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SIMPLE, FAST AND ACCURATE FOUR-POINT ESTIMATORS OF SINUSOIDAL SIGNAL FREQUENCY

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Abstract

In this paper, two new sinusoidal signal frequency estimators calculated on the basis of four equally spaced signal samples are presented. These estimators are called four-point estimators. Simulation and experimental research consisting in signal frequency estimation using the invented estimators have been carried out. Simulation has also been performed for frequency tracking. The simulation research was carried out applying the MathCAD computer program that determined samples of a sinusoidal signal disturbed by Gaussian noise. In the experimental research, sinusoidal signal samples were obtained by means of a National Instruments PCI-6024E data acquisition card and an Agilent 33220A function generator. On the basis of the collected samples, the values of four-point estimators invented by the authors and, for comparison, the values of three- and four-point estimators proposed by Vizireanu were determined. Next, estimation errors of the signal frequency were determined. It has been shown that the invented estimators can estimate a signal frequency with greater accuracy.

Keywords: sinusoidal signal, frequency estimation, frequency tracking.

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1. Introduction

Frequency is one of the most important signal parameters. For its determination we are applying methods that can be divided into two main categories. The spectral methods (interpolated DFT methods [1–6], cepstral method [7], ACOLS method [8], methods in which frequency estimation is carried out by means of maximum likelihood estimators [9]) and the time methods (correlation methods [10–12], threshold methods [13], methods using the digital filters [14], Bayesian methods [15], point methods [16–18]) can be pointed out.

Methods of frequency estimation using appropriate digital estimators are widely applied in technology (power engineering, communication, audio systems, radars, sonars, biological signal processing, speech processing). These methods are characterized by various levels of computational complexity and different accuracy. The accuracy depends largely on the amount of data on the basis of which the frequency is estimated. In general, the consequence of growing amount of data is an increased accuracy of the frequency estimation. However, the growing amount of data causes an increase of the calculation time. It also requires the use of expensive devices for

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acquisition, processing and archiving of measurement data. There is also an increase in power consumption.

In measurement systems (for example, in audio systems), where the priorities are a short signal processing time and a small calculation cost (understood as the cost of devices and power), the frequency estimation methods based on a small amount of data are applied. An example is the point method that enables the frequency estimation in the time domain with the use of several signal samples (points). Compared with other methods, this method enables faster frequency estimation. A frequency estimate is based on a few samples from all samples present in the signal period. If there are initial signal samples, we can determine the frequency before the signal period expires. The last trait also indicates another area of application of point methods, that is, the ability to predict the frequency of slow-motion signals.

In this paper, two new sinusoidal signal frequency estimators, based on a point method, are presented. The signal frequency estimation applying the invented estimators and – belonging to the same class of solutions – three- and four-point frequency estimators proposed by Vizireanu has been researched [16, 17]. The frequency estimation errors have been determined. The signal frequency tracking has also been carried out. The obtained results have shown that – in comparison with the Vizireanu estimators – the invented estimators can estimate frequency with greater accuracy.

2. Sinusoidal signal and its samples

Let $x(t)$, $t \in \mathbf{R}$ be a sinusoidal signal with an amplitude $A \in \mathbf{R}_+ \backslash \{0\}$, a frequency $f \in \mathbf{R}_+ \backslash \{0\}$, and an initial phase $\varphi \in [-\pi;\ \pi]$. Then:

$$x(t) = A\sin(2\pi f t + \varphi). \tag{1}$$

The samples x$[n]$ of the signal $x(t)$ are obtained as a result of sampling the signal with a sampling frequency:

$$f_s = Mf\frac{1}{\Delta_M N}, \tag{2}$$

where $M \in \mathbf{N}\backslash\{0..2\}$ is the number of samples per $N \in \mathbf{N}\backslash\{0\}$ signal periods, while:

$$\Delta_M \in [1-\varepsilon_M,\ 1+\varepsilon_M] \tag{3}$$

is a measurement window whose width depends on the error:

$$\varepsilon_M = \frac{1}{M} \tag{4}$$

of counting the samples in the period. Then:

$$x[n] = A\,\sin\left(2\pi f n\frac{1}{f_s} + \varphi\right) = A\,\sin\left(\frac{2\pi}{M}n\Delta_M N + \varphi\right), \quad n = 0..M-1. \tag{5}$$

The way of generating samples resulting from the change in the width of the measurement window refers to real measurement conditions when to determine the unknown signal frequency samples obtained from an incomplete number of the measured signal periods can be used.

