Group and Phase Velocity of Love Waves Propagating in Elastic Functionally Graded Materials

Piotr KIEŁCZYŃSKI, Marek SZALEWSKI, Andrzej BALCERZAK, Krzysztof WIEJA

Institute of Fundamental Technological Research
Polish Academy of Sciences
Pawińskiego 5B, 02-106 Warszawa, Poland; e-mail: pkielczy@ippt.pan.pl

This paper presents a theoretical study of the propagation behaviour of surface Love waves in nonhomogeneous functionally graded elastic materials, which is a vital problem in acoustics. The elastic properties (shear modulus) of a semi-infinite elastic half-space vary monotonically with the depth (distance from the surface of the material). Two Love wave waveguide structures are analyzed: 1) a nonhomogeneous elastic surface layer deposited on a homogeneous elastic substrate, and 2) a semi-infinite nonhomogeneous elastic half-space. The Direct Sturm-Liouville Problem that describes the propagation of Love waves in nonhomogeneous elastic functionally graded materials is formulated and solved 1) analytically in the case of the step profile, exponential profile and $1cosh^2$ type profile, and 2) numerically in the case of the power type profiles (i.e. linear and quadratic), by using two numerical methods: i.e. a) Finite Difference Method, and b) Haskell-Thompson Transfer Matrix Method.

The dispersion curves of phase and group velocity of surface Love waves in inhomogeneous elastic graded materials are evaluated. The integral formula for the group velocity of Love waves in nonhomogeneous elastic graded materials has been established. The results obtained in this paper can give a deeper insight into the nature of Love waves propagation in elastic nonhomogeneous functionally graded materials.

Keywords: surface Love waves, group velocity, phase velocity, functionally graded materials, profiles of elastic constants, direct Sturm-Liouville problem.

1. Introduction

In microelectronics and other modern branches of industry (e.g. automotive, aviation and aerospace) layered and heterogeneous materials are applied. An example of these materials are Functionally Graded Materials (FGM), wherein the elastic properties change continuously with increasing distance from the treated surface of the material. Determination of the distribution of elastic parameters in heterogeneous Graded Materials is of great practical importance.

For the measurement of physical parameters of materials, ultrasonic methods have been used recently (KRAUTKRAMER, KRAUTKRAMER, 1983; KIEŁCZYŃSKI *et al.*, 1998; 2002; ROSTOCKI *et al.*, 2011).

Love waves are SH (Shear Horizontal) surface acoustic waves that can propagate in layered media and heterogeneous elastic half-space (ACHENBACH,

1973; KIEŁCZYŃSKI, 1981; AULD, 1991; URBANCZYK, JAKUBIK, 1997). Love wave energy is concentrated near the surface of the waveguide. For this reason, all the disturbances of the material properties in the subsurface region, have considerable influence on the dispersion characteristics of the Love wave (i.e. velocity and attenuation as a function of frequency). Love wave penetration depth depends on the frequency. Therefore, Love waves are particularly suitable to the study of the physical properties of heterogeneous graded materials.

Sensors based on SH (Shear Horizontal) surface acoustic waves (e.g. Love waves), due to their high sensitivity, are used for measuring physical properties of the liquid (e.g. viscosity and density) (KIEŁCZYŃSKI *et al.*, 2011a; 2012a, 2012b; 2014) as well as biosensors and chemosensors (WANG *et al.*, 2008). Moreover, these sensors can be used to investigate thin films (KIEŁCZYŃSKI, 1981; KIEŁCZYŃSKI, SZALEWSKI,

2011b) and layers produced in the surface region of the substrate as a result of various technological processes (diffusion, implantation, carburizing, nitriding, shot peening, laser treatment, etc.) (HIRAO *et al.*, 1981), and also for testing of composites (WANG, ZHAO, 2013). The use of layered Love waves waveguides with a nonhomogeneous distribution of physical properties can significantly improve performance (e.g. sensitivity and selectivity) of bio and chemosensors that employ the inhomogeneous elastic waveguides (WATER, CHEN, 2009).

Aim of this paper is to develop a theoretical model of the propagation of SH (Shear Horizontal) surface Love waves in Functionally Graded Materials with a monotonic variation of elastic properties with the depth (distance from the treated surface of the material).

The following profiles of the shear modulus changes were analyzed: 1)linear, 2) quadratic, 3) step, 4) exponential, and 5) profile of the $1/cosh^2$ type. The problem of Love wave propagation in heterogeneous graded materials has been formulated as a Direct Sturm-Liouville Problem. Solving the Direct Sturm-Liouville Problem, the phase velocity dispersion curves and the distribution of the mechanical displacement (into the bulk of material) of Love wave were determined for known elastic parameters of the medium in which the Love wave propagates. The integral dependence linking the group velocity, phase velocity and the distribution of the mechanical displacement of the Love wave was derived.

