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How Budget Deficit Impairs Long-Term Growth and Welfare under Perfect Capital Mobility

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Abstract

This paper investigates the implications of the size of budget deficit in the open economy under perfect mobility of capital. For that purpose we construct a general equilibrium model with consumers maximizing the discounted utility of consumption, and firms maximizing profits. Government sets the size of the deficit relative to GDP and controls the structure of public debt. Using standard methods of optimal control theory we solve the model, i.e. we find explicit formulas for all trajectories and the level of welfare. Finally, we show that the higher the deficit-to-GDP ratio, the lower the welfare of consumers. Similarly, welfare increases with the share of foreign creditors in public debt.

Keywords: budget deficit, optimal fiscal policy, perfect capital mobility

JEL Classification: C68, F43, E62, H3, H60

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1 Introduction

The state-of-the-art, comprehensive presentation of the contemporary growth theory is presented in the impressive, over-1200-page monograph by Daron Acemoglu (2008). It is symptomatic that only 4 chapters (out of 24) touch any open economy issues. Generally, vast majority of endogenous growth theory is based on models of the closed economy, with no foreign trade, no international capital flows and zero foreign debt. In the XXI century integrated world (especially from the European point of view), this is an unacceptable oversimplification, and scientists realize that. Therefore, researchers put more and more effort into extending closed-economy models to incorporate at least some aspects of openness.

Nevertheless, there are very few publications concerning the fiscal policy in the open-economy Ramsey-type endogenous growth theory. Some of the earliest examples are Nielsen and Sorensen (1991), Rebelo (1992), Razin and Yuen (1994), (1996). More recent examples are Lee and Gordon (2005), Agenor (2007), and Dhont and Heylen (2009). Last but not least, it's worth to mention that so far, to the best of our knowledge, there are no publications in placecountry-regionPoland regarding the role of fiscal policy in the open economy growth theory. There are, however, some research papers regarding the role of fiscal policy in the closed economy context. Examples include: Tokarski (2002), Bukowski et.al. (2005), Rzońca (2005), Konopczyński (2012).

One of the latest examples in the world literature is the monograph by Stephen Turnovsky (2009), who examines several versions of the small open economy (SOE) endogenous growth model with productive government expenditures and several types of taxes and perfect or imperfect mobility of capital. Important qualitative differences between closed economy and SOE are exposed by Turnovsky. For example, the capital income tax ceases to have any effect on the long-run growth rate of the economy. Moreover, the equilibrium growth rate is independent of almost all fiscal instruments, including public expenditures. The only tool that has any influence is the tax rate on foreign interest income. However, the government debt and deficit by assumptions do not exist – Turnovsky's analysis only incorporates private foreign debt.

This is not unusual. The assumption of permanently balanced government budget is typical for vast majority of open economy growth models (though it is not in the closed economy growth theory). To the best of our knowledge, there are only a few exceptions, e.g. Greiner and Semmler (2000), Ghosh and Mourmouras (2004a,b), Futagami et.al. (2008). The assumption of a balanced budget is probably inherited from the theory of closed economy, where (under standard assumptions) the Ricardo equivalence holds, and therefore any deficit or surplus does not affect the long-term growth rate of the economy. However, ignoring budget deficit and public debt in the open-economy setting is – in our view – unjustified.

Therefore, in this paper we are building an open economy general equilibrium model with consumers maximizing utility of the consumption stream in the infinite time horizon, and entrepreneurs maximizing profits. This model differs from the models presented by Turnovsky (2009) in several respects. Above all, the government may run budget deficit and finance it by borrowing both domestically and abroad. Also, we apply a different description of technology with the AK aggregate production function. In addition, unlike Turnovsky, we take into account the depreciation of capital. Finally, following Acemoglu (2008), we use a modified utility function with the rate of discount which depends on the rate of growth of population.

The key assumption is the perfect mobility of capital, defined as the unlimited possibility to borrow and invest – both abroad and in the country – at a constant real interest rate. This assumption is de facto the result of two separate assumptions: purchasing power parity (PPP), and the uncovered interest parity (UIP), which implies that investors assign identical risk to foreign and domestic assets. (Clear mathematical exposition of these assumptions can be found in the Burda, Wyplosz (2001), p. 473-476.) For simplicity, all assets are expressed in national currency, and their real interest rate (yield) is equal to r.

The second important assumption is the presence of positive externalities connected to the accumulation of capital, related to learning-by-doing and the spillover effects. The public sector (hereinafter referred to as the government) can actively control the economy using fiscal policy instruments. We include 3 types of income taxes and consumption tax. The government may also freely decide on the level of budget deficit (relative to GDP) and control the method of financing public debt – through the issue of bonds in the country or abroad. On the expenditure side, the government finances public consumption and financial transfers to the private sector.

2 Technological assumptions

The production of a representative (i-th) firm is described with the Cobb-Douglas production function with constant returns to scale:

$$Y_i = F(K_i, L_i) = aK_i^{\alpha}(EL_i)^{\beta}, \qquad \text{with} \quad \alpha + \beta = 1, \alpha, \beta > 0, a > 0, \tag{1}$$

where K_i denotes the stock of physical capital, and L_i represents raw labor in the i-th firm. Constant returns to scale allow a straightforward aggregation. Let N be the number of representative firms in the country. Then, the aggregate output of the country is equal to

$$Y = NY_i = a(NK_i)^{\alpha}(ENL_i)^{\beta} = aK^{\alpha}(EL)^{\beta}, \tag{2}$$

where K denotes the aggregate stock of physical capital, L is the supply of labor in the country, and E is the labor-augmenting technology index. We assume that $L = L_0 e^{nt}$, where $L_0 > 0$ denotes the initial stock of labor, and $t \geq 0$ is a continuous time index.

From a mathematical point of view the production function (1) is identical with (2), so the economy as a whole can be analyzed in such a way as if it were a single firm

in perfect competition, whose production is described with the function (2).
We assume positive externalities related to learning-by-doing and spillover-effects. (These ideas were first introduced by Arrow 1962 and Lucas 1988; an overview of the literature related to these externalities is provided by Romer 1986, and Barro and Sala-i-Martin 2004). These externalities are reflected in the labor-augmenting technology index E, which is proportional to the capital per worker ratio, i.e.

