such a case the steady-state errors can increase (in fact, the error bound is dependent on the selection of φ_r – cf. Eq. (90)). This property can be illustrated taking into account Fig. 5b – the value of error φ_{e3} in the steady state is quite high compared to the assumed lower bound of ε. Additionally, from Fig. 5e it can be seen that norm $\|z_\xi\|$ at some time interval increases which does not correspond to the theoretical considerations. We believe that the observed behavior of the closed-loop system is caused by numerical instability which comes from non-sufficient accuracy of the double floating point precision used in numerical computations.

Next simulations Sim 4 and Sim 5 take advantage of the extended transverse function given by Eq. (60). The reference motion is assumed to be admissible with positive longitudinal velocity $v_{rx} > 0$. The selected transverse function with positive coefficients $\beta_{i,j}$ gives a possibility to achieve asymptotic stability in the original coordinate space. This property is confirmed in simulation Sim 4 (cf. Fig. 6). It should be emphasized that the asymptotic convergence is achieved in spite of input saturation, which illustrates some robustness to this kind of nonlinearity. However, taking into account Fig. 6e one can conclude that input saturation may deteriorate the stability of the controller – the norm of z_ξ does not tend monotically to zero at some time intervals. As a result the stability is local and the region of the convergence is not given accurately. A negative example that illustrate this drawback is illustrated by the results of simulation Sim 5. Considering Fig. 7 one can observe that the control task is not achieved – norm of the auxiliary error becomes unbounded (it can be interpreted that magnitude of orientation error approaches $\frac{\pi}{2}$ and the transformation map (33) becomes not well defined).



(a) Errors: x_e ■, y_e ■ [m], θ_e ■ [rad]

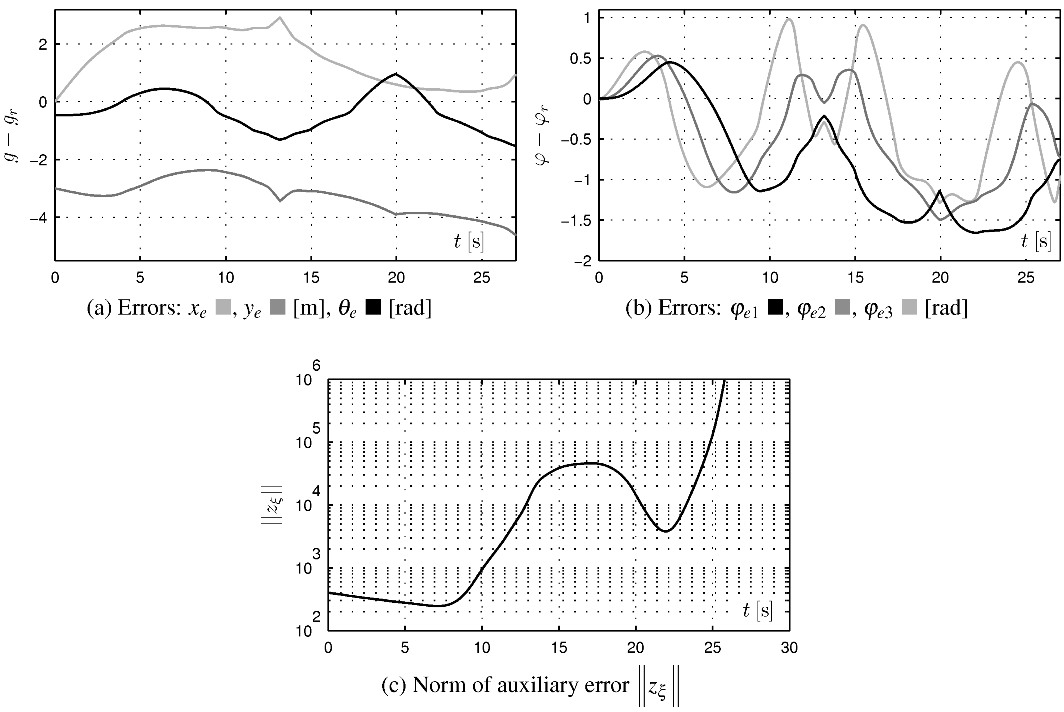
(b) Errors: φ_{e1} ■, φ_{e2} ■, φ_{e3} ■ [rad]

(c) Control input v_1 [m/s]

(d) Control input v_2 [rad/s]

(e) Norm of auxiliary error $\left\|z_\xi\right\|$

Fig. 6. Results of simulation Sim 4

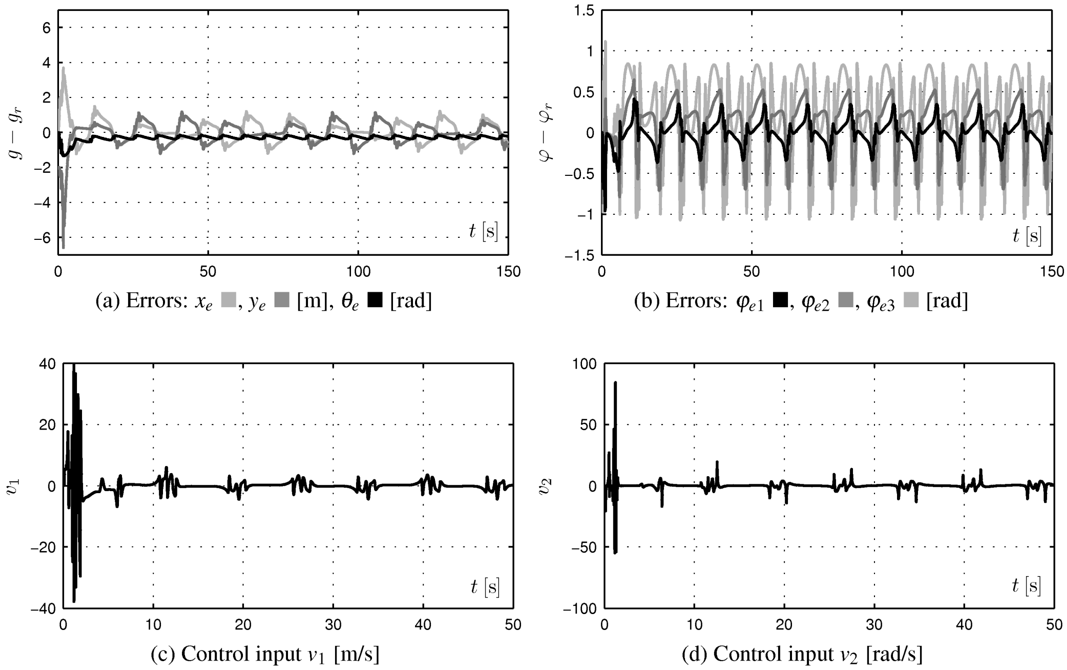


(a) Errors: x_e ■, y_e ■ [m], θ_e ■ [rad]