For profiles: step, exponential and $1/cosh^2$ type, Direct Sturm-Liouville Problem has been solved analytically. For these profiles, analytical formulas for the group velocity have been derived. For power-law type profiles the Direct Sturm-Liouville Problem was solved numerically using the Finite Difference Method and the Transfer Matrix Method. Group velocity for these profiles was determined using a derived integral formula that links the group velocity, phase velocity and the mechanical displacement of the Love wave.

The results obtained in this work will constitute the basis of the inverse procedure to determine profiles (as a function of depth) of the mechanical properties of inhomogeneous FGM resulting from the application of various technological processes of surface treatment. The results of this study also provide a more complete description (than those published in the scientific literature) of the propagation of Love waves in graded materials with various profiles of changes in elastic properties, e.g. in layered inhomogeneous microstructures used in MEMS (Micro Electro Mechanical Systems) and other microelectronic devices, in photonics and in acoustoelectronics (KUZNETSOV, 2010; TASINKEVYCH, DANICKI, 2011; ZHANG *et al.*, 2013).

The results of this study can also find application in geophysics, earthquake engineering (KUZNETSOV, NAFASOV, 2011) and seismology to investigate the internal structure of the Earth. Moreover, they can be very helpful in exploration of natural resources (e.g. natural gas and petroleum) (GUPTA *et al.*, 2013).

Due to the similarity of the mathematical description of the phenomenon of propagation of Love waves in elastic inhomogeneous media and a description of the propagation of light waves in inhomogeneous planar optical waveguides, established in this work the theory of Love waves in elastic inhomogeneous media (FGM) can also be used to analyze performance of inhomogeneous optical planar waveguides (CIPLYS *et al.*, 1995).

The results obtained in this paper are fundamental and can give more profound insight into the nature of Love wave propagation in the elastic nonhomogeneous media (e.g. in functionally graded materials and composites).

2. Direct Sturm-Liouville Problem for Love waves

Love wave propagation in inhomogeneous elastic media can be described in terms of the Sturm-Liouville Direct Problem. Determination of the phase velocity and mechanical displacement distribution with depth of the SH surface Love wave from a knowledge of elastic parameters of a non-homogeneous half-space constitutes a Direct Sturm-Liouville Problem.

2.1. Formulation of the problem

Consider the Love wave that propagates in a non-homogeneous elastic half-space as shown in Fig. 1. The elastic properties of inhomogeneous half-space vary monotonically with depth (distance from the surface).
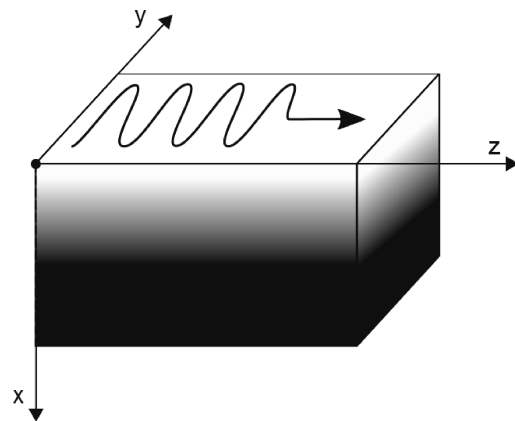

Fig. 1. Love wave waveguide structure (inhomogeneous elastic half-space) and coordinate system. Love wave propagates along the free surface ($x = 0$). The y axis is directed perpendicularly to the plane of the figure and forms with the axes x and z a right-handed Cartesian coordinate system.

Mechanical vibrations of the SH surface Love wave are performed along the y axis perpendicularly to the direction of propagation z and parallel to the propagation surface. The x axis is normal to the waveguide surface $(x = 0)$.

Mathematical description of the propagation of surface shear Love waves in graded media involves the use of continuum mechanics formalism to describe the motion of inhomogeneous elastic half-space.

SH surface wave of the Love type which propagates in a nonhomogeneous waveguide structure of Fig. 1 may be represented in the following form: $v(x, z, t) = f(x) \cdot \exp j(\beta z - \omega t)$, where: $f(x)$ is the distribution of the mechanical displacement of the Love wave with the depth, β is the wave propagation constant, $j = (-1)^{1/2}$, x is the distance from the surface (depth), z is the direction of wave propagation and ω is the angular frequency.

2.2. Boundary conditions

Mechanical field generated by Love waves propagating in an inhomogeneous elastic graded medium satisfies the following boundary conditions:

a) on a free surface $(x = 0)$, the transverse shear stress

$$\tau_{yx} = c_{44}(0) \cdot \frac{\mathrm{d}f(0)}{\mathrm{d}x} \cdot \exp\{j(\beta z - \omega t)\}$$

   is equal to zero, hence $\dfrac{\mathrm{d}f(0)}{\mathrm{d}x} = 0$,

b) at each interface between two layers the condition of continuity of mechanical displacement v and transverse shear stress τ_{yx} is fulfilled,

c) at large distances $(x \rightarrow \infty)$ from the surface $(x = 0)$ the mechanical displacement of the Love wave should tend to zero, i.e. $f(\infty) = 0$.

