

MAREK PAWLIKOWSKI \*

## CORTICAL BONE TISSUE VISCOELASTIC PROPERTIES AND ITS CONSTITUTIVE EQUATION – PRELIMINARY STUDIES

In the paper, preliminary studies on formulation of a new constitutive equation of bone tissue are presented. A bone is modelled as a viscoelastic material. Thus, not only are elastic properties of the bone taken into account, but also both short-term and long-term viscoelastic properties are considered. A potential function is assumed for the bone, constant identification on the basis of experimental stress-strain curve fitting is completed and a preliminary constitutive equation is formulated. The experiments consisted of compressive tests performed on a cuboids-like bone sample of the following dimensions:  $10 \times 5 \times 7.52$  mm. The specimen was compressed along the highest dimension at the strain rates  $0.016 \text{ s}^{-1}$  and  $0.00016 \text{ s}^{-1}$ . In addition to this, stress relaxation test was performed to identify long-term viscoelastic constants of bone. In the experiments, only displacement in the load direction was measured. The bone sample was extracted from a bovine femur. The form of the proposed potential function is such that it models a bone as a transversely isotropic material. For the sake of simplicity, it is assumed that the bone is incompressible. After the material constant identification the strain energy function proved to be adequate to describe bone behaviour under compressive load. Due to the fact that the function is convex, the results of the studies can be utilised in modelling of bone tissue in finite element analyses of an implant-bone system. Such analyses are very helpful in the process of a new prosthesis design as one can preoperatively verify the construction of the new implant and optimise its shape.

### 1. Introduction

Clinical results of orthopaedical surgeries depend on variety of factors, i.e. patient condition, surgical technique, applied artificial components etc. Also the way of patient's living influences significantly whether or not orthopaedical treatment is successful. Surgeons estimate the possible clinical results on the basis of medical examination of a patient. However, there is

---

\* *Warsaw University of Technology, Institute of Mechanics and Printing, ul.Narbutta 85, 02-524 Warszawa, Poland, email: mpawlik1@wip.pw.edu.pl*

always a certain risk, which cannot be pre-operatively predicted, that the organism's response to the surgery will be clinically unfavourable. Bone tissue atrophy is one of such negative reactions of the operated patient's organism [1], [2]. Bone atrophy, i.e. decrease of bone density, is a result of stress shielding phenomenon that takes place in bone after a prosthesis implantation. In normal conditions, non-implanted bone carries out load itself. After a prosthesis implantation, the load is carried out both by bone and the prosthesis. In consequence, stresses in the bone are lower than in non-implanted one. According to Wolff's law, stress reduction in bone in respect of natural state leads to bone functional adaptation, i.e. bone density decrease (internal remodelling) and/or bone volume decrease (external remodelling) [3].

Internal remodelling is the most dangerous phenomenon, which also occurs most often after a prosthesis implantation. The phenomenon may lead to the prosthesis loosening (e.g. the stem loosening in femur or the acetabulum loosening in pelvis in the case of implanted hip joint). Low durability of the prosthesis, which is revealed by early prosthesis loosening due to lack of its firm biofixation in the bone, leads to the revision arthroplasty that is usually more difficult than the primary arthroplasty. A realistic pre-operative modelling of bone behaviour after an orthopaedical surgery becomes, thus, very important, not to say indispensable, in the clinical practice to minimise the risk of failure. The mentioned above phenomenon of bone remodelling can be modelled as bone density change with time [4]. Simulation of bone remodelling allows one to predict, to a certain degree, how bone tissue will react to the changed load conditions after the surgery of prosthesis implantation. This consequently leads to the process of the prosthesis construction optimisation against bone density distribution in the vicinity of the prosthesis [5]. The author would like to go a step further and to create a new constitutive formula for bone tissue. In this paper, the preliminary studies in the range are shown.

The approach of the constitutive equation formulation presented in the paper is based on postulation of a potential function form  $W$  and is well described in the literature [6], [7], [8]. It is a very convenient approach as it makes it possible to take into account various phenomena, such as a material ability to dissipate energy, relation between material reaction and deformation rate, anisotropy of a material by means of structural tensors, etc. Potential energy of elasticity per unit of volume is the physical interpretation of the potential function  $W$ , which is also referred to as strain energy density. The energy must be an invariant of the coordinate system transformation. Thus, potential energy has to depend on invariants of the tensors used to describe material behaviour. In this approach, another method is also involved, namely

the approach of mechanistic models, i.e. springs and dampers. Bone tissue is a viscoelastic material. This is true for both trabecular bone [9], [10], [11] and compact bone [12], [13]. The viscoelastic properties of cortical bone are accounted for the presence of collagen matrix in the structure [14]. In the paper, those properties are modelled by means of the Kelvin-Voigt model, and then the model is applied to formulate the constitutive equations by means of a potential function. The differential equation describing a one-dimensional case is generalised to a tensor form and the proper expressions are replaced with derivatives of the potential function.

The strain energy density  $W$  is a scalar function of one tensorial variable, i.e. the deformation gradient  $\mathbf{F}$ . The second-order tensor  $\mathbf{F}$  represents the mapping of the deformation from the reference state to the deformed configuration. It is assumed that  $W$  vanishes in the reference configuration, i.e. when time  $t = 0$ . Thus, in the initial state the deformation gradient  $\mathbf{F} = \mathbf{I}$  where  $\mathbf{I}$  is the unit tensor. It is known from physical observations that the potential function increases monotonically with deformation. Thus,  $W$  attains its global minimum in the reference configuration which is a stress-free state:

$$W(\mathbf{I}) = 0, \quad W(\mathbf{F}) \geq 0. \quad (1)$$

Another restriction that is placed on the strain energy density is that in order to expand a body infinitely or to compress a body to the volume of zero an infinite amount of energy is needed:

$$W(\mathbf{F}) \rightarrow \infty \text{ as } \det \mathbf{F} \rightarrow \infty, \quad (2)$$

$$W(\mathbf{F}) \rightarrow \infty \text{ as } \det \mathbf{F} \rightarrow 0. \quad (3)$$

It is obvious that the potential function and the resulting constitutive equations must satisfy some requirements. Those requirements result from mathematical theory and the physical nature of the materials:

- i) convexity,
- ii) objectivity,
- iii) material-frame indifference.

The constraint i) is a fundamental one for existence and uniqueness of the solution in the boundary value analysis [15], [16]. In order to obtain a numerical solution in cases where an analytical one is not possible to be obtained, one has to ensure the uniqueness of the solution. This requirement is essential to have confidence in numerical results.

The objectivity constraint, which is also referred to as observer invariance demand, means that the state of deformation of a body cannot depend on the position of the observer registering the motion. This can be put in other words

as follows: two observers in different positions will observe the identical deformation of a body at one instance.

The third constraint is closely related to the previous one. It states that a rigid motion of a deformed body does not influence the value of the energy of the body.

In the paper, the potential function  $W$  for bone tissue is formulated such that it takes into account the instantaneous response of bone  $W_e$  and the viscous properties of the body  $W_v$ . Thus, takes the form:

$$W = W_e + W_v, \quad (4)$$

where  $W_e$  – elastic potential,  $W_v$  – viscous potential.