$$E = xK/L, \qquad \text{with} \quad x = const > 0. \tag{3}$$

Dividing both sides of (2) by L, we get the per capita production function:

$$y = \frac{Y}{L} = ak^{\alpha}(E)^{\beta}. \tag{4}$$

Henceforth, throughout the text, lowercase letters denote variables per capita, e.g. $k = K/L$. Having regard to (3) the function (4) can be written in a simpler form:

$$y = ak^{\alpha}(E)^{\beta} = Ak^{\alpha}(k)^{\beta} = Ak \tag{5}$$

with $A = ax^{\beta} = const > 0$. Similarly, the aggregate output function (2) can be written as

$$Y = aK^{\alpha}(EL)^{\beta} = AK^{\alpha}(K)^{\beta} = AK. \tag{6}$$

Therefore, de facto, we use a production function of the AK type, very popular in the theory of endogenous growth. Its primary advantage is simplicity, but – more importantly – it is consistent with the observed stylized facts. For example, in developed countries, even in the very long periods of time the ratio of (annual) GDP to the stock of capital is approximately constant and equal to about 1/3, which implies $A = 1/3$.
By assumption, firms are maximizing profits in the perfectly competitive markets, which implies that the marginal product of capital must be equal to the real rental rate, and simultaneously the marginal product of labor must be equal to the real wage rate, i.e.

$$\forall t \quad MPK = \frac{\partial Y}{\partial K} = \alpha a K^{\alpha-1}(EL)^{\beta} = \frac{\alpha Y}{K} = \frac{\alpha y}{k} = \alpha A = w_K, \tag{7}$$

$$\forall t \quad MPL = \frac{\partial Y}{\partial L} = \beta a K^{\alpha} E(EL)^{\beta-1} = \frac{\beta Y}{L} = \beta y = w. \tag{8}$$

The accumulation of capital is described in a standard way (per capita):

$$\dot{k} = i - (n+\delta)k, \tag{9}$$

where $\delta > 0$ is the depreciation rate. Investment requires the so-called *adjustment cost*, described by the following equation:

$$C(I,K) = \frac{\chi}{2}\frac{I^2}{K} \qquad \text{with} \quad \chi > 0. \tag{10}$$

The concept of adjustment costs is attributed to Hayashi (1982). In order to achieve net investment equal to I, one needs expenditures equal to

$$\Phi(I,K) = I + C(I,K) = I\left(1 + \frac{\chi}{2}\frac{I}{K}\right) \qquad \text{with} \quad \chi > 0, \tag{11}$$

or, in per capita terms,

$$\phi(i,k) = i\left(1 + \frac{\chi}{2}\frac{i}{k}\right). \tag{12}$$

3 Consumer preferences

The level of happiness of a representative household at any given time t is described by the following instantaneous utility function:

$$u(t) = \frac{1}{\gamma}\left(c_t g_{Ct}^{\kappa}\right)^{\gamma} \qquad \text{with} \quad \gamma < 0,\ \kappa > 0, \tag{13}$$

where c_t is a personal consumption per capita, and g_{Ct} denotes a public consumption per capita at time t. The parameter κ expresses the elasticity of substitution between both types of consumption. A fraction $\gamma/(1-\gamma)$ is equal to the intertemporal elasticity of substitution. The assumption $\gamma < 0$ is justified by empirical research; see e.g. Turnovsky (2009), p. 177. Notice that these assumptions guarantee strict concavity of the utility function given by (13) with respect to both types of consumption.

The level of happiness resulting from the present and future consumption is described by the following intertemporal utility function:

$$U = \int_0^{\infty} u(t)e^{-(\rho-n)t}dt = \int_0^{\infty} \frac{1}{\gamma}\left(c_t g_{Ct}^{\kappa}\right)^{\gamma} e^{-(\rho-n)t}dt, \rho > 0. \tag{14}$$

The parameter $\rho > 0$ is the subjective discount rate (of the future consumption). We must explain an unusual effective rate of discount equal to $\rho - n$, which is rare in the literature. This is adopted from Acemoglu (2008), p. 310. It reflects the assumption that a household derives utility from both its own consumption, and consumption of its future members (children, grandchildren, etc.), the number of which is growing at the rate n. Therefore, the higher the rate of population growth (n) in a country, the smaller the effective discount rate, because the number of children, grandchildren, etc., which will be consuming in the future is greater. Speaking a little more intuitively – the more children (per family), the more we value future consumption. In our opinion this is a realistic assumption – parents of 3 or 4 kids plan their spending flow otherwise (leaving more for the future and drawing satisfaction from bequests that their children get) than parents of a single child, not to mention a family without children or singles. In our view, this assumption brings the standard Ramsey-type theory somewhat closer to the overlapping generations model. We assume that $\rho > n$. Otherwise, the integral in (14) would not be convergent, and it would be impossible to solve any optimization problems involving it.

4 The public sector (government)

The total amount of tax revenue in real terms is as follows:

$$T = \tau_L wL + \tau_K w_K K + \tau_D r_D D_D + \tau_C C, \tag{15}$$

where τ_L, τ_K, τ_D, τ_C are the average tax rates on wages, capital income, interest on government bonds held by domestic investors, and consumption. The real deficit is the difference between government spending and tax receipts, i.e.

$$J = G + r_D D - T, \tag{16}$$

where G is government spending in real terms, and D represents the total public debt. We assume that the budget deficit is a fixed percentage of the GDP, i.e.

$$J = \xi Y, \tag{17}$$

where $\xi = const > 0$ is a decision parameter. Using (16), the budgetary rule (17) can be written in the form:

$$G = T - r_D D + \xi Y. \tag{18}$$

The deficit is financed by government bonds, which increases the public debt according to the equation: $\dot{D} = \xi Y$. Certain part (ω) of bonds is sold to foreign investors, and the rest to domestic creditors:

$$\dot{D}_F = \omega \dot{D} = \omega \xi Y \qquad \text{with} \quad 0 \le \omega \le 1, \tag{19}$$

$$\dot{D}_D = (1-\omega)\dot{D} = (1-\omega)\xi Y, \tag{20}$$

where D_D denotes the domestic debt, and D_F is the foreign debt of the government. Of course, at any time $D = D_D + D_F$. Government spending includes two components:

$$G = G_T + G_C = G_T + \sigma_C C \qquad \text{with} \quad 0 < \sigma_C < 1, \tag{21}$$

where G_T means the cash transfers to the private sector, and G_C is the public consumption which is proportional to the consumption of the private sector. The equations (18) and (21) imply that the real size of the cash transfers is:

$$G_T = G - G_C = T + \xi Y - r_D D - G_C. \tag{22}$$

According to this equation, the collected taxes, plus the planned budget deficit are used to service the public debt and public consumption as planned by the government. The rest of the money is transferred to the private sector.