(b) Errors: φ_{e1} ■, φ_{e2} ■, φ_{e3} ■ [rad]

(c) Norm of auxiliary error $\left\|z_\xi\right\|$

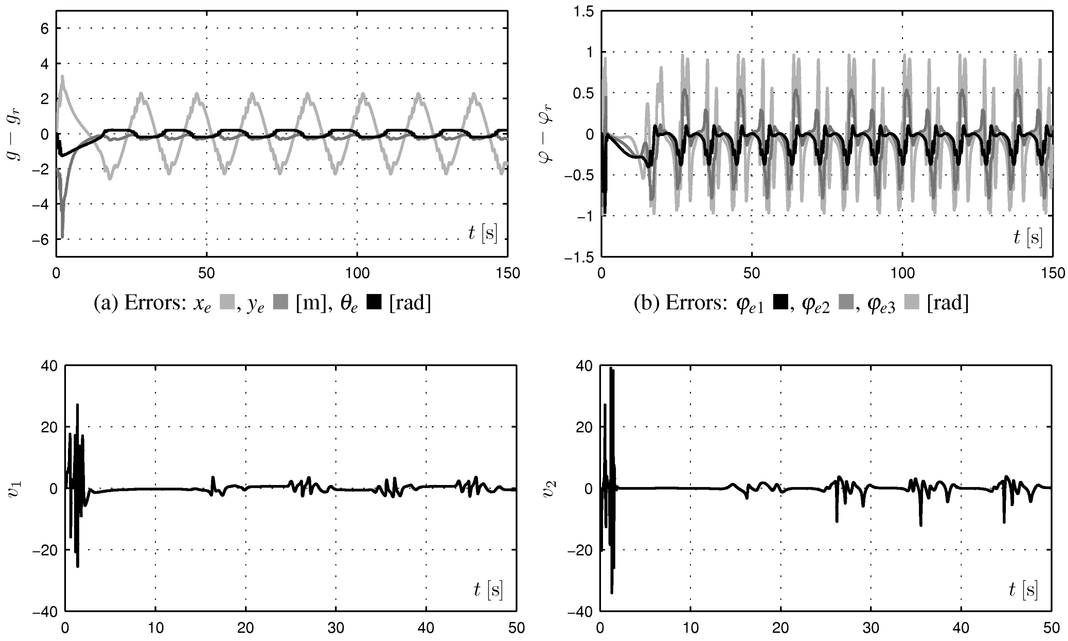
Fig. 7. Results of simulation Sim 5



(a) Errors: x_e ■, y_e ■ [m], θ_e ■ [rad] (b) Errors: φ_{e1} ■, φ_{e2} ■, φ_{e3} ■ [rad]

(c) Control input v_1 [m/s] (d) Control input v_2 [rad/s]

Fig. 8. Results of simulation Sim 6



(a) Errors: x_e ■, y_e ■ [m], θ_e ■ [rad] (b) Errors: φ_{e1} ■, φ_{e2} ■, φ_{e3} ■ [rad]

(c) Control input v_1 [m/s] (d) Control input v_2 [rad/s]

Fig. 9. Results of simulation Sim 7

In simulations Sim 6 and Sim 7 an approximation of non-admissible trajectories is taken into account. The reference trajectory considered in simulation Sim 6 describes pure rotation of the last trailer without changing its position. On the other hand, in simulation Sim 7 the linear reference motion in the y direction with constant orientation is assumed. In both simulations control input is not saturated. From Figs. 8 and 9 it can be seen that the control task is realized properly, namely the configuration errors are bounded and the reference motion is tracked with the assumed accuracy. The control inputs given in Figs. 8c, 8d, 9c and 9d achieve significantly high values which indicates that the reference trajectory is quite demanding for the considered system. However, the magnitude of control input can be quite easily limited by decreasing the velocity of the reference motion.

Comparing position paths of the last trailer and the tractor illustrated in Fig. 10 one can observe that the motion of the tractor is quite extensive with respect to motion of the last trailer. This phenomena can be quite easily explained referring to the complex structure of the considered nonholonomic vehicle. Taking into account high nonholonomy degree of the system the tractor movement has to generate "difficult" directions in highly dimensional coordinate space in order to solve the control task.