As it was mentioned above, the strain energy density depends on invariants of some tensors. The elastic potential  $W_e$  can be dependent on three invariants of the right Cauchy tensor  $\mathbf{C}$ , i.e.  $I_1, I_2$  and  $I_3$ . The tensor  $\mathbf{C}$  is defined:  $\mathbf{C} = \mathbf{F}^T \mathbf{F}$ . For incompressible isotropic materials,  $W_e$  is a function of only two invariants, i.e.  $I_1$  and  $I_2$  or one of them, as the third invariant of  $\mathbf{C}$  defines the volume change of a body.

The viscous potential  $W_v$ , on the other hand, can be a function of  $I_1, I_2, I_3$  and invariants of the right Cauchy deformation rate tensor  $\dot{\mathbf{C}}$ , i.e.  $J_1, J_2$  and  $J_3$ . For incompressible isotropic materials  $W_v$  depends on  $I_1$  and/or  $I_2$  and  $J_1$  and/or  $J_2$  [17].

In a general viscoelastic constitutive equation formulation one has to take into account the principle of fading memory developed in [18]. It takes into consideration the deformation history and states that the deformation occurred in the recent time history influences in a greater degree the actual state of stress than the deformation occurred in the more distant time history. The principle is mathematically expressed as follows:

$$\mathbf{S} = \mathbf{S}_e(\mathbf{C}(t)) + \int_0^t \mathcal{F}(\mathbb{G}(t-s), s; \mathbf{C}(t)) ds, \quad (5)$$

where:  $\mathbf{S}$  – second Piola-Kirchhoff stress tensor,  $\mathbf{S}_e$  – elastic second Piola-Kirchhoff stress tensor,  $\mathcal{F}$  – a general tensor-valued function that depends on variables  $\mathbb{G}(t-\tau)$  and  $\tau$ . In Eq. (5)  $s$  represents the historical time variable, whereas  $t$  is the current time.

The application of a viscoelastic constitutive model indicating the long-term viscoelastic properties is restricted to low strain rates, namely 0.0006–0.0075 s<sup>-1</sup> [19]. For higher low strain rates (up to 0.1 s<sup>-1</sup>) the constitutive model gives inaccurate results [20]. This seems to be quite evident as such viscoelastic models are to capture the whole history of deformation through the integral formulation and are not suited to capture the material behaviour in the very short time.

The first objective of the paper is to propose a potential function for bone tissue and formulate a phenomenological constitutive equation for the tissue. In the studies, the bone tissue is assumed to be incompressible and transversely isotropic. The viscoelastic material constants are identified on the basis of experimental data by means of the author's code realising Levenberg-Marquardt algorithm for least-square curve fitting. In this approach, only short-term viscoelastic effects are taken into account.

The second objective of the paper is to utilise the theory, first described in [21] and developed in [22], concerning long-term viscoelastic formulation of constitutive equation to create a new viscoelastic constitutive equation for bone tissue. The hereditary integral in Eq. (5) can be numerically approximated by a function representing the relaxation process of bone or by means of the theory introduced in [21]. In the process of bone viscoelastic constants identification, three Kelvin-Voigt models are applied to simulate viscoelastic properties of bone tissue.

## 2. Materials and Methods

The paper contains preliminary studies on the formulation of new constitutive equations that will be implemented in a commercial finite element system. Although bone tissue is a highly anisotropic material, it is considered in this work as a transversely isotropic one. In addition to this, it is assumed to be incompressible. This assumption is often applied to various biological tissues, e.g. [23].

### 2.1. Short-term viscoelastic effects

On the basis of thermodynamical considerations [24], a relation between the second Piola-Kirchhoff stress tensor and the arguments of the potential energy function  $W$  can be derived as:

$$\mathbf{S} = 2 \sum_{\alpha=1}^m \left( \frac{\partial W_e}{\partial I_\alpha} \frac{\partial I_\alpha}{\partial \mathbf{C}} \right) + 2 \sum_{\alpha=1}^n \left( \frac{\partial W_v}{\partial J_\alpha} \frac{\partial J_\alpha}{\partial \mathbf{C}} \right), \quad (6)$$

where  $m, n$  denote the number of the Cauchy stress tensor invariants and Cauchy stress rate tensor invariants, respectively. It can be seen from Eq. (6) that the stress response is divided in an elastic  $\mathbf{S}_e$  and a viscous  $\mathbf{S}_v$  contribution. The form of Eq. (6) ensures also fulfilment of requirement (1). Thus, Eq. (6) can be shortly written as follows:

$$\mathbf{S} = \mathbf{S}_e + \mathbf{S}_v \quad (7)$$

In the case of the models that take into account the short-term memory effects, the stress state depends only on a very short part of the deformation history. In those cases  $s$  in Eq. (5) tends to zero, and the component  $\mathbb{G}(t-s)$  in the equation can be approximated by its Taylor expansion. If we consider only the first degree of Taylor expansion, Eq. (5) takes the form of Eq. (7).

In order to calculate the second Piola-Kirchhoff stress tensor, one has to decide the number of elastic invariants  $I_\alpha$  and viscous invariants  $J_\alpha$  that the strain energy functions  $W_e$  and  $W_v$  depend on. Limbert and Middleton provide 12 elastic and viscous invariants for a transversely isotropic material [25]:

$$I_1 = \text{tr} \mathbf{C}, \quad I_2 = \frac{1}{2} \left( (\text{tr} \mathbf{C})^2 - \text{tr} \mathbf{C}^2 \right), \quad I_3 = \det \mathbf{C}, \quad (8)$$

$$I_4 = \mathbf{N}_0 : \mathbf{C}, \quad I_5 = \mathbf{N}_0 : \mathbf{C}^2, \quad (9)$$

$$J_1 = \text{tr} \dot{\mathbf{C}}, \quad J_2 = \text{tr} \dot{\mathbf{C}}^2, \quad J_3 = \det \dot{\mathbf{C}}, \quad (10)$$

$$J_4 = \mathbf{N}_0 : \dot{\mathbf{C}}, \quad J_5 = \mathbf{N}_0 : \dot{\mathbf{C}}^2, \quad (11)$$

$$J_6 = \text{tr} (\mathbf{C} \cdot \dot{\mathbf{C}}), \quad J_7 = \text{tr} (\mathbf{C} \cdot \dot{\mathbf{C}}^2), \quad J_8 = \text{tr} (\mathbf{C}^2 \cdot \dot{\mathbf{C}}), \quad J_9 = \text{tr} (\mathbf{C}^2 \cdot \dot{\mathbf{C}}^2), \quad (12)$$

$$J_{10} = \text{tr} (\mathbf{N}_0 \cdot \mathbf{C} \cdot \dot{\mathbf{C}}), \quad J_{11} = \text{tr} (\mathbf{N}_0 \cdot \mathbf{C} \cdot \dot{\mathbf{C}}^2), \quad J_{12} = \text{tr} (\mathbf{N}_0 \cdot \mathbf{C}^2 \cdot \dot{\mathbf{C}}), \quad (13)$$

where  $\mathbf{N}_0 = \mathbf{n}_0 \otimes \mathbf{n}_0$  is a symmetric second-order structural tensor,  $\mathbf{n}_0$  is a unit vector defined in the reference configuration oriented in the direction perpendicular to the isotropy plane,  $\cdot$  is the dot product of two tensors or vectors,  $:$  is the double contraction of two tensors,  $\otimes$  denotes the dyad or tensor product of two vectors,  $\text{tr}(\mathbf{A})$  represents the trace of the tensor  $\mathbf{A}$ ,  $\det(\mathbf{A})$  is the determinant of the tensor  $\mathbf{A}$ . It is assumed that in the uniaxial compression the vector  $\mathbf{n}_0$  represents the direction of loading.