5 The private sector

The private sector receives income in the form of remuneration of labor and capital, the interest on domestic debt and the income from (net) foreign assets. After taxes, the disposable income of the private sector equals

$$Y_d = (1-\tau_L)wL + (1-\tau_K)w_K K + (1-\tau_D)rD_D + rB. \tag{23}$$

The income of the private sector augmented by the cash transfers from the government is spent on consumption and investment, as well as purchases of government bonds. Any difference is invested in (net) foreign assets. The budget equation is of the form:

$$Y_d + G_T = C(1+\tau_C) + \Phi(I,K) + \dot{D}_D + \dot{B}. \tag{24}$$

Substituting formula (11) and considering equation (20) and (23), the budget constraint (24) can be written in the equivalent form:

$$\begin{aligned} \dot{B} = (1-\tau_L)wL + (1-\tau_K)w_K K + (1-\tau_D)rD_D + rB+ \\ +G_T - C(1+\tau_C) - I\left(1+\tfrac{\chi}{2}\tfrac{I}{K}\right) - (1-\omega)\xi Y, \end{aligned} \tag{25}$$

or, in per capita terms:

$$\begin{aligned} \dot{b} = (1-\tau_L)w + (1-\tau_K)w_K k + (1-\tau_D)rd_D + (r-n)b+ \\ +g_T - c(1+\tau_C) - i\left(1+\tfrac{\chi}{2}\tfrac{i}{k}\right) - (1-\omega)\xi y. \end{aligned} \tag{26}$$

6 The optimal control problem and its solution

The private sector determines the flows of consumption and investment, so as to achieve the highest level of utility described by the function (14), with a budget constraint of (26). That decision problem boils down to the following optimal control problem:

$$\begin{cases} \max \displaystyle\int_0^\infty \frac{1}{\gamma}\left(cg_C^{\kappa}\right)^{\gamma} e^{-(\rho-n)t}dt, \\ \dot{b} = (1-\tau_L)w + (1-\tau_K)w_K k + (r-n)b + (1-\tau_D)rd_D - c(1+\tau_C)+ \\ \qquad -i\left(1+\tfrac{\chi}{2}\tfrac{i}{k}\right) + g_T - (1-\omega)\xi y, \\ \dot{k} = i - (n+\delta)k. \end{cases} \tag{27}$$

Control variables: c, i. State variables: b, k. The initial values of variables (endowments): b_0, $k_0 > 0$, $d_0 \geq 0$, $d_{F0} \geq 0$, $d_{D0} \geq 0$, with $d_{F0} + d_{D0} = d_0$.

The variables treated by an individual decision-maker as exogenous: w, w_K, g_T, g_C, d_D.

We must strongly emphasize that the individual decision-maker treats the five listed variables as exogenous, because alone he does not have any noticeable impact on any of them. However, the individual decisions summed together (aggregated) do affect these variables, which has been described above by respective equations. It's worth to provide a simple example of such a situation. A single firm considering how many people to hire assumes that increasing or reducing the number of its employees has no impact on the labor market, and therefore assumes that the wage rate it must offer is independent of the number of its employees. But if all (representative) firms in the country increase or reduce employment at the same time, the labor market will be influenced, and the wage rate will rise. Similarly, an individual decision-maker assumes that alone he has no impact on the rental rate of capital, as well as the size of public consumption, the volume of cash transfers from the government, and the size of public debt (all of these in per capita terms). Nevertheless, aggregated (summed up) decisions of all representative agents obviously do have influence on these variables, which has been described above with the relevant equations. Therefore, while formulating and solving the problem (27) we assume that the values of the above listed 5 variables are exogenous. But at the same time the solution must satisfy all equations describing the relationships between these variables and aggregated decisions. The solution obtained must therefore satisfy all equations written down in sections 2-5.

A mathematician would probably say that at first we solve the problem (27) taking w, w_K, g_T, g_C, and d_D as exogenous (constants). In this way we obtain a bundle (a set) of potential solutions, among which we have to select these solutions which satisfy all the equations listed in sections 2-5. This process of selecting a feasible solution(s) can in practice be simplified in such a way that while solving the necessary and sufficient conditions for optimality we immediately take advantage of equations from sections 2-5. This is exactly the way we are going to proceed.

In addition, we assume that at any point of time investment and consumption are positive, and consumption per capita is not decreasing, that is

$$\forall t \geq 0 \quad i(t) > 0, \tag{28}$$

$$\forall t \geq 0 \quad c(t) > 0 \quad \text{and} \quad \dot{c}(t) \geq 0. \tag{29}$$

Only in this case the model has a reasonable economic interpretation. For simplicity, we first solve the problem (27) without taking into account these two assumptions. Only after deriving en explicit solution we will investigate, what assumptions regarding parameters and//or endowments are necessary for the obtained solution to meet these two assumptions.

The current value hamiltonian is:

$$H_c = \frac{1}{\gamma}\left(cg_C^{\kappa}\right)^{\gamma} + \lambda_1 \cdot \Big[(1-\tau_L)w + (1-\tau_K)w_K k + [r-n]\, b + (1-\tau_D)\, r\, d_D + \\ -c(1+\tau_C) - i\left(1+\tfrac{\chi}{2}\tfrac{i}{k}\right) + g_T - (1-\omega)\xi y\Big] + \lambda_2 \cdot \Big[i - (n+\delta)\, k\Big]. \tag{30}$$

The optimal solution of (27) must meet the following conditions (necessary and sufficient):

$$\forall t \quad \frac{\partial H_c}{\partial c} = 0. \tag{31}$$

$$\forall t \quad \frac{\partial H_c}{\partial i} = 0. \tag{32}$$

$$\dot{\lambda}_1 = -\frac{\partial H_c}{\partial b} + \lambda_1(\rho - n). \tag{33}$$

$$\dot{\lambda}_2 = -\frac{\partial H_c}{\partial k} + \lambda_2(\rho - n). \tag{34}$$

$$\lim_{t\to\infty} e^{-(\rho-n)t}\lambda_1(t)b(t) = 0. \tag{35}$$

$$\lim_{t\to\infty} e^{-(\rho-n)t}\lambda_2(t)k(t) = 0. \tag{36}$$

The condition (31) has the form:

$$\lambda_1(1+\tau_C) = c^{\gamma-1}g_C^{\kappa\gamma}, \tag{37}$$

which means that the shadow price of wealth (in the form of bonds), adjusted for the size of consumption tax must be (for each t) equal to the marginal utility of private consumption. By differentiating this equation with respect to t, after transformation we get:

$$\frac{\dot{\lambda}_1}{\lambda_1} = (\gamma - 1)\frac{\dot{c}}{c} + \kappa\gamma\frac{\dot{g}_C}{g_C}. \tag{38}$$