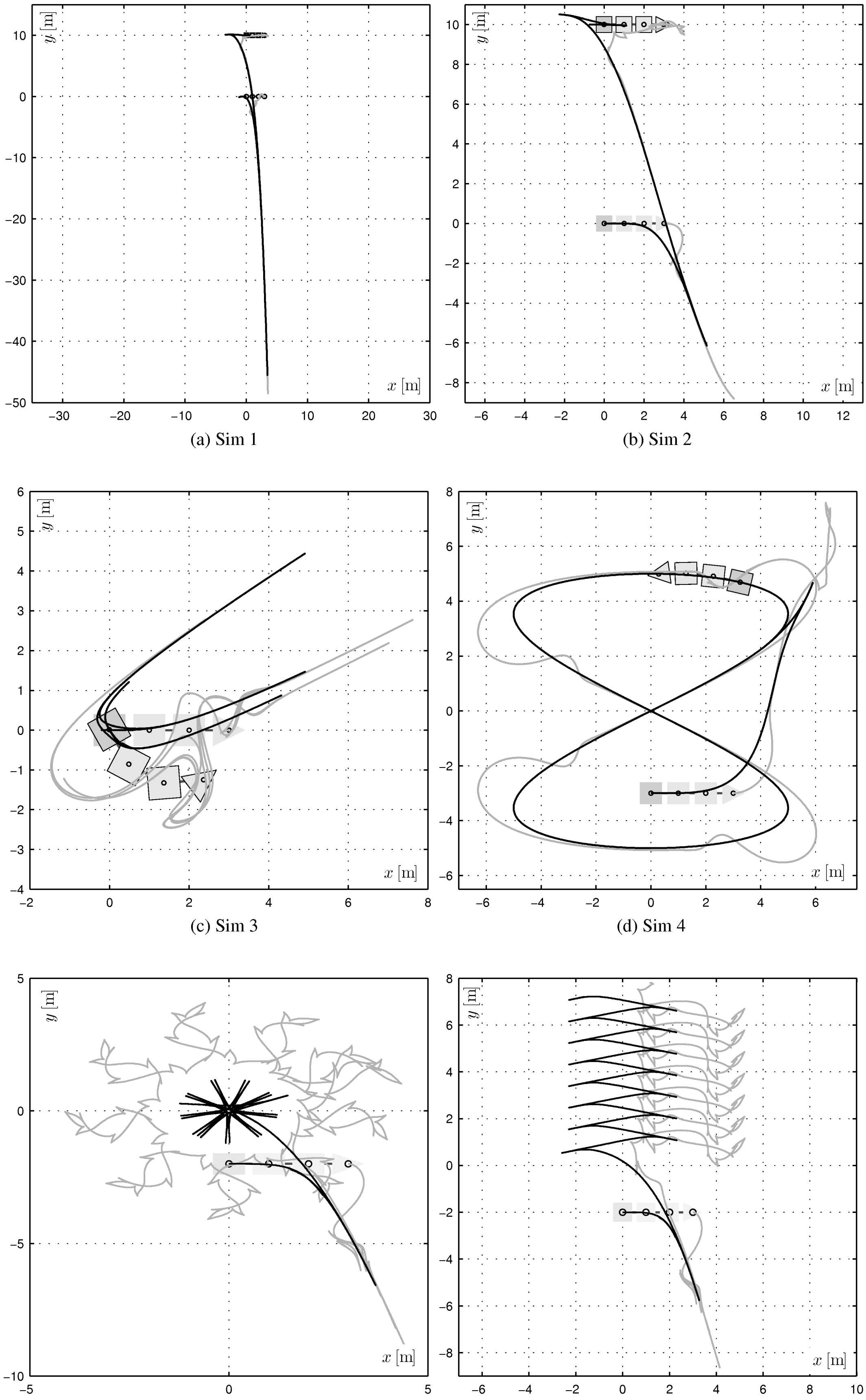


Fig. 10. Path describing position of the last trailer (■) and the tractor (■)

6. Concluding remarks

In this paper the design of the controller for a unicycle-like vehicle with three on-axis trailers is investigated in details. The considered control solution ensures local convergence and practical (or conditionally asymptotic) stabilization in the regulation and trajectory tracking case. The algorithm is based on the transverse function calculated for the IV-order two-input chained system.

Taking into account a possibility of implementation of the controller in practice the optimal control feedback is designed. Moreover, selected tuning techniques which limit oscillatory behavior during transient stage are discussed. In order to meet the assumed saturation of the control input and to improve numerical stability of the algorithm input and time scaling is taken into account.

Examination of the controller performance is realized using numerical simulations. The selected scenarios illustrate properties of the algorithm assuming significant initial configuration errors. In particular, the advantage of the optimal control over the classic approach is shown. Based on the given results one can conclude that a proper scaling of the transverse function gives a possibility to limit oscillations during transient stage at least for some cases. However, it is shown that for some desired configurations, for which no straight kinematic chain of the vehicle is assumed, the performance of the controller is still not satisfactory. It is shown that the saturation of control input can be quite efficiently overcome in the regulation case. Unfortunately, in the trajectory tracking case the saturation may deteriorate system stability (even if the reference trajectory can be tracked by the vehicle with velocities satisfying the limitations), that is illustrated by simulations.

The control solution presented here can be seen as a general approach for the stabilization. However, implementation of this solution in practice may be difficult. Still the open problems include optimization of the control loop in order to obtain better performance. In spite of some progress that has been achieved with respect to tuning of the controller the detailed rules of the tuning are still not formulated precisely, at least with respect to highly dimensional systems.

The future works can be devoted to attenuation of oscillatory motion when it is not necessary for realization of the task. In order to do that one can for example investigate the problem of using α_r as additionally tuning variable and consider the optimal selection of transverse function coefficients. The comparison of the presented algorithm to other variant of the control law for which no explicit coordinate transformation to the chained form is used (an extensive simulations has been reported in [18]) can be also take into account. In particular one can verify which solution ensures better performance for stabilization task at "difficult" desired points (with no straight kinematic chain).

Finally, it is worth to point out the problem directly related to the complexity of the controller. It turns out that further extension to the vehicles with more trailers may be extremely difficult. According to authors' knowledge accuracy of typical numerical representation may be not sufficient to ensure stability of the algorithm. These problems seem to be demanding and still unsolved.

Appendix

Derivation of transformations (5) and (6). Considering the case $N = 3$ and comparing kinematics (2) and (7) one has:

$$\dot{x} = \cos\theta \cos\varphi_1 \cos\varphi_2 \cos\varphi_3 v_1 = \cos\theta \eta_1, \tag{99}$$

where

$$\eta_1 = \cos\varphi_1 \cos\varphi_2 \cos\varphi_3 v_1. \tag{100}$$

Based on definition (100) dynamics of variables φ_1, φ_2 and φ_3 described by Eq. (2) can be rewritten as follows:

$$\begin{bmatrix}\dot{\varphi}_1\\ \dot{\varphi}_2\\ \dot{\varphi}_3\end{bmatrix} = \begin{bmatrix} \dfrac{1}{\cos\varphi_1}\left(\dfrac{\tan\varphi_2}{l_1} - \dfrac{\sin\varphi_1}{l_0}\right) \\ \dfrac{1}{\cos\varphi_1\cos\varphi_2}\left(\dfrac{\tan\varphi_3}{l_2} - \dfrac{\sin\varphi_2}{l_1}\right) \\ -\dfrac{1}{\cos\varphi_1\cos\varphi_2}\dfrac{\tan\varphi_3}{l_2} \end{bmatrix}\eta_1 + \begin{bmatrix}0\\0\\1\end{bmatrix} v_2. \tag{101}$$