The isotropic response of the material is represented by the invariants  $I_1, I_2, I_3, J_1, J_2, J_3, J_6, J_7, J_8$  and  $J_9$  while its transversely isotropic response is taken into account by means of the other complementary invariants.

As collagen, in particular type-I collagen, constitutes a great majority of the organic substances in bone [26], the structure of this biological tissue can be regarded as a composite structure made of the collagen fibres distributed in an isotropic solid matrix. Thus, it is reasonable that the mechanical behaviour of bone be described by means of the transversely isotropic material symmetry approach. In view of this information, the unit vector  $\mathbf{n}_0$  represents the preferred direction of anisotropy in the material [27]. Apart from this, the fact that bone contains a relatively high amount of water justifies the assumption of incompressibility of bone.

Constitutive relationships based on a strain function with fabric tensor taken into account have been proposed by various authors, e.g. [28], [29]. Those constitutive approaches, however, describe only the elastic contribution in a material behaviour. The strain energy function proposed in the paper is a non-linear viscoelastic function. As for other biological tissues, the potential functions that are most often presented in many papers concern soft tissues, such as ligaments or tendons, e.g. [23], [25], [30]. In the paper, the proposed strain energy function is adopted to bone tissue behaviour. For the sake of further possibility to utilise the new constitutive formula for bone tissue in numerical analyses, it is taken care of that the potential function be convex.

The identification of the material parameters is completed (Fig. 1) on the basis of experimental tests consisting in uniaxial compression along the vector  $\mathbf{n}_0$ . One compression test was performed with relatively high strain rate, namely  $0.016 \text{ s}^{-1}$ , in order to formulate a constitutive model including short-term viscoelastic effects. Another one was performed with lower strain rate, namely,  $0.00016 \text{ s}^{-1}$ , to simulate long-term viscoelastic effects. The strain rate  $0.016 \text{ s}^{-1}$  corresponds to the compressive velocity 10 mm/min, whereas  $0.00016 \text{ s}^{-1}$  corresponds to the velocity 0.1 mm/min. In this way, a wide range of the strain rate values is considered in the formulation of the constitutive equation for bone. In addition to this, a relaxation test was executed to simulate the viscous response of bone. During the tests, deformation in the direction of load was measured. The cuboid bone sample was cut out from the cortical tissue of the bovine femur. The dimension of the specimen were  $10 \times 5 \times 7.52 \text{ mm}$ . The specimen was compressed along the highest dimension which corresponded to the direction of the femur long axis. Before the sample formation, the bone was kept in the temperature of  $-20^\circ\text{C}$ . Then, it was left to thaw for 12 hours and, after that, the bone sample was cut out. Between the tests, the sample was kept in distilled water for two hours to avoid drying out. After that, the sample was refrozen. The tests were performed at the time interval of at least 24 hours to let the bone to relax. The sample surfaces perpendicular to the loading direction were covered with a thin layer of lubricant to ensure free sliding against the metal surfaces of the strength machine elements that compressed the sample. Thus, friction between the surfaces in contact and the risk of barrel-like deformation of the sample were minimised.

The state of deformation of the sample is described by the deformation gradient tensor  $\mathbf{F}$ :

$$\mathbf{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad (14)$$

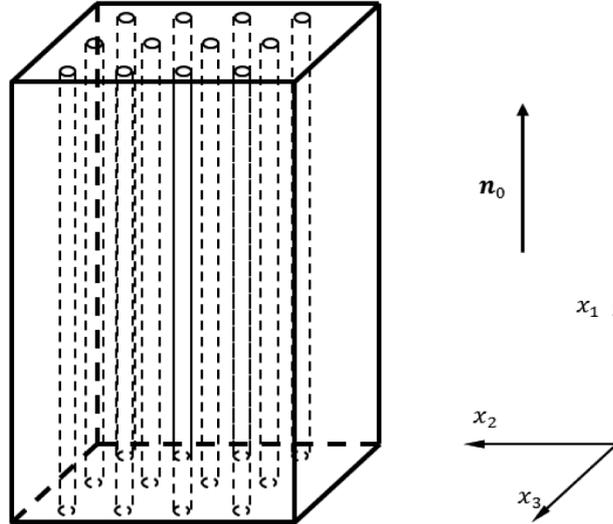


Fig. 1. The bone sample utilised in the experimental tests. Vector  $\mathbf{n}_0$  indicates the preferred direction of anisotropy. In the assumed coordinate system  $\mathbf{n}_0(1,0,0)$ .

where  $\lambda_1, \lambda_2, \lambda_3$  are the stretch ratios along the directions  $x_1, x_2, x_3$ , respectively. Due to incompressibility, the relation  $\lambda_1 \lambda_2 \lambda_3 = 1$  is valid. In addition to this, the transverse isotropy assumption justifies the fact that the stretch ratios in the direction perpendicular to  $\mathbf{n}_0$  are equal, i.e.  $\lambda_2 = \lambda_3$ . Thus, those two assumptions imply:  $\lambda_2 = \lambda_3 = \frac{1}{\sqrt{\lambda_1}}$ . The deformation gradient tensor is now defined as:

$$\mathbf{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda_1}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\lambda_1}} \end{bmatrix}. \quad (15)$$

According to Eq. (4), the strain energy function is represented by the sum of an elastic strain energy function  $W_e(I_1, I_5)$  and a viscous energy function  $W_v(J_2, J_5)$ . This additive hypothesis does not exclude the coupling between elastic and viscous invariants in the function  $W_v(J_2, J_5)$ . In view of this information,  $W_v(J_2, J_5)$  is rewritten as  $W_v(J_2, J_5, I_1, I_5)$ . This approach, i.e. the additive decomposition of stress, corresponds to a Kelvin-Voigt rheological model for bone tissue.

Based on the observations above, it is clearly visible that the strain energy function applied in these studies depends on two elastic invariants, i.e.  $I_1, I_5$ , and two viscous invariants, i.e.  $J_2, J_5$ . As the state of deformation in the material is multi-axial it seems natural to select  $I_1$  as one of the invariants

characterising the energy function, because it represents the sum of the square of stretch ratios  $\lambda_1, \lambda_2, \lambda_3$ . As for the invariant  $I_5$ , it was chosen in order to take into consideration the transverse isotropy. The invariant can be written as follows (see Eq. (9)-2):

$$I_5 = n_0^i C_{ik} C_{kj} n_0^j \quad (16)$$

Taking into account the form of the deformation gradient tensor (Eq. (15)) and the fact that the vector  $\mathbf{n}_0$  has the components  $\mathbf{n}_0(1,0,0)$ ,  $I_5$  represents stretch ratio along the fibre direction to the fourth power:

$$I_5 = \lambda_1^4 \quad (17)$$

The viscous invariant  $J_2$  represents the sum of the square of the eigenvalues of the right Cauchy deformation rate tensor  $\dot{C}_{ij}$ . Moreover, its derivative with respect to  $\dot{C}_{ij}$  is equal to the rate itself:

$$\frac{\partial J_2}{\partial \dot{C}_{ij}} = 2\dot{C}_{ij} \quad (18)$$

It is then ensured that the expression for viscous stress response is linearly dependant on the rate of deformation  $\dot{C}_{ij}$ . The invariant  $J_5$  was selected to take into account the anisotropic response of the material. It can be written in the form (see Eq. (11)-2):

$$J_5 = n_0^i \dot{C}_{ik} \dot{C}_{kj} n_0^j \quad (19)$$

In view of the experiment conditions and the assumption of transverse isotropy with the preferred anisotropy direction defined by  $\mathbf{n}_0(1,0,0)$  (Fig. 1),  $J_5$  can be written as:

$$J_5 = \dot{C}_{11}^2 \quad (20)$$

In order to be able to formulate an expression for stress  $\mathbf{S}$  (Eq. 6), the derivatives of the invariants  $I_1$  and  $I_5$  with respect to  $C_{ij}$  and that of  $J_5$  with respect to  $\dot{C}_{ij}$  have to be provided:

$$\frac{\partial I_1}{\partial C_{ij}} = \delta_{ij}, \quad \frac{\partial I_5}{\partial C_{ij}} = n_0^i C_{ik} n_0^k + n_0^j C_{jk} n_0^k, \quad \frac{\partial J_5}{\partial \dot{C}_{ij}} = n_0^i \dot{C}_{ik} n_0^k + n_0^j \dot{C}_{jk} n_0^k. \quad (21)$$