Note that from the equation (21) it follows that private and public consumption must grow at equal rates. Let this rate of growth be denoted by ψ. Hence:

$$\hat{g}_C = \hat{c} = \psi. \tag{39}$$

The condition (33) can be written as:

$$\frac{\dot{\lambda}_1}{\lambda_1} = \rho - r. \tag{40}$$

Substituting (40) into (38), and taking into account (39), we calculate the growth rate of consumption (private and public) per capita:

$$\psi = \frac{\dot{c}}{c} = \frac{r-\rho}{1-\gamma(1+\kappa)}. \tag{41}$$

Note that the optimal solution is characterized by a constant growth rate of per capita consumption, which depends only on the real interest rate and the parameters that describe the consumer preferences. The optimal trajectory of private consumption per capita is thus:

$$c(t) = c_0 \cdot e^{\psi t}, \tag{42}$$

whereas the trajectory of public consumption per capita has the form:

$$g_C(t) = \sigma_C c(t) = \sigma_C c_0 \cdot e^{\psi t}. \tag{43}$$

The condition (33) can be written in the form:

$$q = \frac{\lambda_2}{\lambda_1} = 1 + \chi \frac{i}{k}. \tag{44}$$

The ratio of the shadow prices $q = \lambda_2/\lambda_1$ can be interpreted as the market price of capital in relation to the market price of foreign bonds. Dividing (9) on both sides by k and having regard to the (44) we get the growth rate of capital per capita:

$$\varphi = \hat{k} = \frac{q-1}{\chi} - (n+\delta). \tag{45}$$

This growth rate is not necessarily constant, because it is related to the trajectory $t \to \infty$. Therefore the trajectory of per capita capital can be written only in a very general form:

$$k(t) = k_0 \exp\left(\int_0^t \varphi(s) ds\right). \tag{46}$$

To determine the path of $q(t)$, we need to use the last of the necessary conditions for optimality, i.e. (31d). Having regard to the (40) and (44), it can be written in the form of the following equation:

$$\dot{q} = (r+\delta) \cdot q - (1-\tau_K)\alpha A + (1-\omega)\xi A - \frac{(q-1)^2}{2\chi}. \tag{47}$$

It is a quadratic autonomous differential equation with constant coefficients. Figure 1 shows the phase diagram of the equation.

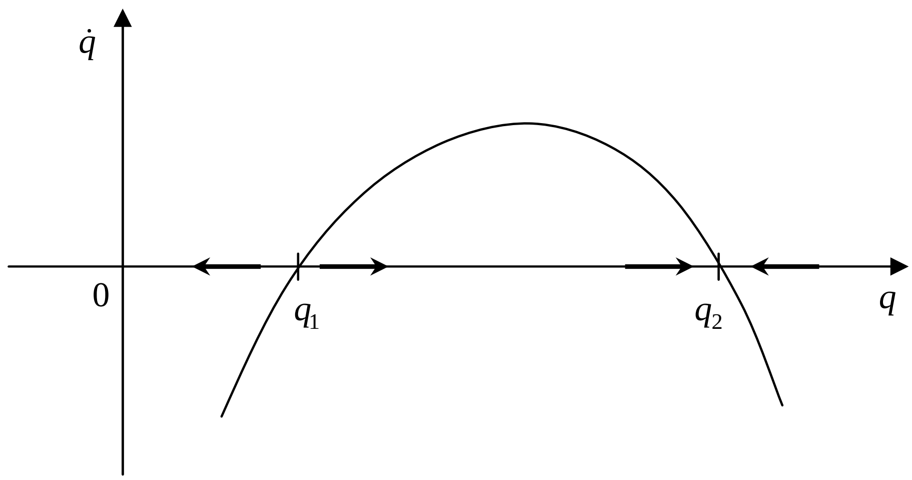
The parabola presented on the graph must have at least one real root − otherwise we would have $\forall q \;\; \dot{q} < 0$, hence $q(t) \xrightarrow[t\to\infty]{} -\infty$, which is unacceptable (infeasible) because of economic meaning of the variable q. Therefore, the equation

$$(r+\delta)q - (1-\tau_K)\alpha A + (1-\omega)\xi A - \frac{(q-1)^2}{2\chi} = 0 \tag{48}$$

must have at least one real root, which occurs if, and only if,

$$\Delta = 2\chi\left[r + \delta - \alpha A(1-\tau_K) + (1-\omega)\xi A\right] + \chi^2(r+\delta)^2 \geq 0. \tag{49}$$

Figure 1: The phase diagram of the equation (47)



The roots of the equation (48) are:

$$q_1 = 1 + \chi(r + \delta) - \sqrt{\Delta}, \tag{50}$$

$$q_2 = 1 + \chi(r + \delta) + \sqrt{\Delta}. \tag{51}$$

Notice that:

1. If $r + \delta = \alpha A(1 - \tau_K) + (1 - \omega)\xi A$, then $q_1 = 1$, and $q_2 = 1 + 2\chi(r + \delta)$.
2. If $r + \delta > \alpha A(1 - \tau_K) + (1 - \omega)\xi A$, then $0 < q_1 < 1 < q_2$. (Note: this case is rejected, because it would imply negative investment.)
3. If $r + \delta < \alpha A(1 - \tau_K) + (1 - \omega)\xi A$, then $1 < q_1 < q_2$.

Generally, in each case $q_2 > q_1 > 0$ and $q_2 > 1$. Both points are stationary states, however q_1 is unstable, and q_2 is locally stable. In the appendix we show that the transversality condition (31f) is satisfied only at the point q_1. Therefore, one can make the following conclusion: the market price of capital must at any time t be in unstable point q_1. In response to any shock (a change of any exogenous variable) the market price of capital must immediately adjust itself (jump). In other words, the process of adjustment to a new steady state is instantaneous. Therefore

$$\forall t \quad q = q_1. \tag{52}$$

As a consequence the capital per capita is growing at a constant rate:

$$\varphi = \frac{q_1 - 1}{\chi} - (n + \delta) = const. \tag{53}$$

The trajectory of capital (46) can therefore be simplified to

$$k(t) = k_0 e^{\varphi t}. \tag{54}$$

To examine the second transversality condition we must first determine the trajectories of public debt (domestic and foreign). First, note that the growth rate of per capita public debt is the difference between the growth rates of public debt and the rate of population growth:

$$\hat{d} = \widehat{\left(\frac{D}{L}\right)} = \hat{D} - n. \tag{55}$$

Since $\hat{D} = \dot{D}\big/D = \xi y/d$, we get:

$$\hat{d} = \xi \frac{y}{d} - n. \tag{56}$$

From the equation $\hat{d} = \dot{d}\big/d$ it follows that $\dot{d} = \hat{d} \cdot d$, which for the sake of (56) leads to the following linear differential equation:

$$\dot{d} = \xi y - nd, \tag{57}$$

where $y(t) = Ak_0 e^{\varphi t}$. It is not difficult to show that the general solution to this differential equation has the following form:

$$d(t) = s_1 e^{-nt} + \frac{\xi A k_0}{n + \varphi} e^{\varphi t}. \tag{58}$$

Knowing the initial size of the government debt $d_0 \geq 0$, from this equation the constant s_1 can be calculated:

$$s_1 = d_0 - \frac{\xi A k_0}{n + \varphi}. \tag{59}$$

Therefore, the trajectory of public debt is expressed by the following formula:

$$d(t) = \left(d_0 - \frac{\xi A k_0}{n + \varphi}\right) e^{-nt} + \frac{\xi A k_0}{n + \varphi} e^{\varphi t}. \tag{60}$$