2.3. Governing equations

The equation of motion (along with the appropriate boundary conditions) for Love waves propagating in an inhomogeneous elastic medium (isotropic and in some specified directions in media of regular and hexagonal symmetry) is represented by the following Differential Problem:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(c_{44}(x)\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right) + \rho\omega^2 f(x) = c_{44}(x)\beta^2 f(x), \quad (1)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}f(0) = 0, \qquad f(\infty) = 0, \qquad (2)$$

where $f(x)$ is the mechanical displacement distribution of the Love wave with the depth x, $c_{44}(x)$ is the elastic shear modulus that depends on the depth, β^2 is an eigenvalue determining the phase velocity of the Love wave, ρ is the density of the medium, and ω is the angular frequency. Differential Problem (1–2) is called

a generalized Sturm-Liouville Problem for Love waves propagating in elastic graded materials.

Using the substitution $z(x) = (c_{44}(x))^{1/2} f(x)$ and assuming small changes in the elastic modulus $c_{44}(x)$, the terms involving derivatives of $c_{44}(x)$, i.e. $\frac{1}{2c_{44}}\left[\frac{\mathrm{d}^2 c_{44}}{\mathrm{d}x^2} - \frac{3}{2}\left(\frac{\mathrm{d}c_{44}}{\mathrm{d}x}\right)^2 \frac{1}{c_{44}^2}\right]$, may be neglected. In this way, we obtain the following Differential Problem (ADAMS, 1981):

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} + \rho\omega^2 s_{44}(x)z = \beta^2 z, \qquad (3)$$

$$\frac{\mathrm{d}z(0)}{\mathrm{d}x} = 0, \qquad z(\infty) = 0, \qquad (4)$$

where $s_{44}(x) = 1/c_{44}(x)$ is the shear compliance.

Differential Problem (3) and (4) is called a Direct Sturm-Liouville Problem for Love waves. The Sturm-Liouville Differential Problem (3–4) constitutes a mathematical model that describes the propagation of SH surface Love waves in the considered in this paper nonhomogeneous elastic graded materials.

The authors verified numerically the above mentioned approximation. For all considered types of profiles of the $s_{44}(x) = 1/c_{44}(x)$ coefficient, the authors solved numerically the Direct Sturm-Liouville Problem for the full (Eq. (1)) and simplified (Eq. (3)) equation for the same set of material parameters of nonhomogeneous elastic medium. It was found that the results of calculations obtained from the solution of full and simplified equation differ insignificantly, e.g. the phase velocity of the Love wave for the linear profile $(Eq. (7)_1)$ derived from the solution of full and simplified equation differed only in sixth significant place.

The solution of this Direct Sturm-Liouville Problem is a set of pairs $\{\beta_i^2, z_i(x)\}$; wherein i is the i-th eigenvalue, $i = 1, 2, \ldots, n_1$, n_1 is the number of modes of Love waves propagating in considered waveguide and $z_i(x)$ is the eigenvector corresponding to this eigenvalue. The eigenvalue corresponds to the phase velocity of the SH surface wave, while the eigenvector describes the distribution of the mechanical displacement of the corresponding mode of the SH surface wave as a function of depth.

In the present paper, we restricted our analysis of the propagation of Love waves in graded materials to the fundamental $(i = 1)$ mode of Love waves. The constant density of the considered graded materials $\rho = \rho_0 = \text{const}$ was assumed throughout the paper.

3. Solution of the Direct Sturm-Liouville Problem

The Direct Sturm Liouville Problem can be solved analytically only for a few specific shapes of the elastic compliance $s_{44}(x)$. In this work, the Direct Sturm Liouville Problem (3–4) has been solved analytically

for the case of the step profile, exponential profile, and $1/cosh^2$ type profile.

Solution of the Sturm-Liouville Direct Problem (3–4) for arbitrary function $s_{44}(x)$ is possible only numerically. Therefore, in the case of power type profiles (linear and quadratic) of the elastic compliance $s_{44}(x)$, the Direct Sturm-Liouville Problem was solved numerically. To this end, we applied two numerical methods, i.e. the Finite Difference Method and the Transfer Matrix Method.

In the Finite Difference Method the Differential Problem is transformed into the Difference Problem. Consequently, the solution of the Direct Sturm-Liouville Problem (1–2) and/or (3–4) is sought in a finite number of discrete points (BAKHVALOV, 1977). At each point x_j of the discretized area, the derivative of unknown function ($f(x)$ and/or $z(x)$), is approximated by the difference operator. In this way, the first and the second derivative of the function in point x_j is presented as a linear combination of the values of this function at the point x_j and in two neighboring points. In this way, the considered Differential Sturm-Liouville Problem transforms into a matrix Sturm-Liouville Problem for eigenvalues and eigenvectors.