Then variable ψ_1 can be derived taking into account that

$$\dot{\theta} = \frac{1}{l_0}\sin\varphi_1\cos\varphi_2\cos\varphi_3 v_1 \overset{(100)}{=} \frac{\tan\varphi_1}{l_0}\eta_1 \overset{(7)}{=} \psi_1\eta_1 \tag{102}$$

which implies that $\psi_1 := \dfrac{\tan\varphi_1}{l_0}$. Calculating the time derivative of ψ_1 yields in

$$\dot{\psi}_1 = \frac{1}{\cos^2\varphi_1 l_0}\dot{\varphi}_1 \overset{(101)}{=} \frac{1}{\cos^3\varphi_1 l_0^2 l_1}\left(l_0\tan\varphi_2 - l_1\sin\varphi_1\right)\eta_1 \overset{(7)}{=} \psi_2\eta_1 \tag{103}$$

and gives $\psi_2 := \dfrac{l_0\tan\varphi_2 - l_1\sin\varphi_1}{\cos^3\varphi_1 l_0^2 l_1}$. In the similar way transformation with respect to variable ψ_3 can be derived. Calculating $\dot{\psi}_2$ gives

$$\begin{aligned}\dot{\psi}_2 &= \frac{3\sin\varphi_1}{l_0^2 l_1\cos^4\varphi_1}\left(l_0\tan\varphi_2 - l_1\sin\varphi_1\right)\dot{\varphi}_1 \\ &+ \frac{1}{\cos^3\varphi_1 l_0^2 l_1}\left(\frac{l_0}{\cos^2\varphi_2}\dot{\varphi}_2 - l_1\cos\varphi_1\dot{\varphi}_1\right) \\ &\overset{(101)}{=} \left(\frac{3\tan\varphi_1\left(l_0\tan\varphi_2 - l_1\sin\varphi_1\right)^2}{l_0^3 l_1^2\cos^4\varphi_1} - \frac{l_1\cos\varphi_1\left(l_0\tan\varphi_2 - l_1\sin\varphi_1\right)}{l_0^3 l_1^2\cos^4\varphi_1} + \frac{l_1\tan\varphi_3 - l_2\sin\varphi_2}{l_0 l_1^2 l_2\cos^4\varphi_1\cos^3\varphi_2}\right)\eta_1 \overset{(7)}{=} \psi_3\eta_1.\end{aligned} \tag{104}$$

Hence, one obtains

$$\psi_3 = \frac{(l_0\tan\varphi_2 - l_1\sin\varphi_1)(3\tan\varphi_1(l_0\tan\varphi_2 - l_1\sin\varphi_1) - l_1\cos\varphi_1)}{l_0^3 l_1^2\cos^4\varphi_1} + \frac{l_1\tan\varphi_3 - l_2\sin\varphi_2}{l_0 l_1^2 l_2\cos^4\varphi_1\cos^3\varphi_2}.$$

Then, calculating the time derivative of ψ_3 gives

$$\dot{\psi}_3 = \sum_{i=1}^{3} \frac{\partial \psi_3}{\partial \varphi_i}\dot{\varphi}_i \overset{(101)}{=} \left(\frac{\partial \psi_3}{\partial \varphi_1} \frac{\frac{\tan\varphi_2}{l_1} - \frac{\sin\varphi_1}{l_0}}{\cos\varphi_1} \right.$$
$$\left. + \frac{\partial \psi_3}{\partial \varphi_2} \frac{\frac{\tan\varphi_3}{l_2} - \frac{\sin\varphi_2}{l_1}}{\cos\varphi_1 \cos\varphi_2} - \frac{\partial \psi_3}{\partial \varphi_2} \frac{\tan\varphi_3}{l_2 \cos\varphi_1 \cos\varphi_2} \right) \eta_1 \tag{105}$$
$$+ \frac{\partial \psi_3}{\partial \varphi_3} v_2 \overset{(7)}{=} \eta_2.$$

Derivation of kinematic error (30). Calculating the time derivative of (29) we have

$$\dot{\tilde{g}} = dl_{g_r^{-1}}(g)\,\dot{g} + dr_g\left(g_r^{-1}\right) \frac{d}{dt} g_r^{-1}. \tag{106}$$

Next, using the following relationship: $\frac{d}{dt} g_r^{-1} := -dr_{g_r^{-1}}(e_g)\,X^E(e_g)\,u_r$ (cf. Eq. (25) and refer to [12]) and taking advantage of (30) give

$$\begin{aligned} \dot{\tilde{g}} &= dl_{g_r^{-1}}(g)\,X^E(g)\,C_g(\psi)\,\eta \\ &- dr_g\left(g_r^{-1}\right) dr_{g_r^{-1}}(e_g)\,X^E(e_g)\,u_r \\ &\overset{(17),(12)}{=} X^E\left(g_r^{-1} g\right) C_g(\psi)\,\eta \\ &- dr_{g_r^{-1} g}(e_g)\,X^E(e_g)\,u_r \overset{(29)}{=} X^E(\tilde{g})\,C_g(\psi)\,\eta \\ &- dr_{\tilde{g}}(e_g)\,X^E(e_g)\,u_r, \end{aligned} \tag{107}$$

where

$$\begin{aligned} &dr_{\tilde{g}}(e_g)\,X^E(e_g) = dl_{\tilde{g}}(e_g)\,Ad\left(\tilde{g}^{-1}\right) X^E(e_g) \\ &= dl_{\tilde{g}}(e_g)\,X^E(e_g)\,Ad^{X^E}\left(\tilde{g}^{-1}\right) = X^E(\tilde{g})\,Ad^{X^E}\left(\tilde{g}^{-1}\right). \end{aligned} \tag{108}$$

Finally, substituting (108) to (107) we obtain Eq. (30).

Derivation of transformation of system (8) to the VI-order chained form. Here we refer to the algorithm proposed by Sørdalen in [7]. The task is to find coordinate and input transformation which transform system (8) (at least locally) to IV-order chained system given by (31).

Following Sørdalen [7] we start the calculations assuming that $w_1 := \eta_1 \cos\theta$. It allows one to rewrite Eq. (8) as

$$\dot{\overline{q}} = X_1^A w_1 + X_2^A \eta_2 \tag{109}$$

with

$$\begin{aligned} X_1^A(\overline{q}) &:= \begin{bmatrix} 1 & \tan\overline{q}_3 & \frac{\overline{q}_4}{\cos\overline{q}_3} & \frac{\overline{q}_5}{\cos\overline{q}_3} & \frac{\overline{q}_6}{\cos\overline{q}_3} & 0 \end{bmatrix}^\top, \\ X_2^A &:= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^\top \end{aligned} \tag{110}$$

being vector fields.