Since $y(t) = Ak_0 e^{\varphi t}$, this trajectory can be written in the equivalent form:

$$d(t) = \frac{\xi}{n + \varphi} y(t) + \left(d_0 - \frac{\xi y_0}{n + \varphi}\right) e^{-nt}. \tag{61}$$

Note that

$$\lim_{t \to \infty} \frac{d(t)}{y(t)} = \frac{\xi}{n + \varphi}. \tag{62}$$

This result is well known in the literature. For example, Paul de Grauwe (2012) on page 138 shows that public debt-to-GDP ratio in the long run stabilizes at a level equal to the ratio of the budget deficit (in % of GDP) and the average rate of growth of nominal GDP. Our model does not take inflation into account, but the formula (62) generally has the same meaning, because the numerator is equal to the deficit-to-GDP ratio, while the denominator is the growth rate of real GDP $(n+\varphi)$.
In the same way by solving equation (19) and (20) transformed to per capita versions we can calculate the trajectories of domestic and foreign government debt:

$$d_D(t) = \frac{(1-\omega)\xi}{n+\varphi} y(t) + \left(d_{D0} - \frac{(1-\omega)\xi y_0}{n+\varphi} \right) e^{-nt}, \tag{63}$$

$$d_F(t) = \frac{\omega\xi}{n+\varphi} y(t) + \left(d_{F0} - \frac{\omega\xi y_0}{n+\varphi} \right) e^{-nt}. \tag{64}$$

It's straightforward to demonstrate that

$$\lim_{t\to\infty} \frac{d_D(t)}{y(t)} = \frac{(1-\omega)\xi}{n+\varphi}, \tag{65}$$

$$\lim_{t\to\infty} \frac{d_F(t)}{y(t)} = \frac{\omega\xi}{n+\varphi}. \tag{66}$$

Now we can take a look at the second transversality condition (31e), which determines the initial consumption. We must first determine the trajectory $c(t) = c_0 \cdot e^{\psi t}$ by solving the equation (26), that must first be slightly transformed. From (15) it follows that

$$t = \tau_L w + \tau_K w_K k + \tau_D r d_D + \tau_C c. \tag{67}$$

Therefore, the equation (26) can be reduced to the form:

$$\dot{b} = w + w_K k + r d_D + (r-n)b - rd - g_C - c - i\left(1 + \frac{\chi}{2}\frac{i}{k}\right) + \omega\xi y. \tag{68}$$

From equations (5), (7) and (8) it follows that

$$w + w_K k = \beta y + \alpha A k = (\alpha+\beta)y = y, \tag{69}$$

and $d - d_D = d_F$. Therefore equation (68) can be written in the form:

$$\dot{b} = (1+\omega\xi)y + (r-n)b - i\left(1 + \frac{\chi}{2}\frac{i}{k}\right) - c - r d_F - \sigma_C c. \tag{70}$$

Using (42) (44), (54) and (64), it can be transformed to:

$$\dot{b} = (r-n)b + \upsilon k_0 e^{\varphi t} - c_0(1+\sigma_C)\cdot e^{\psi t} - \left[r d_{F0} - \frac{r\omega\xi A k_0}{n+\varphi} \right] e^{-nt} \tag{71}$$

where

$$\upsilon = A(1 + \omega\xi) - \frac{q^2 - 1}{2\chi} - \frac{r\omega\xi A}{n + \varphi}. \tag{72}$$

In the appendix we show that the general solution to the equation (71) has the form:

$$b(t) = Se^{(r-n)t} - \frac{\upsilon k_0}{r - n - \varphi} \cdot e^{\varphi t} + \frac{c_0(1 + \sigma_C)}{r - n - \psi} \cdot e^{\psi t} + \left(d_{F0} - \frac{\omega\xi A k_0}{n + \varphi}\right) e^{-nt}. \tag{73}$$

Knowing the initial (endowment) stock of net foreign assets b_0, we can calculate the constant:

$$S = b_0 - d_{F0} + \frac{\upsilon k_0}{r - n - \varphi} - \frac{c_0(1 + \sigma_C)}{r - n - \psi} + \frac{\omega\xi A k_0}{n + \varphi}. \tag{74}$$

Knowing the trajectory $b(t)$, we can analyze the transversality condition (31e). The equation (40) implies that

$$\lambda_1(t) = \lambda_1(0)e^{(\rho - r)t}. \tag{75}$$

In view of this fact and (112) together with (74), the condition (31e) can be written in the form:

$$\begin{aligned} &\lambda_1(0) \cdot \lim_{t\to\infty} \left\{ S - \left(\frac{\upsilon k_0}{r - n - \varphi} \cdot e^{\varphi t} + \frac{c_0(1 + \sigma_C)}{r - n - \psi} \cdot e^{\psi t} \right) e^{n-r} + \right. \\ &\left. + \left[d_{F0} - \frac{\omega\xi A k_0}{n + \varphi} \right] \right\} = 0, \end{aligned} \tag{76}$$

which occurs if and only if the 3 conditions are met:

$$S = b_0 - d_{F0} + \frac{\upsilon k_0}{r - n - \varphi} - \frac{c_0(1 + \sigma_C)}{r - n - \psi} + \frac{\omega\xi A k_0}{n + \varphi} = 0, \tag{77}$$

$$r - n - \varphi > 0, \tag{78}$$

$$r - n - \psi > 0. \tag{79}$$

Using equations (50) and (53) it can be easily demonstrated that the condition (78) is always satisfied. The inequality (79) can be converted to an equivalent form:

$$\rho > n + \gamma(1 + \kappa)(r - n), \tag{80}$$

which means that, the transversality conditions require that the discount rate must be sufficiently high. Notice that in the special case of stable population ($n = 0$), this condition can be reduced to: $\rho > \gamma(1 + \kappa)r$, and is satisfied for any $\rho > 0$, since $\gamma < 0$. The equality (77) determines the initial consumption:

$$c_0 = \left(b_0 - d_{F0} + \frac{\upsilon k_0}{r - n - \varphi} + \frac{\omega\xi A k_0}{n + \varphi} \right) \frac{r - n - \psi}{1 + \sigma_C}, \tag{81}$$

which having regard to (72) can be written in the form:

$$c_0 = \left(b_0 - d_{F0} + \frac{Ak_0}{r - n - \varphi} \left(1 - \frac{q^2 - 1}{2A\chi} \right) \right) \frac{r - n - \psi}{1 + \sigma_C}. \tag{82}$$