Solving this matrix Sturm-Liouville Problem we obtain a set of pairs (eigenvalue, eigenvector). The eigenvalue (β^2) determines the phase velocity of the Love wave ($v_p = \omega/\beta$), while the eigenvector ($f(x)$ and/or $z(x)$), determine the distribution of the mechanical displacement of the Love wave with depth.

In the Transfer Matrix Method the nonhomogeneous elastic waveguide is divided (along the vertical x axis) into a finite number of homogeneous layers (HASKELL, 1953; THOMPSON, 1950). In each homogeneous layer, an ordinary differential equation of second order (Eq. (1)) occurring in the differential Sturm-Liouville Problem is replaced by the system of two ordinary differential equations of the first order. Here, the mechanical displacement of the Love wave and shear stress are the unknowns. Solving this set of differential equations in a layer, from knowledge of the mechanical displacement and shear stress on the upper surface of the layer, we can determine the mechanical displacement and shear stress on the lower surface of the layer. By performing this operation for each layer we can link the mechanical displacement and shear stress on the upper surface of the domain with the mechanical displacement and shear stress on the lower surface of the domain.

Imposing the appropriate boundary conditions on this two boundary surfaces leads to the dispersion equation for the Love wave. This equation is nonlinear algebraic equation for the unknown β^2, where β is the wave number of the Love wave. Thus, the phase velocity of the Love wave amounts to: $v_p = \omega/\beta$. The set of pairs (v_p, ω) determines the phase velocity dispersion curves of the Love wave.

4. Group velocity of Love waves

The group velocity of Love wave was calculated by means of a method employed in the theory of planar optical waveguides (ADAMS, 1981). A similar relationship between the phase velocity and the group velocity can be developed by using formulas for the potential and kinetic energy of Love waves resulting from Analytical Mechanics (ACHENBACH, 1973).

The Differential Problem Eqs. (3), (4) can be formulated in integral (variational) form in terms of the Rayleigh quotient:

$$\beta^2 = \frac{\int\limits_0^\infty \left[-\left(\frac{\mathrm{d}z(x)}{\mathrm{d}x} \right)^2 + \rho\omega^2 s_{44}(x)z^2(x) \right] \mathrm{d}x}{\int\limits_0^\infty z^2(x)\,\mathrm{d}x}. \quad (5)$$

By differentiating the Rayleigh quotient (Eq. (5)) with respect to the angular frequency ω (ADAMS, 1981), we arrive at the following formula:

$$\frac{v_p v_g}{v_0^2} = \frac{\int\limits_0^\infty z^2(x)\,\mathrm{d}x}{\int\limits_0^\infty \frac{s_{44}(x)}{s_0} z^2(x)\,\mathrm{d}x}, \quad (6)$$

where s_0 is the shear elastic modulus in the substrate, and $v_0 = \sqrt{1/\rho_0 s_0}$ is the phase velocity of bulk SH waves in the substrate, $(x \to \infty)$.

Equation (6) links v_p and v_g for Love waves propagating in elastic graded materials. Knowing v_p (for given values of v_0 and $z(x)$) one can calculate the group velocity v_g and *vice versa*.

5. Various elastic shear compliance profiles in graded materials

Profiles of elastic properties of the surface layers in the graded materials are produced due to the use of various technological processes such as rolling, laser hardening (parabolic profile), shot peening, nitriding, carburizing (linear profile), boronizing. Moreover, the processes typical for the microelectronics and integrated optics, such as ion implantation and diffusion, lead to exponential and Gaussian profiles (KIEŁCZYŃSKI, PAJEWSKI, 1989).

In the present study the following profiles of elastic properties (shear modulus $s_{44}(x)$) in heterogeneous graded materials were examined (see Fig. 2a,b).

1. Profiles of the power-law type (n is the exponent)
   a) linear profile $n = 1$ (profile no 1 in Fig. 2a)

$$s_{44}(x)/s_0 = 1 + (\Delta s/s_0)[1 - x/D] \\ \cdot [H(x - D) - H(x)], \quad (7)_1$$

b) quadratic profile $n = 2$ (profile no 2 in Fig. 2a)

$$s_{44}(x)/s_0 = 1 + (\Delta s/s_0)[1 - (x/D)^2]$$
$$\cdot [H(x - D) - H(x)], \qquad (7)_2$$

c) step profile $n = \infty$ (typical for classical Love wave, profile no 3 in Fig. 2a)

$$s_{44}(x)/s_0 = 1 + (\Delta s/s_0)[H(x - D) - H(x)], \quad (7)_3$$

where $H(x)$ is the Heaviside step function, D is the depth of an inhomogeneous elastic layer.