The strain energy function proposed for bone tissue is of the form:

$$W = c_1 (I_1 - 3)^2 + c_2 (I_5 - 1) \cdot \left( e^{c_2(I_5-1)} - 1 \right) + \mu_1 J_2 (I_1 - 3) + \mu_2 J_5 \cdot \ln \left( 1 + \mu_3 (I_5 - 1)^2 \right), \quad (22)$$

where:  $c_1, c_2, c_3$  are elastic material constants,  $\mu_1, \mu_2, \mu_3$  are viscous material constants to be identified on the basis of experimental tests. First, the elastic constants were identified and then the viscous ones. The constants were determined using a code written in MATLAB that realised the Levenberg-Marquardt algorithm. The method of least squares is utilised in the algorithm for best theoretical curve fitting to the experimental results.

Incorporating Eq. (22) into Eq. (6) one finally obtains:

$$\begin{aligned}
 S_{ij} = & 4c_1 (I_1 - 3) \delta_{ij} + 2 \left( c_2 \left( e^{c_2(I_5-1)} - 1 \right) + \right. \\
 & \left. + c_2 (I_5 - 1) c_3 e^{c_2(I_5-1)} \right) \left( n_0^i C_{ik} n_0^k + n_0^j C_{jk} n_0^k \right) + \\
 & + 2 \left( \mu_1 (I_1 - 3) 2C_{ij} + \mu_2 \cdot \ln \left( 1 + \mu_3 (I_5 - 1)^2 \right) \left( n_0^i C_{ik} n_0^k + n_0^j C_{jk} n_0^k \right) \right) - p C_{ij}^{-1},
 \end{aligned}
 \tag{23}$$

where  $p$  plays the role of a Lagrangean multiplier and has to be entered into the expression for the second Piola-Kirchhoff stress tensor when incompressibility is assumed. The physical meaning of the quantity  $p$  is a hydrostatic pressure. It is determined by means of the equilibrium equations and the boundary conditions.

## 2.2. Long-term viscoelastic effects

In the second approach, a relaxation test was performed in order to take into account long-term viscoelastic effects and, in addition, a uniaxial compression test with the strain rate  $0.00166 \text{ s}^{-1}$  was realised. On the basis of the compression test, the elastic constants in the elastic strain energy function were determined. The results of the relaxation test were utilised to determine viscous constants of the material.

In the relaxation test, the bone sample was subjected to the load corresponding to the stress in the sample 52MPa. The relaxation process was registered for approx. 20 min. The test was a ramp strain test, i.e. both the loading phase to the strain 0.0164 and the relaxation stage was taken into account for the viscous constant calibration. The time of the whole test could not be long to prevent the sample from excessive drying out.

The general constitutive model used in this approach consists of a strain-dependant function having dimension of stress  $T_0(\lambda_1)$  and a dimensionless time-dependant function  $g(t)$ :

$$T(\lambda_1, t) = T_0(\lambda_1) * g(t). \tag{24}$$

In Eq. (24),  $T(\lambda_1, t)$  denotes the first Piola-Kirchhoff stress at time  $t$  and strain corresponding to stretch ratio  $\lambda_1$ . The strain dependant function  $T_0(\lambda_1)$  is in

fact the first Piola-Kirchhoff stress derived from the elastic strain function  $W_e$ . The time-dependant function  $g(t)$  takes the form:

$$g(t) = g_\infty + \sum_{i=1}^3 g_i \cdot e^{-\frac{t}{\tau_i}}, \quad (25)$$

where  $g_\infty$  and  $g_i$  are constants,  $\tau_i$  represent the relaxation times in the Prony series [31]. The three constants  $g_i$  are to be calibrated from the experimental data and  $g_\infty$  is a function of them:

$$g_\infty = 1 - \sum_{i=1}^3 g_i. \quad (26)$$

It has to be noted that  $*$  in Eq. (24) denotes the convolution of  $T_0$  and  $g$ . Thus, Eq. (24) takes the form:

$$T(\lambda_1, t) = \int_0^t g(t-s) \frac{\partial T_0(\lambda_1)}{\partial \lambda_1} \frac{\partial \lambda_1}{\partial s} ds, \quad (27)$$

which, in turn, can be split into a long-term elastic response and a viscoelastic contribution:

$$T(t) = g_\infty T_0(t) + \frac{\sum_{i=1}^3 \int_0^t g_i \cdot e^{-\frac{t-s}{\tau_i}} (\partial T_0(s))}{\partial s} ds. \quad (28)$$

The stress  $T(t)$  is now a function of only time  $t$  if the strain history  $\lambda_1(t)$  is known.

The integral in Eq. (28) may be computed using the algorithm presented in [22], which is based on finite increments of time. Following the derivation introduced in [22], Eq. (28) can be written in the form:

$$T(t+1) = g_\infty T_0(t+1) + \sum_{i=1}^3 \left( e^{-\frac{\Delta t}{\tau_i}} \cdot h_i(t) + g_i \frac{1 - e^{-\frac{\Delta t}{\tau_i}}}{\frac{\Delta t}{\tau_i}} (T_0(t+1) - T_0(t)) \right), \quad (29)$$

where  $\Delta t$  is the time increment,  $h_i(t)$  represents the stress at the previous time step. As the initial stress and strain in the material are known, the stress at time  $t > 0$  can be easily calculated.

The constant calibration in the constitutive modelling with long-term viscoelastic effects taken into consideration was completed according to the following plan: first the elastic constants  $c_1$ ,  $c_2$  and  $c_3$  and viscous ones  $g_1$ ,  $g_2$  and  $g_3$  were determined on the basis of the relaxation compression test. In the second step, the uniaxial compression test data were used to calibrate once again the elastic constants. The relaxation times  $\tau_1$ ,  $\tau_2$ ,  $\tau_3$  were assumed to be equal to 1, 10, 100 s, respectively.

### 3. Results

The viscoelastic constitutive model for bone tissue was firstly determined taking into account only short-term viscoelastic effects. In this step, the potential function given in Eq. (22) was chosen. The constants were calculated using the monotonic stress-strain data obtained from the uniaxial compression test.

The second approach consisted in determination of a constitutive model for bone tissue that took into account long-term viscoelastic effects. The constants were calibrated using the stress-strain data and the relaxation data.

#### 3.1. Short-term viscoelastic effects

The elastic and viscoelastic constants were identified using the Levenberg-Marquardt algorithm realising the least-square optimisation procedure [32]. The computer code utilising the algorithm was written in the Matlab script language.