3. Three- and four-point frequency estimators

In the point methods, estimators of a frequency f calculated on the basis of the samples $x[n]$ of a signal $x(t)$ are used.

Let us assume $M \geq 4$, and take into account the first, second and fourth samples $x[n]$ of the signal $x(t)$. Then, on the basis of (5), we obtain:

$$\begin{cases} x[0] = A\sin(\varphi), \\ x[1] = A\left(\sin\left(2\pi\dfrac{f}{f_s}\right)\cos(\varphi) + \cos\left(2\pi\dfrac{f}{f_s}\right)\sin(\varphi)\right), \\ x[3] = A\left(\sin\left(6\pi\dfrac{f}{f_s}\right)\cos(\varphi) + \cos\left(6\pi\dfrac{f}{f_s}\right)\sin(\varphi)\right). \end{cases} \tag{6}$$

Determining the amplitude:

$$A = \frac{x[0]}{\sin(\varphi)} \tag{7}$$

from the first equation of (6), and substituting (7) in the remaining equations of (6), we obtain:

$$\begin{cases} x[1] = x[0]\dfrac{\cos(\varphi)}{\sin(\varphi)}\sin\left(2\pi\dfrac{f}{f_s}\right) + x[0]\cos\left(2\pi\dfrac{f}{f_s}\right), \\ x[3] = x[0]\dfrac{\cos(\varphi)}{\sin(\varphi)}\sin\left(6\pi\dfrac{f}{f_s}\right) + x[0]\cos\left(6\pi\dfrac{f}{f_s}\right). \end{cases} \tag{8}$$

Determining the expression:

$$x[0]\frac{\cos(\varphi)}{\sin(\varphi)} = \frac{x[1] - x[0]\cos\left(2\pi\dfrac{f}{f_s}\right)}{\sin\left(2\pi\dfrac{f}{f_s}\right)} \tag{9}$$

from the first equation of (8), and substituting (9) in the second equation of (8), we obtain:

$$x[3] = \frac{x[1] - x[0]\cos\left(2\pi\dfrac{f}{f_s}\right)}{\sin\left(2\pi\dfrac{f}{f_s}\right)}\sin\left(6\pi\frac{f}{f_s}\right) + x[0]\cos\left(6\pi\frac{f}{f_s}\right). \tag{10}$$

Tidying up (10), we obtain the following equation:

$$4x[1]\cos^2\left(2\pi\frac{f}{f_s}\right) - 2x[0]\cos\left(2\pi\frac{f}{f_s}\right) - x[1] - x[3] = 0. \tag{11}$$

If:

$$x^2[0] + 4x^2[1] + 4x[1]x[3] > 0, \tag{12}$$

then (11) has two equivalent solutions which can be described with the formula:

$$\cos\left(2\pi\frac{f}{f_s}\right) = \frac{x[0] \pm \sqrt{x^2[0] + 4x^2[1] + 4x[1]x[3]}}{4x[1]}. \tag{13}$$

Equation (13) can be replaced by a formula which enables to calculate $\cos(2\pi f/f_s)$ in an unambiguous manner. To this end, it has been proposed that in (13) the sign of a sample, *i.e.*:

$$s_1 = \mathrm{sign}(x[1]) \tag{14}$$

is to be taken into account, as well as the sign of the expression:

$$s_2 = \mathrm{sign}\left(\frac{x[0]+2x[2]}{4x[1]}\right), \tag{15}$$

where $\mathrm{sign}(\cdot)$ is the sign function. The formula (15) is arrived at by establishing the sign of the comparison between $\cos(2\pi f/f_s)$ and $x[0]/(4x[1])$, replacing $\cos(2\pi f/f_s)$ with a formula that makes it possible to calculate this expression based on three samples $x[n]$ of the signal $x(t)$ [16]. Since for any a, $b \in \mathbf{R}$ it follows that $\mathrm{sign}(a)\cdot\mathrm{sign}(b) = \mathrm{sign}(a \cdot b)$, then:

$$s^{(1)} = s_1\, s_2 = \mathrm{sign}\left(x[0]+2x[2]\right). \tag{16}$$

Thus,

$$\cos\left(2\pi\frac{f}{f_s}\right) = \frac{x[0]+s^{(1)}\sqrt{x^2[0]+4x^2[1]+4x[1]x[3]}}{4x[1]}. \tag{17}$$