Substituting the initial consumption (82), the trajectory $b(t)$ can finally be expressed as

$$\begin{aligned} b(t) &= \left(b_0 - d_{F0} + \frac{\upsilon k_0}{r - n - \varphi} + \frac{\omega \xi A k_0}{n + \varphi} \right) \cdot e^{\psi t} + \\ &- \frac{\upsilon k_0}{r - n - \varphi} \cdot e^{\varphi t} + \left(d_{F0} - \frac{\omega \xi A k_0}{n + \varphi} \right) e^{-nt} \end{aligned} \tag{83}$$

or in equivalent form:

$$b(t) = \frac{c_0(1 + \sigma_C)}{r - n - \psi} \cdot e^{\psi t} - \frac{\upsilon k_0}{r - n - \varphi} \cdot e^{\varphi t} + \left(d_{F0} - \frac{\omega \xi A k_0}{n + \varphi} \right) e^{-nt}. \tag{84}$$

7 The feasibility conditions

The model has a reasonable economic interpretation, if conditions (28) and (29) are satisfied. Using the equation (44) it can easily be shown that (28) is satisfied if, and only if $q > 1$. Since $\forall t \;\; q = q_1$, using (50) and (49) it can easily be proved that $q > 1$ if, and only if,

$$r < \alpha A(1 - \tau_K) - (1 - \omega)\xi A - \delta. \tag{85}$$

This condition has a clear economic interpretation. It means that the real rate of return on net foreign assets must be lower than the marginal profitability of investment in productive capital from the point of view of an individual agent.

Consumption in the entire time horizon must also be positive and non-decreasing in per capita terms. From (42) it follows that this condition is satisfied when $\psi \geq 0$ and simultaneously $c_0 > 0$. Equation (41) implies that $\psi \geq 0$ if, and only if,

$$\rho \leq r. \tag{86}$$

On the other hand, (79) implies that $r - n - \psi > 0$, hence $c_0 > 0$ if, and only if,

$$b_0 + \frac{\upsilon k_0}{r - n - \varphi} - d_{F0} + \frac{\omega \xi A k_0}{n + \varphi} > 0, \tag{87}$$

which can be written in the following equivalent form:

$$\frac{d_{F0} - b_0}{k_0} < \frac{\upsilon}{r - n - \varphi} + \frac{\omega \xi A}{n + \varphi}. \tag{88}$$

The right-hand side of this inequality does not depend on the initial state. Therefore, this inequality means that the initial net debt of the entire country, which is equal to the difference between foreign debt of the government and net foreign assets of the private sector, compared to the initial stock of productive capital must be less than a certain number.

8 The dynamic equilibrium

To summarize, the solution of the optimal control problem (27) is of the following form:

$$k(t) = k_0 e^{\varphi t} \quad \text{with} \quad \varphi = \frac{q-1}{\chi} - (n+\delta) = const., \quad q = 1 + \chi(r+\delta) - \sqrt{\Delta},$$

$$\Delta = 2\chi\left[r + \delta - \alpha A(1-\tau_K) + (1-\omega)\xi A\right] + \chi^2(r+\delta)^2,$$

$$i(t) = \frac{q-1}{\chi} \cdot k(t), \quad \text{i.e.} \quad \frac{i}{y} = \frac{q-1}{A\chi} = const.,$$

$$y(t) = Ak(t),$$

$$c(t) = c_0 \cdot e^{\psi t}, \quad \text{with} \quad \psi = \frac{r-\rho}{1-\gamma(1+\kappa)} = const,$$

$$g_C(t) = \sigma_C c_0 \cdot e^{\psi t}, \quad \text{with} \quad c_0 = \left(b_0 - d_{F0} + \frac{Ak_0}{r-n-\varphi}\left(1 - \frac{q^2-1}{2A\chi}\right)\right)\frac{r-n-\psi}{1+\sigma_C},$$

$$b(t) = \frac{c_0(1+\sigma_C)}{r-n-\psi} \cdot e^{\psi t} - \frac{\upsilon k_0}{r-n-\varphi} \cdot e^{\varphi t} + \left(d_{F0} - \frac{\omega\xi A k_0}{n+\varphi}\right)e^{-nt},$$

where $\upsilon = A(1+\omega\xi) - \dfrac{q^2-1}{2\chi} - \dfrac{r\omega\xi A}{n+\varphi}$,

$$d(t) = \frac{\xi}{n+\varphi} y(t) + \left(d_0 - \frac{\xi y_0}{n+\varphi}\right)e^{-nt}$$

,

$$d_F(t) = \frac{\omega\xi}{n+\varphi} y(t) + \left(d_{F0} - \frac{\omega\xi y_0}{n+\varphi}\right)e^{-nt},$$

$$d_D(t) = d(t) - d_F(t) = \frac{(1-\omega)\xi}{n+\varphi} y(t) + \left(d_{D0} - \frac{(1-\omega)\xi y_0}{n+\varphi}\right)e^{-nt},$$

with the following assumptions: (49), (80), (85), (86), (88). These are (successively) a condition to ensure that $\Delta \geq 0$, the transversality condition, and

three conditions that ensure that $i(t) > 0$, $\psi \geq 0$, and $c_0 > 0$.
It is worth noting that the growth rate of consumption ψ can differ from the rate of growth of the GDP. Moreover, it can be so permanently – in the infinitely long time horizon. This property represents a key difference compared to the standard closed economy models in which consumption is determined by the accumulation of capital and the rate of GDP growth, and all real variables – including production, capital, investment, consumption must (at least at the equilibrium) grow at the same rate.
Another important feature of the model is the neutrality of all tax rates except for the tax on capital income. Notice that neither the tax rate on consumption, nor the tax rate on labor, nor the tax rate on interest on government bonds show up in the solution to the model. It means that they are all neutral – they do not affect the optimal trajectories and welfare. Only the tax rate on capital has an impact on the economy.

9 The welfare

Taking into account the trajectories of private and public consumption together with (41), welfare measured by the value of (14) can be written in the form:

$$\Omega = \frac{1}{\gamma}\sigma_C^{\kappa\gamma} c_0^{\gamma(1+\kappa)} \int_0^{\infty} e^{(\psi\gamma(1+\kappa)-\rho+n)t}dt. \tag{89}$$

It is not difficult to show that:

$$\psi\gamma\,(1+\kappa) - \rho + n = -(r - n - \psi). \tag{90}$$

Hence

$$\Omega = \frac{1}{\gamma}\sigma_C^{\kappa\gamma} c_0^{\gamma(1+\kappa)} \int_0^{\infty} e^{-(r-n-\psi)t}dt. \tag{91}$$

The transversality conditions – in particular (79) – ensure that the integral in (91) converges. Therefore (91) can be reduced to:

$$\Omega = \frac{\sigma_C^{\kappa\gamma} c_0^{\gamma(1+\kappa)}}{\gamma(r-n-\psi)}. \tag{92}$$

The formula (92) seems to be very simple. However, after appropriate substitutions, the full parametric form of the above (where Ω is expressed as a function of parameters and the initial endowments) is:

$$\Omega = \frac{\sigma_C^{\kappa\gamma}}{\gamma}\left(b_0 - d_{F0} + \frac{Ak_0}{r+\delta-\dfrac{q-1}{\chi}}\left(1-\frac{q^2-1}{2A\chi}\right)\right)^{\gamma(1+\kappa)} \cdot \left(\frac{r-n-\dfrac{r-\rho}{1-\gamma(1+\kappa)}}{1+\sigma_C}\right)^{\gamma(1+\kappa)} \cdot \left(r-n-\frac{r-\rho}{1-\gamma(1+\kappa)}\right)^{-1}, \tag{93}$$

where

$$q = 1+\chi\,(r+\delta)-\sqrt{2\chi\,[r+\delta-\alpha A(1-\tau_K)+(1-\omega)\xi A]+\chi^2(r+\delta)^2}.$$

Notice that the level of welfare in the economy depends on the size of budget deficit ξ, and on the structure of public debt expressed by the parameter ω. In the next section we investigate these relationships.