2. Exponential profile (profile no 4 in Fig. 2b)

$$s_{44}(x)/s_0 = 1 + (\Delta s/s_0) \cdot \exp(-2x/D). \quad (7)_4$$

3. Profile of the $1/cosh^2(x)$ type – similar to the Gaussian profile (profile no 5 in Fig. 2b)

$$s_{44}(x)/s_0 = 1 + (\Delta s/s_0) \cdot 1/cosh^2(2x/D). \quad (7)_5$$
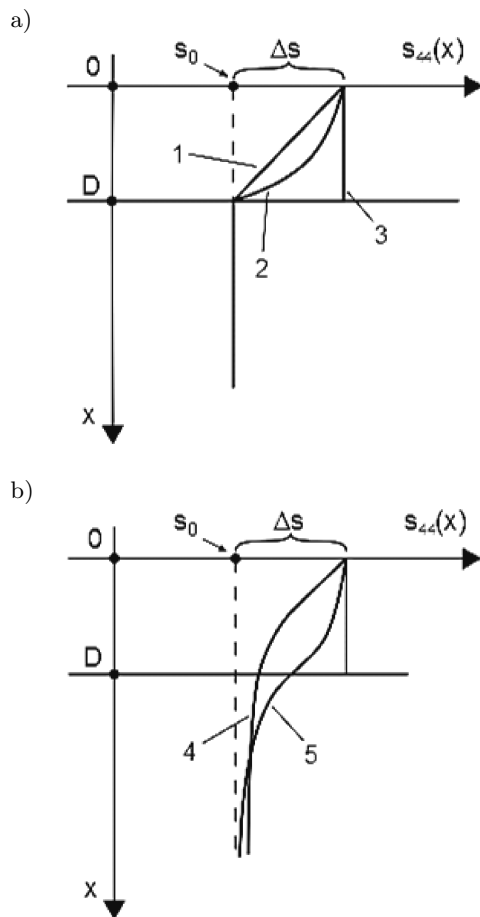
a)

b)

Fig. 2. Considered profiles of the elastic shear compliance $s_{44}(x)$ of graded materials as a function of depth x, a) profiles in a nonhomogeneous surface layer $x \in [0, D]$, b) profiles in a nonhomogeneous half-space $x \in [0, \infty]$.

6. Phase and group velocity of Love waves for various graded profiles. Results of numerical calculations and discussion

To study the propagation behaviour of Love waves in non-homogeneous graded materials from Fig. 2a,b, the following material parameters are assumed in the numerical calculations:

$$s_0 = 0.39 \cdot 10^{-10} \ \frac{m^2}{N}, \qquad v_0 = 1849 \ m/s,$$

$$\rho_0 = 7.5 \cdot 10^3 \ kg/m^3, \qquad \Delta s/s_0 = 0.0966.$$

These parameters are typical for PZT-4 ceramics with elastic properties perturbed in the vicinity of the treated surface.

Applying the Transfer Matrix Method and the Finite Difference Method the Direct Sturm-Liouville Problem (3–4) was solved and consequently the dispersion curves of Love waves propagating in the graded materials with profiles no 1 and 2 presented in Fig. 2a were evaluated numerically.

The group velocity of Love waves for that profiles has been evaluated by using the integral formula (6).

The dispersion curves of the phase and group velocity for Love waves propagating in an elastic waveguide with profiles no 3, 4 and 5 in Fig. 2a,b were calculated analytically.

6.1. Step Profile

The dispersion equation that characterizes the propagation of Love waves in the layered waveguide from Fig. 2a (profile no 3) is given by the formula (ACHENBACH, 1973):

$$F(\omega, \beta) = \tan\left[\left(\sqrt{\left(\frac{\omega}{v_L}\right)^2 - \beta^2}\right) D\right]$$

$$-\frac{(s_0 + \Delta s)\left(\sqrt{\beta^2 - \left(\frac{\omega}{v_0}\right)^2}\right)}{s_0\left(\sqrt{\left(\frac{\omega}{v_L}\right)^2 - \beta^2}\right)} = 0, \quad (8)$$

where $v_L^2 = 1/((s_0 + \Delta s)\rho_0)$ is the phase velocity of bulk shear waves corresponding to the point $x = 0$, $\beta = \frac{\omega}{v_p}$ is the propagation constant of the Love wave, v_p is the phase velocity of the Love wave.

Differentiating formula (8) with respect to β and ω and employing the theorem for the derivative of an implicit function we get the following formula for the group velocity (TOURNOIS, LARDAT, 1973):

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}\beta} = -\frac{\partial F/\partial \beta}{\partial F/\partial \omega}$$

$$= \frac{1}{v_p} \frac{\left\{ \frac{(s_0+\Delta s)}{s_0 a^*} + \frac{\tan(Db^*)}{b^*} + \frac{D}{\cos^2(Db^*)} \right\}}{\left\{ \frac{(s_0+\Delta s)}{s_0 v_0^2 a^*} + \frac{\tan(Db^*)}{v_L^2 b^*} + \frac{D}{v_L^2 \cos^2(Db^*)} \right\}}, \quad (9)$$