The proposed change of (13) into (17) enables to obtain an appropriate form of the frequency estimator.

From (17) we obtain the following estimator:

$$f_1^{(1)} = \frac{f_s}{2\pi}\mathrm{acos}\left(\frac{x[0]+s^{(1)}\sqrt{x^2[0]+4x^2[1]+4x[1]x[3]}}{4x[1]}\right) \tag{18}$$

of a frequency f of the signal $x(t)$. Since to calculate $f_1^{(1)}$ four samples $x[n]$ of the signal $x(t)$ ought to be used, then we shall call such an estimator a four-point estimator. If we assume $k = 1..M-3$, then $f_1^{(1)}$ can be generalized to the following form:

$$f_1^{(1)}(k) = \frac{f_s}{2\pi}\mathrm{acos}\left(\frac{x[k-1]+s^{(1)}(k)\sqrt{x^2[k-1]+4x^2[k]+4x[k]x[k+2]}}{4x[k]}\right), \tag{19}$$

where:

$$s^{(1)}(k) = \mathrm{sign}\left(x[k-1]+2x[k+1]\right). \tag{20}$$

Taking into account the first, third and fourth samples $x[n]$ of the signal $x(t)$, we obtain:

$$\begin{cases} x[0] = A\sin(\varphi), \\ x[2] = A\left(\sin\left(4\pi\dfrac{f}{f_s}\right)\cos(\varphi)+\cos\left(4\pi\dfrac{f}{f_s}\right)\sin(\varphi)\right), \\ x[3] = A\left(\sin\left(6\pi\dfrac{f}{f_s}\right)\cos(\varphi)+\cos\left(6\pi\dfrac{f}{f_s}\right)\sin(\varphi)\right). \end{cases} \tag{21}$$

Then,

$$x[3] = \frac{x[2]-x[0]\cos\left(4\pi\dfrac{f}{f_s}\right)}{\sin\left(4\pi\dfrac{f}{f_s}\right)}\sin\left(6\pi\frac{f}{f_s}\right)+x[0]\cos\left(6\pi\frac{f}{f_s}\right). \tag{22}$$

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Equation (22) can be brought to the following form:

$$4x[2]\cos^2\left(2\pi\frac{f}{f_s}\right)-2x[3]\cos\left(2\pi\frac{f}{f_s}\right)-x[0]-x[2]=0. \tag{23}$$

If:

$$x^2[3]+4x^2[2]+4x[0]x[2]>0, \tag{24}$$

then (23) has two equivalent solutions which can be described with the formula:

$$\cos\left(2\pi\frac{f}{f_s}\right)=\frac{x[3]\pm\sqrt{4x^2[2]+x^2[3]+4x[0]x[2]}}{4x[2]}. \tag{25}$$

Equation (25) can be replaced by a formula which makes it possible to calculate $\cos(2\pi f/f_s)$ in an unambiguous manner. To this end, in (25), we ought to take into account the sign of a sample $x[2]$, *i.e.*:

$$s_3=\mathrm{sign}(x[2]), \tag{26}$$

as well as the sign of the expression:

$$s_4=\mathrm{sign}\left(\frac{2x^2[2]+2x[0]x[2]-x[1]x[3]}{4x[1]x[2]}\right). \tag{27}$$

On the basis of (26) and (27), we obtain:

$$s^{(2)}=s_3s_4=\mathrm{sign}\left(2\left(x[0]+x[2]\right)\frac{x[2]}{x[1]}-x[3]\right). \tag{28}$$