10 The importance of the deficit and the structure of public debt

Let us examine the relationship between ξ and ω and the obtained optimal solution, as well as the welfare. Because we have the explicit formulas describing the optimal solution (dynamic equilibrium), we can calculate the appropriate partial derivatives and specify their signs. The results of the calculations are presented in the table.
The symbol W_1 represents the following sum:

$$W_1 = -\tau_K\alpha A - (1-\omega)\xi A - (1-\alpha)A < 0. \tag{94}$$

The signs of derivatives reported in the table are determined by the assumptions about the individual parameters of the model and transversality conditions, in particular the inequality (79).
On the one hand, the budget deficit negatively influences the rate of growth of GDP, i.e. the higher the deficit relative to GDP, the lower the rate of growth of GDP. It follows from the fact that deficit reduces the rate of private investment (as percentage of GDP). On the other hand, the rate of growth of consumption per capita (both private and public) is independent of ξ, whereas the higher the ξ, the lower the initial consumption per capita c_0. It follows that the higher the deficit-to-GDP ratio, the lower is the whole trajectory of consumption per capita. This implies that the higher the deficit-to-GDP ratio, the lower is the welfare achieved by consumers, which is confirmed by the negative derivative of $\partial\Omega/\partial\xi$.

Table 1: The sensitivity of the optimal solution to the size of the budget deficit and the share of foreign debt in the public debt

the variable (.)	$\frac{\partial(.)}{\partial\xi}$	sign	$\frac{\partial(.)}{\partial\omega}$	sign
Δ	$2\chi(1-\omega)A$	+	$-2\chi\xi A$	?
q	$-\frac{\chi(1-\omega)A}{\sqrt{\Delta}}$	?	$\frac{\chi\xi A}{\sqrt{\Delta}}$	+
i/y	$-\frac{(1-\omega)}{\sqrt{\Delta}}$	?	$\frac{\xi}{\sqrt{\Delta}}$	+
φ	$-\frac{(1-\omega)A}{\sqrt{\Delta}}$	?	$\frac{\xi A}{\sqrt{\Delta}}$	+
ψ	0	0	0	0
c_0	$\frac{(1-\omega)(r-n-\psi)y_0}{(1+\sigma_C)(r-n-\varphi)^2\sqrt{\Delta}}\cdot W_1$	−	$\frac{-\xi(r-n-\psi)y_0}{(1+\sigma_C)(r-n-\varphi)^2\sqrt{\Delta}}\cdot W_1$	+
Ω	$\frac{\sigma_C^{\kappa\gamma}(1+\kappa)c_0^{\gamma(1+\kappa)-1}}{(r-n-\psi)}\cdot\frac{\partial c_0}{\partial\xi}$	−	$\frac{\sigma_C^{\kappa\gamma}(1+\kappa)c_0^{\gamma(1+\kappa)-1}}{(r-n-\psi)}\cdot\frac{\partial c_0}{\partial\omega}$	+

The share of foreign creditors in the public debt (ω) also has an impact on almost all of the trajectories. The higher is ω, the higher the rate of growth of GDP. It is due to the fact that the lower the share of domestic creditors in public debt, the higher the rate of investment relative to GDP. The rate of growth of consumption per capita (both private and public) is independent of ω, whereas the higher the ω, the higher the initial consumption per capita c_0. It follows that the higher the ω, the higher is the whole trajectory of consumption per capita. This implies that the higher the share of foreign creditors in public debt, the higher is the welfare, which is reflected in the positive derivative of $\partial\Omega/\partial\omega$.

11 Conclusion

The general equilibrium model presented in this paper leads to the conclusion that under perfect capital mobility, the society is able to achieve the higher welfare, the lower is the deficit-to-GDP ratio and the higher is the share of foreign investors in the financing of public debt. Naturally so unambiguous conclusion is conditional on strong assumptions, in particular, the assumption of perfect capital mobility, which de facto means the ability to borrow from abroad (as well as invest abroad) unlimited amounts of funds with a fixed interest rate.

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A proof that the transversality condition (31f) is satisfied only at the point q_1

The equation (40) implies that

$$\lambda_2(t) = q(t) \cdot \lambda_1(0) \cdot e^{(\rho - r)t}. \tag{95}$$

From (45) and (46) it follows that

$$\begin{aligned} k(t) &= k_0 \exp\left(\int_0^t \left(\tfrac{q(s)-1}{\chi} - n - \delta\right) ds\right) = \\ &= k_0 \exp\left(\int_0^t \left(\tfrac{q(s)-1}{\chi}\right) ds\right) \cdot e^{-(n+\delta)t} \end{aligned} \tag{96}$$

Therefore, condition (36) can be written in the form:

$$\lambda_1(0) k_0 \cdot \lim_{t\to\infty} \left\{ q(t) e^{-(r+\delta)t} \cdot \exp\left(\int_0^t \left(\frac{q(s)-1}{\chi} \right) ds \right) \right\} = 0. \tag{97}$$

The fixed point q_1 is unstable, so any transition dynamics is excluded. If the economy is to reach this point, it must be there from the very beginning, that is, there must be:

$$\forall t \quad q_1(t) = q_1 = const, \tag{98}$$

where q_1 is given by (50). Therefore, in this case it follows that

$$\begin{aligned} &\int_0^t \left(\frac{q(s)-1}{\chi}\right) ds = \frac{q_1 - 1}{\chi} \cdot t = \left(r + \delta - \sqrt{\frac{\Delta}{\chi^2}} \right) \cdot t = \\ &= \left(r + \delta - \sqrt{\frac{2(r + \delta - \alpha A(1-\tau_K) + (1-\omega)\xi A)}{\chi} + (r+\delta)^2} \right) \cdot t = \\ &= \left(r + \delta - \sqrt{\dots} \right) \cdot t, \end{aligned} \tag{99}$$

which implies that the condition (97) is satisfied, because we have:

$$\begin{aligned} &\lambda_1(0) k_0 \cdot \lim_{t\to\infty} \left\{ q_1 e^{-(r+\delta)t} \cdot \exp\left(\int_0^t \left(\frac{q_1 - 1}{\chi} \right) ds \right) \right\} = \\ &= \lambda_1(0) k_0 q_1 \cdot \lim_{t\to\infty} \left\{ e^{-(r+\delta)t} \cdot e^{\left(r + \delta - \sqrt{\dots}\right) t} \right\} = \\ &= \lambda_1(0) k_0 q_1 \cdot \lim_{t\to\infty} \left\{ e^{\left(-\sqrt{\dots}\right) t} \right\} = 0. \end{aligned} \tag{100}$$