Utilising only the elastic part of Eq. (23) and taking advantage of the fact that stress in the perpendicular directions was equalled to zero, the elastic constants were first calibrated. In the process of the constant calibration, the relationship between the first and the second Piola-Kirchhoff stress tensors was utilised:

$$T_{ij} = F_{ik} S_{kj}. \quad (30)$$

The values of the identified constants are presented in Table 1.

Table 1.

The elastic constants obtained by identification of the proposed transversely isotropic viscoelastic model with the experimental data (uniaxial compression with strain rate  $0.016 \text{ s}^{-1}$ ). The elastic strain energy function  $W_e$  from Eq. (23) was utilised in the calibration process

$c_1$ [MPa]	33.5
$c_2$ [MPa]	104.8
$c_3$ [-]	2.6

From the numerical point of view it is crucial that the elastic potential  $W_e$  in Eq. (22) be convex. In Fig. 2, the potential  $W_e$  is shown as a function of  $C_{11}$  and  $C_{22}$ . The value of  $C_{33}$  was assumed to be equal 1.1. Under the compressive test along the 1<sup>st</sup> direction (Fig. 1) the value of the stretch ratio in the 3<sup>rd</sup> direction will be increasing, as well as that in the 2<sup>nd</sup> direction. The idea of Fig. 2 is to show qualitatively the convexity of the potential function. The value of  $C_{33}$  can be assumed arbitrarily with the restriction that it should be greater than 1. Fig. 2 clearly demonstrates that the elastic potential of the proposed form is convex.

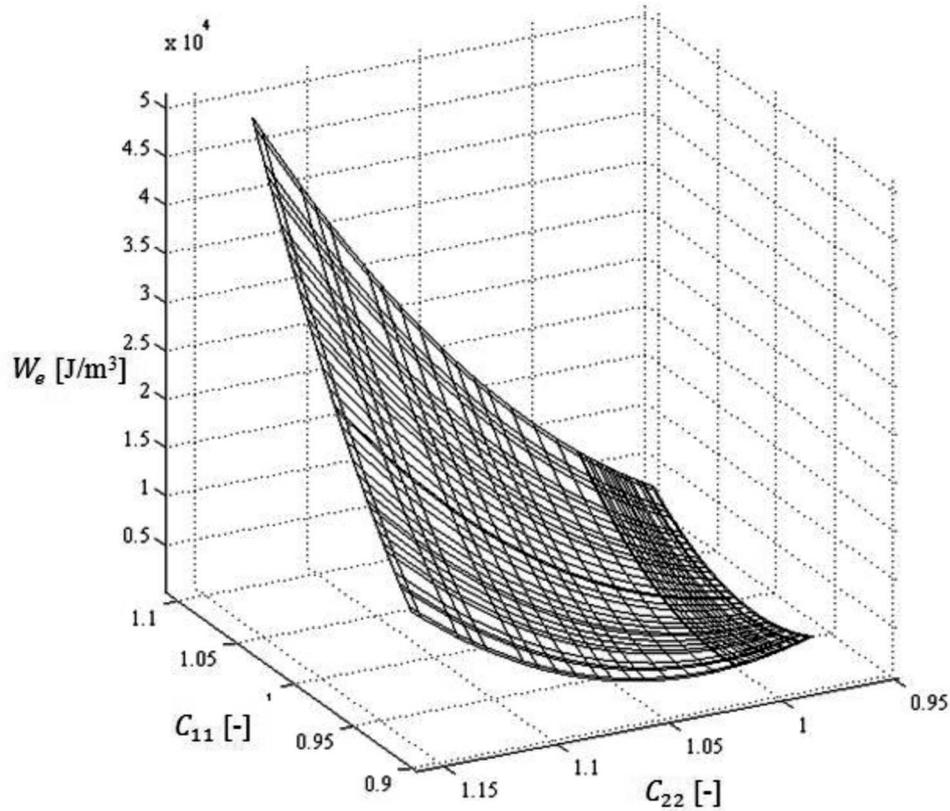


Fig. 2. Convexity of the elastic potential  $W_e$  in function of  $C_{11}$  and  $C_{22}(C_{33}=1.1)$

Thus, the identification of the viscous constants could be performed. Through this calibration process, the values of the elastic constants were kept unchanged and equal, as it is presented in Table 1. The full forms of Eq. (22) and Eq. (23) were used, as well as Eq. (30). The identification of the constants  $\mu_1, \mu_2$  and  $\mu_3$  were performed on the same experimental data obtained from uniaxial compression test. In Fig. 3 the theoretical curve matched to the experimental data is presented. The values of the viscous constants are listed in Table 2.

Table 2.

The viscous constants obtained by identification of the proposed transversely isotropic viscoelastic model with the experimental data (uniaxial compression with strain rate  $0.016 \text{ s}^{-1}$ ). The full strain energy function  $W$  from Eq. (23) was utilised in the calibration process

$\mu_1$ [MPa·s]	82433.5
$\mu_2$ [MPa·s]	126392.95
$\mu_3$ [-]	-0.12

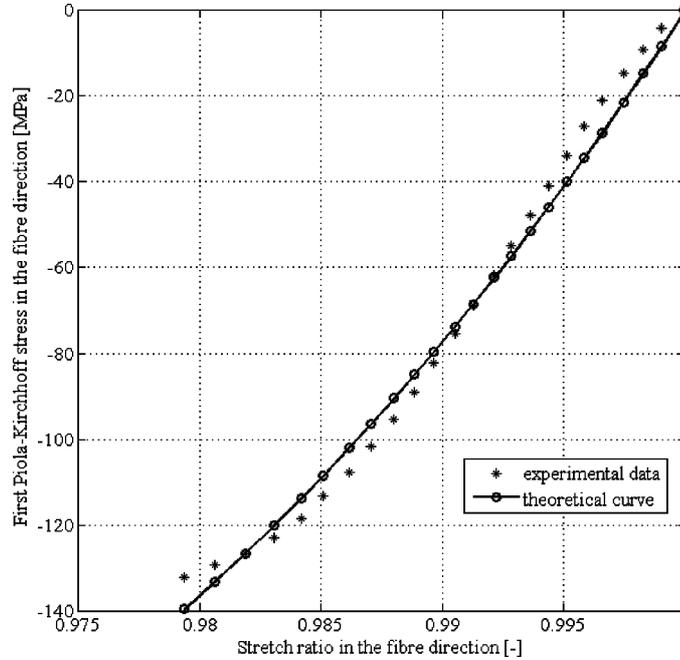


Fig. 3. Analytical stress-stretch ratio curve obtained at the compression rate  $0.016 \text{ s}^{-1}$ . The curve match corresponds to the identification of the elastic and viscous constants  $c_1, c_2, c_3, \mu_1, \mu_2, \mu_3$  in Eq. (23)

In view of the identified viscoelastic constants, the expression (22) for strain energy function  $W$  is written as follows:

$$\begin{aligned}
 W = & \underbrace{33.5 (I_1 - 3)^2 + 104.8 (I_5 - 1) \cdot (e^{2.6(I_5 - 1)} - 1)}_{W_e} + \\
 & + \underbrace{82433.5 \cdot J_2 (I_1 - 3) + 126392.95 \cdot J_5 \cdot \ln(1 - 0.12 (I_5 - 1)^2)}_{W_v} \quad (31)
 \end{aligned}$$

From Eq. (6), the second Piola-Kirchhoff stress can be now calculated. The hydrostatic pressure  $p$  can be determined from the condition that  $S_{22} = 0$ . Utilising the relationship between  $S$  and the Cauchy stress  $\sigma$ :

$$\sigma_{ij} = F_{ik} S_{kl} F_{jl} J^{-1}, \quad (32)$$

where  $J$  is the Jacobian of the deformation tensor  $\mathbf{F}$  defined as  $J = \det \mathbf{F}$ , one can easily formulate the viscoelastic constitutive equation for bone tissue.