Next, consider two outputs functions $h_1(\overline{q}) := \overline{q}_1$ and $h_2(\overline{q}) := \overline{q}_2$ and assume that $\xi_1 := h_1$, $\xi_6 := h_2$. Then one has

$$\xi_k = L_{X_1^A}^{N-k} h_2, \tag{111}$$

where $L_X^i f$ denotes iterative Lie derivative of a scalar function f in direction determined by vector field X, while in the case considered it is assumed that $N = 6$ and $k = 2, \ldots, 5$. Consequently, taking advantage of (111) it follows that

$$\begin{aligned} \xi_5 &= L_{X_1^A} h_2 = \frac{\partial h_2}{\partial \overline{q}} X_1^A = \tan\overline{q}_3, \\ \xi_4 &= L_{X_1^A}^2 h_2 = \frac{\partial \xi_5(\overline{q})}{\partial \overline{q}} X_1^A = \frac{\overline{q}_4}{\cos^3\overline{q}_3}, \\ \xi_3 &= L_{X_1^A}^3 h_2 = \frac{\partial \xi_4(\overline{q})}{\partial \overline{q}} X_1^A = \frac{3\overline{q}_4^2 \tan\overline{q}_3 + \overline{q}_5}{\cos^4\overline{q}_3} \end{aligned} \tag{112}$$

and

$$\begin{aligned} \xi_2 &= L_{X_1^A}^4 h_2 = \frac{\partial \xi_3(\overline{q})}{\partial \overline{q}} X_1^A \\ &= \frac{3\overline{q}_4^3 + 12\overline{q}_4^3 \sin^2\overline{q}_3 + 5\overline{q}_4\overline{q}_5 \sin(2\overline{q}_3) + \overline{q}_6 \cos^2\overline{q}_3}{\cos^7\overline{q}_3}. \end{aligned} \tag{113}$$

Finally one can write the following map

$$\xi := \begin{bmatrix} \overline{q}_1 \\ \dfrac{3\overline{q}_4^3 + 12\overline{q}_4^3 \sin^2\overline{q}_3 + 5\overline{q}_4\overline{q}_5 \sin(2\overline{q}_3) + \overline{q}_6 \cos^2\overline{q}_3}{\cos^7\overline{q}_3} \\ \dfrac{3\overline{q}_4^2 \tan\overline{q}_3 + \overline{q}_5}{\cos^4\overline{q}_3} \\ \dfrac{\overline{q}_4}{\cos^3\overline{q}_3} \\ \tan\overline{q}_3 \\ \overline{q}_2 \end{bmatrix}. \tag{114}$$

The input transformation is derived taking time derivative of ξ_1 and ξ_2 which yields to

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} \cos\overline{q}_3 \eta_1 \\ \sum_{i=1}^{3} \frac{\partial \xi_2(\overline{q})}{\partial \overline{q}_{i+2}} \dot{\psi}_i \eta_1 + \frac{\partial \xi_2(\overline{q})}{\partial \overline{q}_6} \eta_2 \end{bmatrix} \overset{(31)}{=} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}. \tag{115}$$

Description of VI-order chained system on Lie group. Consider control Lie algebra of 6D chained system (31) generated by two vector fields X_1^C and X_2^C. Calculating Lie brackets $X_3^C := \left[X_2^C, X_1^C\right] = e_3$, $X_i^C := \left[X_{i-1}^C, X_1^C\right] = e_i$ with $i := 4, 5, 6$, where $e_i \in \mathbb{R}^6$ stands for i^{th} base vector of $\mathbb{R}^6$ space, one can define the following distribution

$$\Delta^C = \operatorname{span}\left\{X_1^C, X_2^C, X_3^C, X_4^C, X_5^C, X_6^C\right\}, \tag{116}$$

which spans six-dimensional space for any $\xi \in G^C$. As a result system (31) satisfies LARC and it is controllable.

The control Lie algebra of the system is nilpotent, hence the system can be defined on Lie group G^C with the group operation given by

$$\xi_a\star\xi_b=\begin{bmatrix}\xi_{a1}+\xi_{b1}\\ \xi_{a2}+\xi_{b2}\\ \xi_{a3}+\xi_{b3}+\xi_{b1}\xi_{a2}\\ \xi_{a4}+\xi_{b4}+\frac{1}{2}\xi_{b1}^2\xi_{a2}+\xi_{b1}\xi_{a3}\\ \xi_{a5}+\xi_{b5}+\frac{1}{6}\xi_{b1}^3\xi_{a2}+\frac{1}{2}\xi_{b1}^2\xi_{a3}+\xi_{b1}\xi_{a4}\\ \xi_{a6}+\xi_{b6}+\frac{1}{24}\xi_{b1}^4\xi_{a2}+\frac{1}{6}\xi_{b1}^3\xi_{a3}+\frac{1}{2}\xi_{b1}^2\xi_{a4}+\xi_{b1}\xi_{a5}\end{bmatrix}\tag{117}$$

where $\xi_a,\xi_b\in G^C$.The differentials of the left and right translations become

$$dl_{\xi_a}(\xi_b)=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\ \xi_{a2}&0&1&0&0&0\\ \xi_{b1}\xi_{a2}+\xi_{a3}&0&0&1&0&0\\ \frac{1}{2}\xi_{b1}^2\xi_{a2}+\xi_{b1}\xi_{a3}+\xi_{a4}&0&0&0&1&0\\ \frac{1}{6}\xi_{b1}^3\xi_{a2}+\frac{1}{2}\xi_{b1}^2\xi_{a3}+\xi_{b1}\xi_{a4}+\xi_{a5}&0&0&0&0&1\end{bmatrix},\tag{118}$$

$$dr_{\xi_a}(\xi_b)=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&\xi_{a1}&1&0&0&0\\0&\frac{1}{2}\xi_{a1}^2&\xi_{a1}&1&0&0\\0&\frac{1}{6}\xi_{a1}^3&\frac{1}{2}\xi_{a1}^2&\xi_{a1}&1&0\\0&\frac{1}{24}\xi_{a1}^4&\frac{1}{6}\xi_{a1}^3&\frac{1}{2}\xi_{a1}^2&\xi_{a1}&1\end{bmatrix},\tag{119}$$

while adjoint operator is given by

$$Ad(\xi):=dl_\xi\left(\xi^{-1}\right)dr_{\xi^{-1}}(e_\xi)=$$
$$\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\ \xi_2&-\xi_1&1&0&0&0\\ \xi_3-\xi_1\xi_2&\frac{1}{2}\xi_1^2&-\xi_1&1&0&0\\ \frac{1}{2}\xi_1^2\xi_2-\xi_1\xi_3+\xi_4&-\frac{1}{6}\xi_1^3&\frac{1}{2}\xi_1^2&-\xi_1&1&0\\ -\frac{1}{6}\xi_1^3\xi_2+\frac{1}{2}\xi_1^2\xi_3-\xi_1\xi_4+\xi_5&\frac{1}{24}\xi_1^4&-\frac{1}{6}\xi_1^3&\frac{1}{2}\xi_1^2&-\xi_1&1\end{bmatrix}.\tag{120}$$