Then,

$$\cos\left(2\pi\frac{f}{f_s}\right)=\frac{x[3]+s^{(2)}\sqrt{4x^2[2]+x^2[3]+4x[0]x[2]}}{4x[2]}. \tag{29}$$

From (29) we obtain the following estimator:

$$f_1^{(2)}=\frac{f_s}{2\pi}\mathrm{acos}\left(\frac{x[3]+s^{(2)}\sqrt{4x^2[2]+x^2[3]+4x[0]x[2]}}{4x[2]}\right) \tag{30}$$

of a frequency f of the signal $x(t)$. We shall call such an estimator a four-point estimator. The general form of the estimator $f_1^{(2)}$, for $k=1..M-3$, can be described by the formula:

$$f_1^{(2)}(k)=\frac{f_s}{2\pi}\mathrm{acos}\left(\frac{x[k+2]+s^{(2)}(k)\sqrt{4x^2[k+1]+x^2[k+2]+4x[k-1]x[k+1]}}{4x[k+1]}\right), \tag{31}$$

where:

$$s^{(2)}(k)=\mathrm{sign}\left(2(x[k-1]+x[k+1])\frac{x[k+1]}{x[k]}-x[k+2]\right). \tag{32}$$

Vizireanu has invented their own estimators of a frequency f calculated on the basis of three and four samples $x[n]$ of the signal $x(t)$. If the samples are of the form (5), then, as Vizireanu shows in [16], a frequency f can be estimated using the three-point estimator:

$$f_1^{(3)}=\frac{f_s}{2\pi}\mathrm{acos}\left(\frac{x[0]+x[2]}{2x[1]}\right). \tag{33}$$

Assuming $k = 1..M-2$ we receive the generalized form:

$$f_1^{(3)}(k) = \frac{f_s}{2\pi}\mathrm{acos}\left(\frac{x[k-1]+x[k+1]}{2x[k]}\right) \tag{34}$$

of the estimator $f_1^{(3)}$. If in (1) the constant component is allowed, then Vizireanu proposes in [17] that a frequency f can be estimated using the four-point estimator:

$$f_1^{(4)} = \frac{f_s}{2\pi}\mathrm{acos}\left(\frac{x[0]-x[1]+x[2]-x[3]}{2\,(x[1]-x[2])}\right). \tag{35}$$

If we assume $k = 1..M-3$, then $f_1^{(1)}$ can be generalized to the following form:

$$f_1^{(4)}(k) = \frac{f_s}{2\pi}\mathrm{acos}\left(\frac{x[k-1]-x[k]+x[k+1]-x[k+2]}{2\,(x[k]-x[k+1])}\right). \tag{36}$$

Although the estimators $f_1^{(1)-(4)}$ belong to the same class of solutions, they have different properties. Several of them are characterized below.

We should note that the estimators $f_1^{(1)-(4)}$ are calculated on the basis of several signal samples. This means that we cannot expect high accuracy in the case of strongly disturbed sinusoidal signals. The research results presented in the further part of the paper confirm their usefulness for SNR larger than 40 dB. For smaller SNR values, prediction estimators can be used, which are characterized by greater immunity to noise. In the case of the estimators $f_1^{(3),(4)}$, it is possible to construct linear prediction algorithms. It is due to the linear relationship between the signal samples. The linearity of $f_1^{(3),(4)}$ results from the possibility of such converting the formulas (33) and (35), that the last signal sample would be a sample function linearly dependent on the previous samples. The consequence of this is the possibility of constructing, on the basis of $f_1^{(3),(4)}$, linear prediction filters, for example lattice filters [18], whose coefficients can be determined using the least squares method [19]. In the case of $f_1^{(1),(2)}$, the linear relationship between the samples does not occur. This means that it is not possible to construct linear prediction filters. Nevertheless, the equations (10) and (22), from which the estimators $f_1^{(1),(2)}$ result, may be the basis for the construction of new algorithms.