#### *Long-term viscoelastic effects*

Combining Eq. (6) and Eq. (30), we can write the elastic part of the first Piola-Kirchhoff stress in the direction of the fibres in the form:

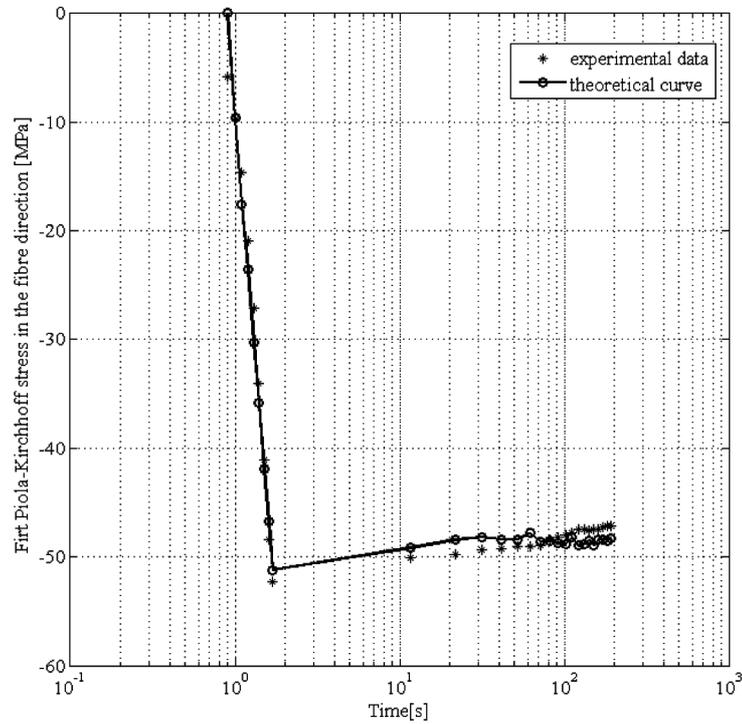


Fig. 4. Analytical curve obtained from the relaxation test data. The curve match corresponds to the identification of the elastic and viscous constants  $c_1, c_2, c_3, g_1, g_2, g_3$  in Eq. (29)

$$T_0 = \frac{1}{\lambda_1} \left( 4 \cdot \left( 1 + 2.6 \left( \lambda_1^4 - 1 \right) \right) \cdot 104.8 \lambda_1^4 e^{2.6 \cdot \left( \lambda_1^4 - 1 \right)} - 419.2 \lambda_1^4 + 134 \left( \lambda_1^2 + \frac{2}{\lambda_1} - 3 \right) \lambda_1^2 - \frac{134}{\lambda_1} \left( \lambda_1^2 + \frac{2}{\lambda_1} - 3 \right) \right). \quad (33)$$

Now, utilising the algorithm based on finite increments of time and Eq. (29) the six constants  $c_1, c_2, c_3, g_1, g_2$  and  $g_3$  were calibrated. In this case, also a code in Matlab was written. The theoretical curve (29) was matched to the relaxation compression test data (Fig. 4).

The elastic constants were then recalculated by matching Eq. (29) with the experimental data obtained from the uniaxial compression test at the strain rate  $0.00016 \text{ s}^{-1}$ . The recalibrated values of the elastic and viscous constants are presented in Table 3.

The relaxation function (25) is, thus, defined and the constitutive equation (24) for bone tissue can be fully formulated. The equation takes into account long-term viscoelastic effects.

Table 3.

The elastic and viscous constants obtained by recalibration of the proposed transversely isotropic viscoelastic model with the experimental data (compression test at compression rate  $0.00016 \text{ s}^{-1}$  and relaxation test)

$c_1$ [MPa]	27.25
$c_2$ [MPa]	426.09
$c_3$ [-]	0.46
$g_1$ [-]	0.054
$g_2$ [-]	0.044
$g_3$ [-]	0.034

#### 4. Discussion

The main objective of the paper was to utilise the method of a strain energy function postulation to derive a constitutive equation for bone tissue. Such a method is widely used to formulate a constitutive equation for various biological tissues [23], [26], [33], [34]. In the paper, the method was adopted to create a constitutive model for bone tissue.

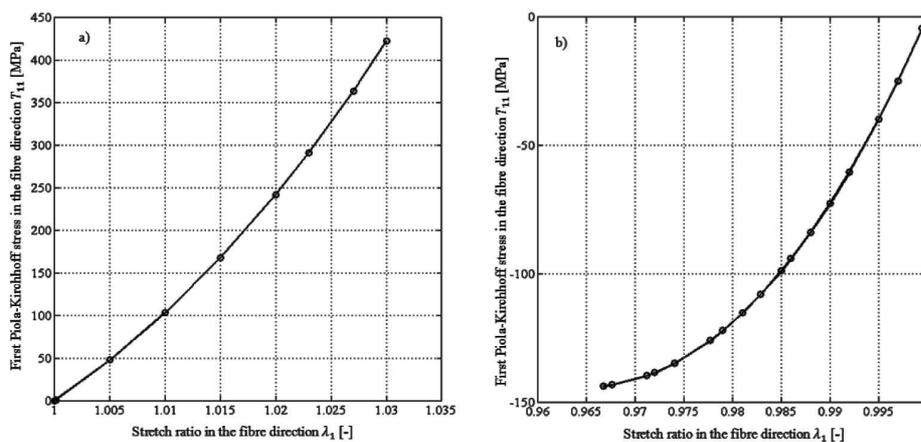


Fig. 5. The response of the material described by the short-term viscoelastic effects constitutive model derived from the potential function (22) to the tensile (a) and compressive (b) simulated loads

In the paper, two approaches of viscoelasticity were presented. In the first one, bone tissue was loaded with a relatively high strain rate, i.e.  $0.016 \text{ s}^{-1}$ . In such a case, the material can be modelled with short-term viscoelastic effects taken into account. The proposed potential  $W$  (22) fulfils the restrictions (1), (2) and (3) and the requirements relating to convexity, objectivity and material-frame indifference. The stress in the fibre direction was derived for

compressive load. In order to verify how it describes the material behaviour under a tensile load, a simulation of a uniaxial tensile test was performed. The result was compared to that of a simulated compressive test (Fig. 5). Fig. 5a shows a prediction of the material behaviour to tensile load ( $\lambda_1$  corresponds to the stretch ration in the simulated tensile test). Fig. 5b shows a prediction of the material behaviour to compressive load ( $\lambda_1$  corresponds to the stretch ration in the simulated compressive test). The curves  $T_{11}(\lambda_1)$  were derived from Eq. (22). It can be seen that the constitutive model describes the material behaviour under compressive load sufficiently well until the moment when stretch ratio achieves the value of 0.97. The corresponding stress is equal to approx. 130 MPa. Under such a compressive load the bone breaks, which is simulated by the theoretical  $T_{11}$  by the change of the curve slope. It does not describe correctly the material behaviour under tensile load, see e.g. [35]. The potential function has to be examined in this matter more thoroughly.