Calculation of the transverse function for VI-order chained system. Taking into account definition (48) we refer to control Lie algebra and distribution (116). Here, we show details of calculations of the first component of transverse function f^ξ – cf. Eq. (47). In order to calculate exponential map ${}^1f^\xi(\alpha_1)=\exp\left(X_1^C\varepsilon\beta_{1,1}\sin\alpha_1+X_2^C\varepsilon\beta_{1,2}\cos\alpha_1\right)$ the following differential equation can be taken into account

$$\frac{d}{ds}{}^1f^\xi=\frac{d}{ds}\begin{bmatrix}{}^1f_1^\xi\\{}^1f_2^\xi\\{}^1f_3^\xi\\{}^1f_4^\xi\\{}^1f_5^\xi\\{}^1f_6^\xi\end{bmatrix}=\begin{bmatrix}1\\0\\{}^1f_2^\xi\\{}^1f_3^\xi\\{}^1f_4^\xi\\{}^1f_5^\xi\end{bmatrix}\varepsilon\beta_{1,1}\sin\alpha_1+\begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix}\varepsilon\beta_{1,2}\cos\alpha_1.\tag{121}$$

Integrating Eq. (121) with respect to variable s under assumption that α_1 is a parameter (it does not depend on s) one has in the sequel

$$\begin{aligned}
{}^1f_1^\xi(s)&=\int_0^s\varepsilon\beta_{1,1}\sin\alpha_1 d\tau=s\varepsilon\beta_{1,1}\sin\alpha_1,\\
{}^1f_2^\xi(s)&=\int_0^s\varepsilon\beta_{1,2}\cos\alpha_1 d\tau=s\varepsilon\beta_{1,2}\cos\alpha_1,\\
{}^1f_3^\xi(s)&=\int_0^s\varepsilon\beta_{1,1}\sin\alpha_1{}^1f_2^\xi(s)\,d\tau\\
&=s^2\frac{\varepsilon^2\beta_{1,1}\beta_{1,2}}{4}\sin(2\alpha_1),\\
{}^1f_4^\xi(s)&=\int_0^s\varepsilon\beta_{1,1}\sin\alpha_1{}^1f_3^\xi(s)\,d\tau\\
&=s^3\frac{\varepsilon^3\beta_{1,1}^2\beta_{1,2}}{6}\sin^2\alpha_1\cos\alpha_1,\\
{}^1f_5^\xi(s)&=\int_0^s\varepsilon\beta_{1,1}\sin\alpha_1{}^1f_4^\xi(s)\,d\tau\\
&=s^4\frac{\varepsilon^4\beta_{1,1}^3\beta_{1,2}}{24}\sin^3\alpha_1\cos\alpha_1,\\
{}^1f_6^\xi(s)&=\int_0^s\varepsilon\beta_{1,1}\sin\alpha_1{}^1f_5^\xi(s)\,d\tau\\
&=s^5\frac{\varepsilon^5\beta_{1,1}^4\beta_{1,2}}{120}\sin^4\alpha_1\cos\alpha_1.
\end{aligned}\tag{122}$$

Next, evaluating ${}^1f^\xi(s)$ at $s=1$ yields in definition of the first component of the transverse function, namely ${}^1f^\xi(\alpha_1):={}^1f^\xi(s)|_{s=1}$. This procedure is repeated with respect to other components of f^ξ. Finally one can obtain the transverse function given by Eq. (47).

Taking the time derivative of (47) and using differential operators we have

$$\begin{aligned}\dot{f}^{\xi} &= \frac{d}{dt}\left({}^{4}f^{\xi}\left(\alpha_4\right)\star{}^{3}f^{\xi}\left(\alpha_3\right)\star{}^{2}f^{\xi}\left(\alpha_2\right)\star{}^{1}f^{\xi}\left(\alpha_1\right)\right)\\
&= \frac{d}{dt}\left(l_{{}^{4}f^{\xi}\star{}^{3}f^{\xi}}\left({}^{2}f^{\xi}\star{}^{1}f^{\xi}\right)\right)\\
&= dl_{{}^{4}f^{\xi}\star{}^{3}f^{\xi}}\left({}^{2}f^{\xi}\star{}^{1}f^{\xi}\right)\frac{d}{dt}\left({}^{2}f^{\xi}\star{}^{1}f^{\xi}\right)\\
&\quad+dr_{{}^{2}f^{\xi}\star{}^{1}f^{\xi}}\left({}^{4}f^{\xi}\star{}^{3}f^{\xi}\right)\frac{d}{dt}\left({}^{4}f^{\xi}\star{}^{3}f^{\xi}\right)\\
&= dl_{{}^{4}f^{\xi}\star{}^{3}f^{\xi}}\left({}^{2}f^{\xi}\star{}^{1}f^{\xi}\right)\left(dl_{{}^{2}f^{\xi}}\left({}^{1}f^{\xi}\right){}^{1}\dot{f}^{\xi}+dr_{{}^{1}f^{\xi}}\left({}^{2}f^{\xi}\right){}^{2}\dot{f}^{\xi}\right)\\
&\quad+dr_{{}^{2}f^{\xi}\star{}^{1}f^{\xi}}\left({}^{4}f^{\xi}\star{}^{3}f^{\xi}\right)\left(dl_{{}^{4}f^{\xi}}\left({}^{3}f^{\xi}\right){}^{3}\dot{f}^{\xi}+dr_{{}^{3}f^{\xi}}\left({}^{4}f^{\xi}\right){}^{4}\dot{f}^{\xi}\right)\\
&= \frac{\partial f^{\xi}}{\partial\alpha}\dot{\alpha}+\frac{\partial f^{\xi}}{\partial\epsilon}\dot{\varepsilon},\end{aligned}\tag{123}$$

where $\frac{\partial f^{\xi}}{\partial\alpha}=\left[\frac{\partial f^{\xi}}{\partial\alpha_1}\ \frac{\partial f^{\xi}}{\partial\alpha_2}\ \frac{\partial f^{\xi}}{\partial\alpha_3}\ \frac{\partial f^{\xi}}{\partial\alpha_4}\right]\in\mathbb{R}^{6\times4}$, with