where

$$a^* = \sqrt{\beta^2 - \left(\frac{\omega}{v_0}\right)^2},$$

$$b^* = \sqrt{\left(\frac{\omega}{v_L}\right)^2 - \beta^2}.$$

The variation of the phase and group velocity of the Love wave that propagates in layered materials represented by profile no 3 in Fig. 2a versus normalized depth D/L_0 (normalized frequency) is plotted in Fig. 3.
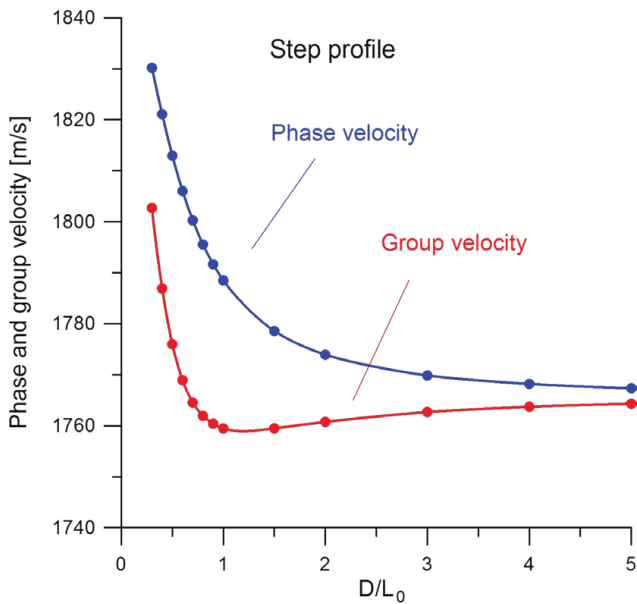

Fig. 3. Phase velocity v_p and group velocity v_g of Love waves propagating in layered waveguide with step profile (plot no 3 in Fig. 2a). D is the thickness of the surface elastic layer, L_0 is the wavelength of the shear bulk wave in the substrate ($x \to \infty$).

6.2. Power type profiles

6.2.1. Linear profile

Figure 4 shows the Love wave dispersion curves of the phase and group velocity for linear profile.

6.2.2. Quadratic profile

Figure 5 demonstrates the plot of the dispersion curves of the phase and group velocity of Love waves for quadratic profile (profile no 2 in Fig. 2a).
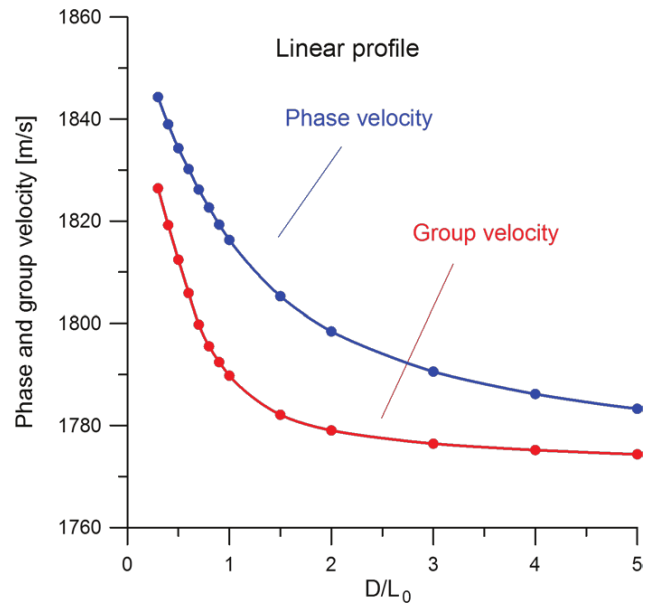

Fig. 4. Love wave dispersion curves of phase velocity v_p and group velocity v_g for linear profile $n = 1$ (plot no 1 in Fig. 2a). D is the thickness of the nonhomogeneous graded elastic layer.
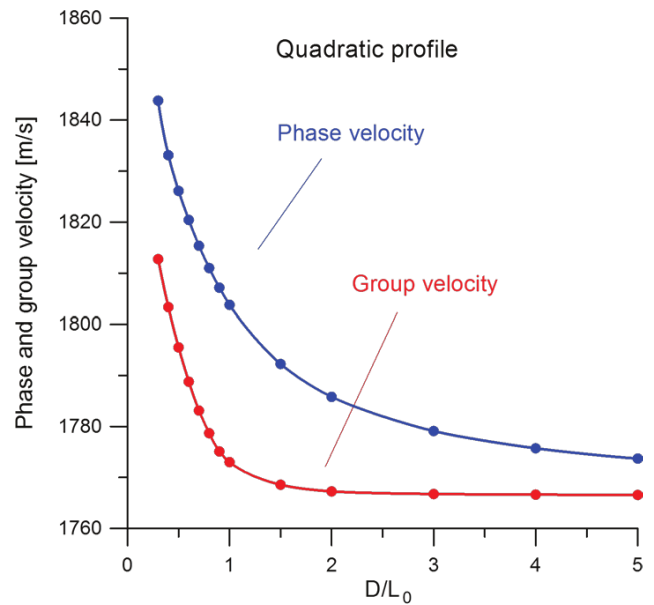

Fig. 5. Love wave dispersion curves of phase velocity v_p and group velocity v_g for quadratic profile $n = 2$ (plot no 2 in Fig. 2a).