Now, we take care of the second fixed point: q_2. Unlike q_1, it is locally stable, therefore the economy converges to that point along some specified trajectory of the following form:

$$q_2(t) = q_2 + (q_2(0) - q_2) \cdot e^{\mu t}, \tag{101}$$

where the rate of growth $\mu < 0$ is a negative eigenvalue of appropriate matrix. Hence

$$\begin{aligned} &\int_0^t \left(\frac{q_2(s) - 1}{\chi} \right) ds = \frac{q_2 - 1}{\chi} \cdot t + \frac{q_2(0) - q_2}{\chi} \cdot \int_0^t e^{\mu s} ds = \\ &= \frac{q_2 - 1}{\chi} \cdot t + \frac{q_2(0) - q_2}{\chi\mu} \cdot (e^{\mu t} - 1), \end{aligned} \tag{102}$$

which after using (51) can be written as:

$$\int_0^t \left(\frac{q_2(s) - 1}{\chi} \right) ds = (r + \delta)t + \frac{1}{\chi}\sqrt{\ldots} \cdot t + \frac{q_2(0) - q_2}{\chi\mu} \cdot \left(e^{\mu t} - 1\right). \tag{103}$$

The left-hand side of (97) is therefore equal to:

$$\begin{aligned} &\lambda_1(0)k_0 \cdot \lim_{t\to\infty} \left\{ q_2(t)e^{-(r+\delta)t} \cdot \exp\left(\int_0^t \left(\frac{q_2(s) - 1}{\chi} \right) ds \right) \right\} = \lambda_1(0)k_0 \cdot \\ &\cdot \lim_{t\to\infty} \left(q_2(t) \exp\left(-(r+\delta)t + (r+\delta)t + \tfrac{1}{\chi}\sqrt{\ldots} \cdot t + \frac{q_2(0) - q_2}{\chi\mu} \left(e^{\mu t} - 1\right) \right) \right) = \\ &= \lambda_1(0)k_0 \cdot \lim_{t\to\infty} \left\{ \left[q_2 + (q_2(0) - q_2) \cdot e^{\mu t}\right] \exp\left(\frac{1}{\chi}\sqrt{\ldots} \cdot t + \frac{q_2(0) - q_2}{\chi\mu} \left(e^{\mu t} - 1\right) \right) \right\} = \\ &\lambda_1(0)k_0 \cdot \left(\lim_{t\to\infty} \left\{ q_2 \exp\left(\frac{1}{\chi}\sqrt{\ldots} \cdot t + \tfrac{q_2(0)-q_2}{\chi\mu} \left(e^{\mu t} - 1\right) \right) \right\} \right) + \\ &+\lambda_1(0)k_0 \cdot \left(\lim_{t\to\infty} \left\{ (q_2(0) - q_2) \cdot \exp\left(\left(\mu + \tfrac{1}{\chi}\sqrt{\ldots}\right) \cdot t + \frac{q_2(0) - q_2}{\chi\mu} \left(e^{\mu t} - 1\right) \right) \right\} \right). \end{aligned} \tag{104}$$

This equation contains the sum of two limits. The first is equal to $+\infty$, whereas the second is either 0 (for $\mu + \frac{1}{\chi}\sqrt{\ldots} < 0$), or $+\infty$ (for $\mu + \frac{1}{\chi}\sqrt{\ldots} > 0$). It follows that the left-hand side of the equation (97) is equal to $+\infty$, so the transversality condition (36) is violated.

The general solution of the equation 71

The equation (71) is of the following type:

$$\dot{y} = a \cdot y + f(t), \tag{105}$$

where $a = const$. The general solution for this type of equation is (see e.g. Blume and Simon 1994, p. 639):

$$y(t) = \left[S + \int^t f(s)e^{-as}ds\right] \cdot e^{at}. \tag{106}$$

Using the formula (106) for the equation (71) we get

$$b(t) = \left[S + \int^t f(s)e^{-(r-n)s}ds\right] \cdot e^{(r-n)t}, \tag{107}$$

where

$$f(t) = \upsilon k_0 e^{\varphi t} - c_0(1+\sigma_C) \cdot e^{\psi t} - \left[rd_{F0} - \frac{r\omega\xi Ak_0}{n+\varphi}\right] e^{-nt}. \tag{108}$$

Substituting (108) into (107) after transformation we get:

$$b(t) = \left[S + \int^t \left(\upsilon k_0 e^{\varphi s} - c_0(1+\sigma_C) \cdot e^{\psi s} - \left[rd_{F0} - \tfrac{r\omega\xi Ak_0}{n+\varphi}\right] e^{-ns}\right) e^{-(r-n)s}ds\right] \cdot$$
$$\cdot e^{(r-n)t}, \tag{109}$$

which can be written in the form:

$$b(t) = Se^{(r-n)t}\left(\int^t \upsilon k_0 e^{(\varphi-r+n)s}ds - \int^t c_0(1+\sigma_C)e^{(\psi-r+n)s}ds+\right) \cdot e^{(r-n)t}+$$
$$-\left(\int^t \left[rd_{F0} - \frac{r\omega\xi Ak_0}{n+\varphi}\right] e^{-rs}ds\right) \cdot e^{(r-n)t}, \tag{110}$$

that is

$$\begin{aligned} b(t) &= Se^{(r-n)t}+ \\ &+\left(\frac{-\upsilon k_0}{r-n-\varphi} \cdot e^{(\varphi-r+n)t} + \frac{c_0(1+\sigma_C)}{r-n-\psi} \cdot e^{(\psi-r+n)t}\right) \cdot e^{(r-n)t}+ \\ &+\left(d_{F0} - \frac{\omega\xi Ak_0}{n+\varphi}\right) e^{-rt} \cdot e^{(r-n)t}, \end{aligned} \tag{111}$$

which can be written as:

$$b(t) = Se^{(r-n)t} - \frac{\upsilon k_0}{r-n-\varphi} \cdot e^{\varphi t} + \frac{c_0(1+\sigma_C)}{r-n-\psi} \cdot e^{\psi t} + \left(d_{F0} - \frac{\omega\xi Ak_0}{n+\varphi}\right) e^{-nt}. \tag{112}$$