Also the constitutive model taking into account long-term viscoelastic effects was verified in the same manner. The response of the material described by such a constitutive equation is presented in Fig. 6. The derived constitutive equation describes almost linearly the response of bone tissue to tensile (Fig. 6a) and compressive (Fig. 6b) loads. Comparing the theoretical curve shown in Fig. 6b corresponding to compression test to the experimental data presented in Fig. 3 one can conclude that the constitutive model (29) with the proposed elastic potential function is adequate for compressive loads up to the destructive load.

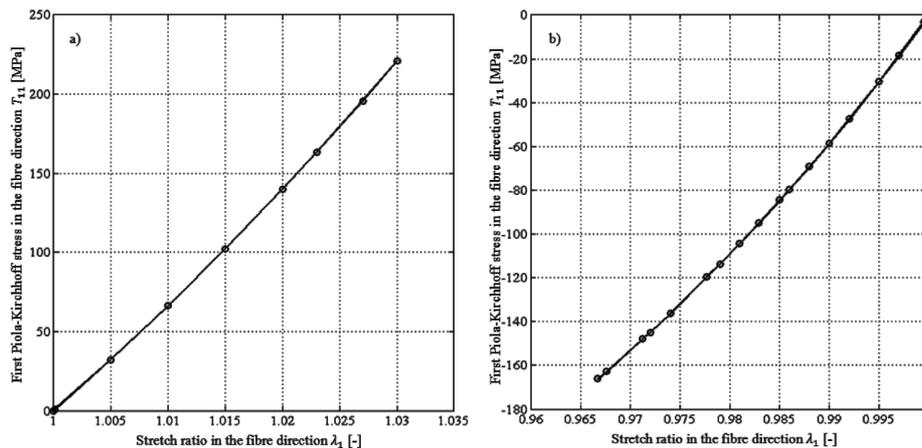


Fig. 6. The response of the material described by the long-term viscoelastic effects constitutive model (29) to the tensile (a) and compressive (b) simulated loads

The constants in the proposed strain energy function have their physical meaning. The constants  $c_1$  and  $c_2$  are isotropic and anisotropic elastic

constants, respectively. The constant  $c_3$  is a material parameter that warrants convexity of the potential function and good matching to the experimental results. The constants  $\mu_1$  and  $\mu_2$  are isotropic and anisotropic viscosity coefficients, respectively. The constant  $\mu_3$  has the similar meaning as  $c_3$ .

The bone sample was subjected to strength tests of various types, i.e. compression test at the two strain rates and a relaxation test. The tests were performed in the time interval 24 hours. After each test, it was kept in distilled water for two hours and then refrozen. The preservation of bone samples is very important for their mechanical properties. Unger et al. concluded that the mechanical properties of cortical bone are preserved best when they are fresh-frozen [36]. They suggest that bone samples kept in a special liquid can be utilised only in pilot tests. There are other factors that influenced the obtained results, such as: i) the tests were performed in room temperature and humidity, ii) the “history” of the animal bone, from which the sample was cut out, is not known, iii) the process of the sample cutting out was also performed in room temperature.

The experimental tests were done in room temperature and humidity. The displacement along the load direction was measured. In the case when incompressibility of the material is assumed, the deformation measurement is sufficient.

To minimise the friction between the sample and the machine plates the surfaces in contact were lubricated. The lubrication of the frontal surfaces of the sample could have had but a positive influence on the character of the stress distribution in the compressed specimen. Thus, it seems that the surface lubrication is reasonable and correct.

The relaxation test could not be performed longer than 20 minutes because it was done in room temperature and humidity. Thus, longer period of stress relaxation would lead to excessive drying out of the sample and make the results less accurate.

In order to fully describe mathematically the behaviour of bone under various loads, one has to study both short- and long-term viscoelastic properties. The former effects are predominant in the case of cyclic loads that are exerted in bones during, for instance, the gait cycle. The studies on the long-term viscoelasticity of bone are undertaken in the context of bone remodelling phenomenon. Mathematical models of the adaptation phenomenon incorporate internal energy that changes during the process of deformation. The energy can be identified with the potential energy. The method of constitutive equation formulation, presented in the paper, is based on the postulation of the potential energy form.

Studies on long-term viscoelasticity of bone are also justified by the fact that the results of the paper are going to be implemented in numerical

analyses of bone behaviour and included in prosthesis design process. The fact that viscoelastic properties of bone will be also included makes the analyses more reliable as the load transfer between the prosthesis and bone is, thus, simulated in a more realistic manner [37]. Hence, pre-operative optimisation of the prosthesis as well as follow-up prediction of its clinical functioning will be possible to perform.

## 5. Conclusions

The study presented in the paper is a preliminary investigation leading to the further studies aiming at formulation of constitutive equations for bone tissue and their application into finite element analyses. The results were obtained from experimental tests performed on one bone sample. As bone tissue, especially that of long bones, is mainly subjected to compressive loads, it has been decided to examine the tissue at this stage of the researches only under compression.

Generally, the proposed strain energy function proved to be adequate to simulate bone behaviour subjected to compressive load. Since the function is convex, the derived constitutive equations can be adopted in numerical analyses. However, the work on the constitutive model formulation is not yet finished. To create a full and reliable set of constitutive equations for bone, more experiments have to be done. Moreover, the experiments should be done on samples extracted from different regions of one bone. Also the fact that the derived model does not describe bone behaviour under tensile load has to be dealt with.

The methodology of constitutive equation formulation presented in the paper proved to be adequate in application to bone tissue. The special emphasis has to be put on the potential function form. It influences the final result, i.e. the proper form of the constitutive equation that can be utilised in numerical analyses.

In the future it is also planned to include the remodelling phenomenon of bone into advanced numerical analyses of bone-implant system with the new constitutive equation of bone taken into account. It is crucial for the prosthesis construction optimisation process.