$$\begin{aligned}\frac{\partial f^{\xi}}{\partial\alpha_1}&=dl_{{}^{4}f^{\xi}{}^{3}f^{\xi}}\left({}^{2}f^{\xi}{}^{1}f^{\xi}\right)dl_{{}^{2}f^{\xi}}\left({}^{1}f^{\xi}\right)\frac{\partial\left({}^{1}f^{\xi}\right)}{\partial\alpha_1},\\
\frac{\partial f^{\xi}}{\partial\alpha_2}&=dl_{{}^{4}f^{\xi}{}^{3}f^{\xi}}\left({}^{2}f^{\xi}{}^{1}f^{\xi}\right)dr_{{}^{1}f^{\xi}}\left({}^{2}f^{\xi}\right)\frac{\partial\left({}^{2}f^{\xi}\right)}{\partial\alpha_2},\\
\frac{\partial f^{\xi}}{\partial\alpha_3}&=dr_{{}^{2}f^{\xi}{}^{1}f^{\xi}}\left({}^{4}f^{\xi}{}^{3}f^{\xi}\right)dl_{{}^{4}f^{\xi}}\left({}^{3}f^{\xi}\right)\frac{\partial\left({}^{3}f^{\xi}\right)}{\partial\alpha_3},\\
\frac{\partial f^{\xi}}{\partial\alpha_4}&=dr_{{}^{2}f^{\xi}f^{\xi}_{I}}\left({}^{4}f^{\xi}{}^{3}f^{\xi}\right)dr_{{}^{3}f^{\xi}}\left({}^{4}f^{\xi}\right)\frac{\partial\left({}^{4}f^{\xi}\right)}{\partial\alpha_4}\end{aligned}\tag{124}$$

and

$$\begin{aligned}\frac{\partial f^{\xi}}{\partial\varepsilon}&=dl_{{}^{4}f^{\xi}{}^{3}f^{\xi}}\left({}^{2}f^{\xi}{}^{1}f^{\xi}\right)\\
&\cdot\left(dl_{{}^{2}f^{\xi}}\left({}^{1}f^{\xi}\right)\frac{\partial\left({}^{1}f^{\xi}\right)}{\partial\varepsilon}+dr_{{}^{1}f^{\xi}}\left({}^{2}f^{\xi}\right)\frac{\partial\left({}^{2}f^{\xi}\right)}{\partial\varepsilon}\right)\\
&+dr_{{}^{2}f^{\xi}{}^{1}f^{\xi}}\left({}^{4}f^{\xi}{}^{3}f^{\xi}\right)\\
&\cdot\left(dl_{{}^{4}f^{\xi}}\left({}^{3}f^{\xi}\right)\frac{\partial\left({}^{3}f^{\xi}\right)}{\partial\varepsilon}+dr_{{}^{3}f^{\xi}}\left({}^{4}f^{\xi}\right)\frac{\partial\left({}^{4}f^{\xi}\right)}{\partial\varepsilon}\right)\in\mathbb{R}^{6}.\end{aligned}\tag{125}$$

Derivation of open-loop dynamics (73). Calculating the term $\dot{z}_{\xi}$ from relationship (72) gives

$$\begin{aligned}\dot{z}_{\xi}&=\left(dr_{\overline{f}^{\xi}}\left(z_{\xi}\right)\right)^{-1}\dot{\tilde{\xi}}-\left(dr_{\overline{f}^{\xi}}\left(z_{\xi}\right)\right)^{-1}dl_{z_{\xi}}\left(\overline{f}^{\xi}\right)\dot{\overline{f}}^{\xi}\\
&\overset{(16)}{=}dr_{\overline{f}^{\xi\,-1}}\left(z_{\xi}\star\overline{f}^{\xi}\right)\dot{\tilde{\xi}}-dr_{\overline{f}^{\xi\,-1}}\left(z_{\xi}\star\overline{f}^{\xi}\right)dl_{z_{\xi}}\left(\overline{f}^{\xi}\right)\dot{\overline{f}}^{\xi}\\
&\overset{(42),(68)}{=}dr_{\overline{f}^{\xi\,-1}}\left(\tilde{\xi}\right)X^{C}\left(\tilde{\xi}\right)\left(C_{\xi}w+\tilde{p}\right)\\
&-dr_{\overline{f}^{\xi\,-1}}\left(\tilde{\xi}\right)dl_{z_{\xi}}\left(\overline{f}^{\xi}\right)X^{C}\left(\overline{f}^{\xi}\right)\left(A_{\alpha}\dot{\alpha}+A_{\alpha_r}\dot{\alpha_r}+A_{\varepsilon}\dot{\varepsilon}\right)\\
&\overset{(12)}{=}dr_{\overline{f}^{\xi\,-1}}\left(\tilde{\xi}\right)dl_{z_{\xi}}\left(\overline{f}^{\xi}\right)X^{C}\left(\overline{f}^{\xi}\right)\\
&\cdot\left(C_{\xi}w+\tilde{p}-A_{\alpha}\dot{\alpha}-A_{\alpha_r}\dot{\alpha_r}-A_{\varepsilon}\dot{\varepsilon}\right).\end{aligned}\tag{126}$$

Next we consider term $dr_{\overline{f}^{\xi\,-1}}\left(\tilde{\xi}\right)dl_{z_{\xi}}\left(\overline{f}^{\xi}\right)X^{C}\left(\overline{f}^{\xi}\right)$ more thoroughly. Taking into account that $dl_{z_{\xi}}\left(\overline{f}^{\xi}\right)X^{C}\left(\overline{f}^{\xi}\right)=dl_{z_{\xi}}\left(\overline{f}^{\xi}\right)dl_{\overline{f}^{\xi}}\left(e_{\xi}\right)X^{C}\left(e_{\xi}\right)$ and $dl_{\overline{f}^{\xi}}\left(e_{\xi}\right)=dr_{\overline{f}^{\xi}}\left(e_{\xi}\right)Ad\left(\overline{f}^{\xi}\right)$ gives

$$\begin{aligned}&dr_{\overline{f}^{\xi\,-1}}\left(\tilde{\xi}\right)dl_{z_{\xi}}\left(\overline{f}^{\xi}\right)X^{C}\left(\overline{f}^{\xi}\right)\\
&=dr_{\overline{f}^{\xi\,-1}}\left(\tilde{\xi}\right)dl_{z_{\xi}}\left(\overline{f}^{\xi}\right)dr_{\overline{f}^{\xi}}\left(e_{\xi}\right)Ad\left(\overline{f}^{\xi}\right)X^{C}\left(e_{\xi}\right)\\
&=dr_{\overline{f}^{\xi\,-1}}\left(\tilde{\xi}\right)dl_{z_{\xi}}\left(\overline{f}^{\xi}\right)dr_{\overline{f}^{\xi}}\left(e_{\xi}\right)X^{C}\\
&\cdot\left(e_{\xi}\right)\underbrace{X^{C}\left(e_{\xi}\right)^{-1}Ad\left(\overline{f}^{\xi}\right)X^{C}\left(e_{\xi}\right)}_{Ad^{X^{C}}\left(\overline{f}^{\xi}\right)}.\end{aligned}\tag{127}$$