6.3. Exponential profile

Analytical solution of the Direct Sturm-Liouville Problem (3–4) for the exponential profile (Eq. $(7)_4$) leads to the following formula for the dispersion equation (TAMIR, 1975):

$$F_1(\omega, \beta) = \beta^2 - \frac{\omega^2}{v_0^2} - \frac{p^2(\omega)}{D^2} = 0, \quad (10)$$

where value of parameter $p(\omega)$ is determined from the equality $\frac{d}{dx} J_p(x)\big|_{x=V} = 0$, here variable $V = \frac{\omega D}{v_0}\sqrt{\frac{\Delta s}{s_0}}$, and $J_p(x)$ is the Bessel function of the order p.

Differentiation of the formula (10) with respect to ω and β gives the following formula for the group velocity:

$$v_g = \frac{d\omega}{d\beta} = -\frac{\partial F_1/\partial \beta}{\partial F_1/\partial \omega} = \frac{v_0^2}{v_p} \frac{1}{\left\{1 - \frac{v_0^2 p}{\omega D^2} c^*\right\}}, \quad (11)$$

where

$$c^* = \frac{\frac{-p}{V^2}\left(\frac{dV}{d\omega}\right) J_p(V) + \frac{p}{V}\frac{d}{d\omega}(J_p(V)) - \frac{d}{d\omega}(J_{p+1}(V))}{\frac{1}{V}J_p(V) + \frac{p}{V}\frac{dJ_p(V)}{dp} - \frac{d}{dp}(J_{p+1}(V))}.$$

Figure 6 displays the dispersion curves of phase velocity v_p and group velocity v_g (calculated by using Eqs. (10) and (11)) of the Love wave propagating in the graded materials with exponential profile (plot no 4 in Fig. 2b).
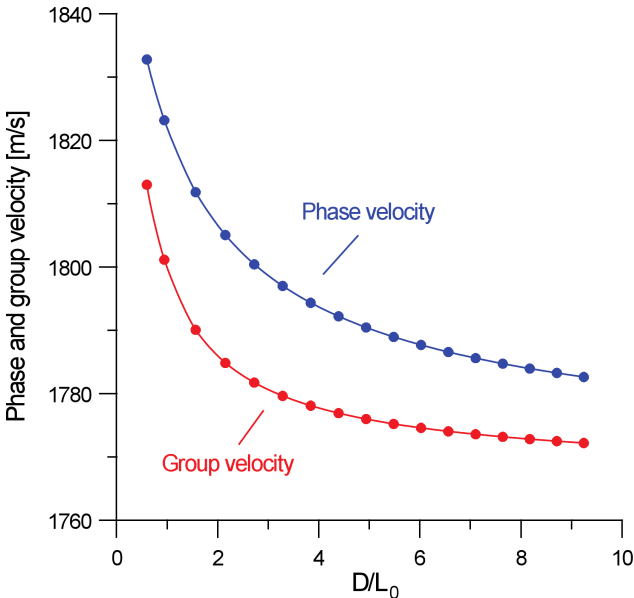

Fig. 6. Love wave dispersion curves of phase v_p and group v_g velocity for the shear compliance profile that follows the exponential profile (plot no 4 in Fig. 2b).

6.4. Profile of the $1/cosh^2$ type

Analytical solution of the Direct Sturm-Liouvillle Problem (3–4) for the case of $1/cosh^2$ type profile gives rise to the following dispersion equation (TAMIR, 1975):

$$F_2(\omega, \beta) = \beta^2 - \frac{\omega^2}{v_0^2} - \frac{4s^2(\omega)}{D^2} = 0, \quad (12)$$

where $s(\omega) = \frac{1}{2}\left\{\sqrt{1 + V^2} - 1\right\}$, and $V = \frac{\omega D}{v_0}\sqrt{\frac{\Delta s}{s_0}}$.

By differentiating the formula (12) with respect to ω and β, we arrive at the following formula for the group velocity in the case of $1/cosh^2$ type profile:

$$v_g = \frac{d\omega}{d\beta} = -\frac{\partial F_2/\partial \beta}{\partial F_2/\partial \omega}$$

$$= \frac{v_0^2}{v_p} \frac{\sqrt{1 + \frac{\omega^2}{v_0^2} h^2 \frac{\Delta s}{s_0}}}{\left\{\left(1 + \frac{\Delta s}{s_0}\right)\sqrt{1 + \frac{\omega^2}{v_0^2} h^2 \frac{\Delta s}{s_0}} - \frac{\Delta s}{s_0}\right\}}. \quad (13)$$