## REFERENCES

- [1] Aamodt A. et al.: Changes in proximal femoral strain after insertion of uncemented standard and customised femoral stems. An experimental study in human femora, *J. Bone Joint Surg.*, 2001, 83, 6, 921-929.
- [2] Doblare M., Garcia J.M.: Anisotropic bone remodeling model based on a continuum damage-repair theory, *J. Biomechanics*, 2001, 35, 1-17.
- [3] Mullender M.G., Huiskes R.: Proposal for the regulatory mechanisms of Wolff's law, *J. Orthop. Res.*, 1995, 13, 1059-1062.
- [4] Hart R.T.: Bone modeling and remodeling: theories and computation, [in:] *Bone Mechanics*, S.C. Cowin (Ed.), CRC Press, Boca Raton, 2001.
- [5] Pawlikowski M., Skalski K., Haraburda M.: Process of hip joint prosthesis design including bone remodeling phenomenon, *Computers and Structures*, 2003, 81(8-11), 887-893.
- [6] Truesdell C., Noll W.: *The Non-linear Field Theories*, Springer, 1992.
- [7] Coleman B.D.: Thermodynamics of Materials with Memory, *Arch. Rat. Mech. Analysis*, 1964, 17: 3-46.
- [8] Skalski K., Pawlikowski M., Suchocki C.: Constitutive equations and bone functional adaptation ([in Polish]: Równania konstytutywne i adaptacja funkcjonalna kości), *Technical Mechanics v. XII, Biomechanics*, Polish Academy of Sciences, 517-614, 2011.
- [9] Carter D.R., Hayes W.C.: The compressive behavior of bone as a two-phase porous structure. *J Bone Joint Surg [Am]*, 1977, 59:954-62.
- [10] Linde F.: Elastic and viscoelastic properties of trabecular bone by a compression testing approach., *Dan Med Bull.*, 1994, 41(2):119-38.
- [11] Guedes R.M., Simoes J.A., Morais J.L.: Viscoelastic behaviour and failure of bovine cancellous bone under constant strain rate, *Journal of Biomechanics*, 2006, 39, 49-60.
- [12] Lakes R.S., Katz J.L.: Viscoelastic properties of wet cortical bone – II. Relaxation Mechanisms, *J. Biomech*, 1979, 12, 679-687.
- [13] Keaveny T.M., Morgan E.F., Yeh O.C.: *Bone Mechanics*, in *Biomechanics of the Human Body*, McGraw-Hill, 2009.
- [14] Knetts I.: Peculiarities of the Structure and Mechanical Properties of Biological Tissues, *Meccanica*, 2002, 37, 375-384.
- [15] Chen W. F., Han D.J.: *Plasticity for structural engineers*. Springer-Verlag, 1988.
- [16] Curnier A.: *Computational methods in solid mechanics*. Kluwer Academic Publishers, 1994.
- [17] Limbert G., Middleton J.: A transversely isotropic viscohyperelastic material. Application to the modeling of biological soft connective tissues, *Int. J Solid and Structures*, 2004, 41, 4237-4260.
- [18] Coleman B.D., Noll W.: Foundations of linear viscoelasticity. *Rev Mod Phys*, 1961; 3(2):239-49.
- [19] Haut R.C.: Little R.W.A. A constitutive equation for collagen fibers. *J Biomech* 1972, 5, 423-30.
- [20] Woo S.L.Y., Gomez M.A., Akeson W.H.: The time and history-dependent viscoelastic properties of the canine medial collateral ligament. *J Biomech Eng*, 1981, 103, 293-298.
- [21] Taylor R.L., Pister K.S., Goudreau G.L.: 'Thermomechanical analysis of viscoelastic solids', *Int. J. Numerical Methods Eng.*, 1970, 2, 45-59.
- [22] Goh S.M., Charalambides M.N., Williams J.G.: Determination of the Constitutive Constants of Non-Linear Viscoelastic Materials, *Mechanics of Time-Dependent Materials*, 2004, 8: 255-268.
- [23] Pioletti D.P., Rakotomanana L.R., Benvenuti J.F., Leyvraz P.F.: Viscoelastic constitutive law in large deformations: application to human knee ligaments and tendons. *Journal of Biomechanics*, 1998b, 31 (8), 753-757.
- [24] Coleman B.D., Gurtin M.E.: Thermodynamics with internal state variables. *Journal of Chemistry and Physics*, 1967, 47, 597-613.

- [25] Limbert G., Middleton J.: A transversely isotropic viscohyperelastic material. Application to the modelling of biological soft connective tissues, *Int. J. of Solids and Structures*, 2004, 41, 4237-4260.
- [26] Viguet-Carrin S., Garnero P., Delmas P.D.: The role of collagen in bone strength, *Osteoporos. Int.*, 2006, 17, 319-336.
- [27] Spencer A.J.M.: *Continuum theory of the mechanics of fibre-reinforced composites*. New York: Springer-Verlag; 1992.
- [28] Jemioło S., Telega J.J.: Fabric tensor in bone mechanics, *Eng. Trans.*, 1998, 46, 3-26.
- [29] Zysset P.K., Curnier A.: An alternative model for anisotropic elasticity based on fabric tensors, *Mech. Mat.*, 1995, 21, 243-250.
- [30] Limbert G., Middleton J.: A constitutive model of the posterior cruciate ligament, *Medical Engineering and Physic*, 2006, 28, 99-113.
- [31] Fung Y.C.: *Foundations of Solid Mechanics*, Prentice-Hall, Inc., 1965.
- [32] Sun W., Yuan Y.-X.: *Optimization theory and methods. Nonlinear Programming*. Springer, 2006.
- [33] Drapaca C.S., Tenti G., Rohlf K., Sivaloganathan S.: A Quasi-linear Viscoelastic Constitutive Equation for the Brain: Application to Hydrocephalus, *J Elasticity*. 2006, 85, 65-83.
- [34] Galle B., Ouyang H., Shi R., Nauman E.: A transversely isotropic constitutive model of excised guinea pig spinal cord white matter, *J Biomechanics*, 2010, 43, 2839-2843.
- [35] Joa W., Jepsen K.J., Davy D.T.: The effect of recovery time and test conditions on isoelectric measures of tensile damage in cortical bone, *J Biomechanics*, 2007, 40, 2731-2737.
- [36] Unger S., Blauth M., Schmoelz W.: Effects of three different preservation methods on the mechanical properties of human and bovine cortical bone, *Bone*, 2010, 47, 1048-1053.
- [37] Piszczatowski S., Skalski K., Świążkowski W.: Load transfer between elastic hip Implant and Viscoelastic Bone, in: *Computer Methods in Biomechanics& Biomedical Engineering – 2*, Gordon and Breach Science Publishers, 123-130, 1998.

### **Równanie konstytutywne i lepkosprężyste właściwości korowej tkanki kostnej – badania wstępne**

#### **Streszczenie**

W pracy przedstawiono wstępne badania nad sformułowaniem nowego równania konstytutywnego dla tkanki kostnej. Kość zamodelowano jako materiał lepkosprężysty. W badaniach uwzględniono więc, nie tylko właściwości sprężyste tkanki kostnej, ale także krótko- i długoterminowe właściwości lepkosprężyste. Zaproponowano funkcję potencjalną, na podstawie badań eksperymentalnych wyznaczono stałe materiałowe metodą najlepszego dopasowania do krzywej naprężenie-odkształcenie oraz sformułowano wstępne równanie konstytutywne dla tkanki kostnej. Badania eksperymentalne polegały na przeprowadzeniu testów ściskania prostopadłościenną próbki kostnej o wymiarach 10×5×7,52 mm. Próbka była ściskana wzdłuż największego wymiaru z prędkościami odkształcenia 0,016 s<sup>-1</sup> i 0,00016 s<sup>-1</sup>. Dodatkowo wykonano jeszcze próbę relaksacji w celu zidentyfikowania długoterminowych właściwości lepkosprężystych kości. Podczas badań doświadczalnych mierzono tylko przemieszczenie w kierunku siły ściskającej. Próbka kostna została wycięta z wołowej kości udowej. Postać funkcji potencjalnej dobrano w ten sposób, żeby modelowała ona tkankę kostną jako materiał poprzecznie izotropowy. Dla uproszczenia założono, że kość zachowuje się jak materiał nieściśliwy. Postać funkcji energii odkształcenia, po wyznaczeniu stałych, była adekwatna do opisu zachowania się tkanki kostnej pod wpływem obciążenia ściskającego. Ze

względu na to, że funkcja ta jest wypukła, wyniki badań mogą być wykorzystane w modelowaniu tkanki kostnej w analizach układu implant-kość wykonanych metodą elementów skończonych. Tego rodzaju analizy są bardzo pomocne w procesie projektowania nowej endoprotezy, ponieważ można jeszcze przed operacją zweryfikować jej konstrukcję i dokonać optymalizacji jej kształtu.