Expanding the following terms $dr_{\overline{f}^{\xi\,-1}}\left(\tilde{\xi}\right)=dr_{\tilde{\xi}^{-1}\star z_{\xi}}\left(\tilde{\xi}\right)=dr_{z_{\xi}}\left(e_{\xi}\right)dr_{\tilde{\xi}^{-1}}\left(\tilde{\xi}\right)$ and $dl_{z_{\xi}}\left(\overline{f}^{\xi}\right)=dl_{\tilde{\xi}\star\overline{f}^{\xi\,-1}}\left(\overline{f}^{\xi}\right)=dl_{\tilde{\xi}}\left(e_{\xi}\right)dl_{\overline{f}^{\xi\,-1}}\left(\overline{f}^{\xi}\right)$ and using them in (127) yields in

$$\begin{aligned}&dr_{\overline{f}^{\xi\,-1}}\left(\tilde{\xi}\right)dl_{z_{\xi}}\left(\overline{f}^{\xi}\right)dr_{\overline{f}^{\xi}}\left(e_{\xi}\right)X^{C}\left(e_{\xi}\right)Ad^{X^{C}}\left(\overline{f}^{\xi}\right)\\
&=dr_{z_{\xi}}\left(e_{\xi}\right)\underbrace{dr_{\tilde{\xi}^{-1}}\left(\tilde{\xi}\right)dl_{\tilde{\xi}}\left(e_{\xi}\right)}_{Ad\left(\tilde{\xi}\right)}\underbrace{dl_{\overline{f}^{\xi\,-1}}\left(\overline{f}^{\xi}\right)dr_{\overline{f}^{\xi}}\left(e_{\xi}\right)}_{Ad\left(\overline{f}^{\xi\,-1}\right)}\\
&\cdot X^{C}\left(e_{\xi}\right)Ad^{X^{C}}\left(\overline{f}^{\xi}\right)\\
&=dr_{z_{\xi}}\left(e_{\xi}\right)Ad\left(\tilde{\xi}\star\overline{f}^{\xi\,-1}\right)X^{C}\left(e_{\xi}\right)Ad^{X^{C}}\left(\overline{f}^{\xi}\right)\\
&=\underbrace{dr_{z_{\xi}}\left(e_{\xi}\right)Ad\left(z_{\xi}\right)}_{dl_{z_{\xi}}\left(e_{\xi}\right)}X^{C}\left(e_{\xi}\right)Ad^{X^{C}}\left(\overline{f}^{\xi}\right)\\
&\overset{(12)}{=}X^{C}\left(z_{\xi}\right)Ad^{X^{C}}\left(\overline{f}^{\xi}\right).\end{aligned}\tag{128}$$

Substituting (127) to (126) implies

$$\begin{aligned}\dot{z}_{\xi}&=X^{C}\left(z_{\xi}\right)Ad^{X^{C}}\left(\overline{f}^{\xi}\right)\\
&\cdot\left(C_{\xi}w+\tilde{p}-A_{\alpha}\dot{\alpha}-A_{\alpha_r}\dot{\alpha_r}-A_{\varepsilon}\dot{\varepsilon}\right).\end{aligned}\tag{129}$$

Optimization of control law. Assume that control effort is given by Eq. (79). Taking into account input transformations (35) and (6) with (10) gives

$$w=U\left(\tilde{\theta},\Psi\left(\psi\right)\right)T\left(\psi\right)v.\tag{130}$$

Then, one can consider the following map

$$\overline{w}=\overline{T}\overline{v},\tag{131}$$

with

$$\overline{T}:=\begin{bmatrix}U\left(\tilde{\theta},\Psi\left(\psi\right)\right)T\left(\psi\right) & 0_{2\times4}\\ 0_{4\times2} & I_{4\times4}\end{bmatrix}\in\mathbb{R}^{6\times6}.\tag{132}$$

In order to simplify calculations the closed-loop dynamics (73) is rewritten as follows

$$\dot{z}_{\xi}=H\left(\overline{C}_{\xi}\overline{w}+b\right),\tag{133}$$

where $H:=X^{C}\left(z_{\xi}\right)Ad^{X^{C}}\left(\overline{f}^{\xi}\right)$ and $b:=\tilde{p}-A_{\varepsilon}\dot{\varepsilon}-A_{\alpha_r}\dot{\alpha}_r$.

We consider minimization of performance index $J\left(\overline{v}\right)$ under constraint (80). Substituting (133) to (80) gives

$$z_{\xi}^{\top}Q\overline{v}+z_{\xi}^{\top}Hb+z_{\xi}^{\top}W_2z_{\xi}=0,\tag{134}$$

with

$$Q := H\overline{C}_{\xi}\overline{T}. \tag{135}$$

The necessary condition for optimal solution taking into account constraint (80) can be written as follows

$$\frac{\partial}{\partial \overline{v}}\left(\frac{1}{2}\overline{v}^{\top}W_1\overline{v} + \lambda z_{\xi}^{\top}\left(Q\overline{v} + Hb + W_2 z_{\xi}\right)\right) = 0, \tag{136}$$

where $\lambda \in \mathbb{R}$ stands for Lagrange multiplier. Consequently, one obtains

$$W_1^{\top}\overline{v} + \lambda Q^{\top}z_{\xi} = 0 \tag{137}$$

which yields in

$$\overline{v} = -\lambda {W_1^{\top}}^{-1} Q^{\top} z_{\xi}. \tag{138}$$

Next using (138) in (134) one can calculate λ as follows

$$\lambda = \frac{z_{\xi}^{\top}\left(W_2 z_{\xi} + Hb\right)}{z_{\xi}^{\top} Q {W_1^{\top}}^{-1} Q^{\top} z_{\xi}}. \tag{139}$$

Finally one can formulate optimal control defined by Proposition 2.

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