The dispersion curves of the phase and group velocity for the $1/cosh^2$ type profile, that were evaluated by using Eqs. (12) and (13), are presented in Fig. 7.
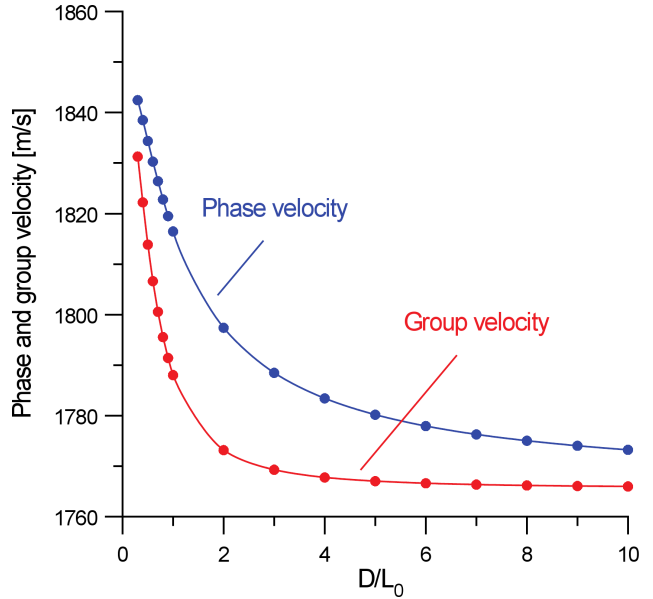

Fig. 7. Love wave dispersion curves of phase velocity v_p and group velocity v_g in the case of the profile of the $1/cosh^2$ type (plot no 5 in Fig. 2b).

As shown in Figs. 3–5 with the increase in the exponent n the phase velocity dispersion curves approach the classical Love wave dispersion curve ($n = \infty$).

As follows from Figs. 3–7, Love wave phase velocity tends to a value of the phase velocity of the bulk shear wave at ($x = 0$) with increasing D/L_0 (i.e. with the frequency increase). As can be seen from Figs. 6 and 7, the Love wave waveguide with the profile of the $1/cosh^2$ type has better waveguide properties (higher slope of the dispersion curve in the region of the steepest descent) than the waveguide with the exponential profile. This indicates higher sensitivity of phase velocity to changes in frequency. This is of great importance in the design and construction of sensors of the physical parameters that use the Love wave.

The calculation of the phase velocity dispersion curves were carried out by using two numerical methods, i.e. Finite Difference Method and Transfer Matrix Method. From the numerical calculations of authors follows that the results obtained using these two methods are identical (with accuracy of 5 decimal places).

This indicates that these two numerical methods are essentially equivalent in applications to describe the propagation of Love waves in considered elastic graded materials.

Numerical calculations have been performed by using the software package Scilab. In the numerical calculations, we assumed the value of thickness $D = 0.4$ mm. In this case, the value of $D/L_0 = 1$ corresponds to frequency 4.626 MHz, and $D/L_0 = 10$ corresponds to frequency 46.26 MHz. Measurement of surface waves velocity in the MHz frequency range are usually performed in quantitative nondestructive evaluation experiments (QNDE). Hence, the results obtained in this paper can be important to the interpretation of experimental dispersion curves of surface Love waves propagating in elastic graded materials.

7. Conclusions

In this paper the Direct Sturm-Liouville Problem that describes the propagation of Love waves in non-homogeneous elastic graded materials has been solved. The integral relationship that relates the group velocity with phase velocity and the distribution of mechanical displacement of the Love wave has been derived.

It was stated that the SH surface Love waves can propagate in nonhomogeneous Functionally Graded half-space, not only in elastic layered structures. It was observed that the group velocity of Love waves is lower than the corresponding phase velocity of Love waves. The results of this study can be useful in the design and construction of sensors of physical quantities as well as in non-destructive testing (NDT) and in geophysics.

Solution of the Direct Sturm-Liouville Problem may also provide a basis for formulating and solving the Inverse Sturm-Liouville Problem. This Inverse Problem consists in determining the unknown material coefficients of graded medium, for example the distribution of elastic compliance $s_{44}(x)$ from a knowledge of the wave dispersion curves of Love waves propagating in an investigated graded material. Formulation and solution of Inverse Problems forms the basis of mathematical methods employed in non-destructive testing of materials (NDT).

The results of this work may also find application in geophysics, seismology and underground acoustics to investigate the internal structure of the Earth (crustal and subcrustal region near the Earth surface). These results can be also applied in exploration of natural resources (e.g. mineral oils, natural gases and minerals) (GUPTA *et al.*, 2013). Moreover, Love waves may also be used to investigate planar optical waveguides (CIPLYS *et al.*, 1995), and layered sensors and resonators of the MEMS (Micro Electro Mechanical Systems) type (KUZNETSOV, 2010; ZHANG *et al.*, 2013).

The results obtained in this paper are fundamental and can give a deeper insight into the behavior of the propagation of Love waves in inhomogeneous elastic media (e.g. in functionally graded materials and composites).

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