Comparing the forms of estimators $f_1^{(1)-(4)}$ with each other, it can be found that the Vizireanu estimators have a simpler structure than those proposed by us. A more complex form of estimators may be important in some applications (for example in implementation on digital signal processors), the result of which is an increase of the number of mathematical operations. First of all, the estimators $f_1^{(1),(2)}$ require calculation of the square root. The conditions (12) and (24) result from the operation of square root calculation. If the conditions (12) and (24) have not been satisfied, *i.e.* when:

$$\varphi^{(1)} = -2\pi\frac{f}{f_s} - \mathrm{atan}\left(\frac{1}{3}\tan\left(2\pi\frac{f}{f_s}\right)\right) + l\pi, \qquad \varphi^{(2)} = -6\pi\frac{f}{f_s} - \varphi^{(1)} + 2l\pi, \qquad l \in \mathbf{Z}, \tag{37}$$

then calculating the estimators $f_1^{(1),(2)}$ will not be possible.

It also ought to be taken into account that calculating the estimators $f_1^{(1)-(4)}$ will not be possible if $x[1] = 0$, $x[2] = 0$ and $x[1] = x[2]$, *i.e.* when:

$$\varphi^{(1)-(3)} = -2\pi\frac{f}{f_s} + l\pi, \qquad \varphi^{(2)} = -4\pi\frac{f}{f_s} + l\pi, \qquad \varphi^{(4)} = \frac{\pi}{2} - 3\pi\frac{f}{f_s} + l\pi. \tag{38}$$

Owing to the fact that not for each value of φ is calculation of $f_1^{(1)-(4)}$ possible, it was assumed that the estimators $f_1^{(1)-(4)}$ would be calculated in the situation when $\varphi = 0$. There are important reasons justifying making such an assumption. Firstly, besides the cases when $f_1^{(4)}$ is calculated for $M = 5$ or $M = 6$, assuming $\varphi = 0$ makes it possible to calculate $f_1^{(1)-(4)}$ for any number of samples M, with $\varphi \neq \varphi^{(1)-(4)}$. Secondly, the assumption $\varphi = 0$ also results in the fact that $f_1^{(1)-(4)}$ will be calculated on the basis of the same signal samples. It means that a frequency f will be estimated using different estimators, but in the same measurement conditions. Choosing $\varphi = 0$ has yet another practical justification resulting from the hardware capabilities of performing measurements by means of commercially available measurement devices. In such devices, commencing a measurement (for $\varphi = 0$) can be triggered off by a clock signal from a sinusoidal signal source. An example of such a source is an Agilent 33220A function generator used in the experimental research. Another example is a Keysight 3458A sampling device in which commencing a measurement can be program-controlled.

It should be stressed that the estimators $f_1^{(1),(2)}$ will be useful only when the errors of the estimators are significantly smaller than the errors of the Vizireanu estimators $f_1^{(3),(4)}$. It will be shown by the simulation and experimental research results presented below.

4. Discussion of research results

Simulations and experimental research consisting in comparing the accuracy of the frequency f estimation with the invented estimators, as well as the Vizireanu estimators, have been carried out. Each of the estimators $f_1^{(1)-(4)}$ was assessed based on a formula for the expressed percentagewise relative error:

$$\varepsilon_f^{(1)-(4)} = \frac{\left| f_1^{(1)-(4)} - f \right|}{f} 100. \tag{39}$$

The calculations of $f_1^{(1)-(4)}$ and then $\varepsilon_f^{(1)-(4)}$ were repeated K times, and – on the basis of the obtained sets of errors – the maximum errors:

$$\varepsilon^{(1)-(4)} = \max \left\{ \varepsilon_f^{(1)-(4)}(k) \right\}_{k=1}^{K}, \tag{40}$$

were determined. The maximum errors were determined and compared for different parameters of a sinusoidal signal with Gaussian noise, and for the parameters of its A–D processing. The determination of maximum errors is justified because the estimators presented in this paper are calculated based on several signal samples. Usually, the accuracy of a particular parameter is assessed based on the bias and the standard deviation of the estimator. In the case of multi-point estimators, the interpretation of such accuracy measures would still be difficult or ambiguous due to widely scattered estimation results.

Simulations were also carried out consisting in the modulated and unmodulated estimation of a sinusoidal signal frequency $f(k)$, by means of $f_1^{(1)-(4)}(k)$. In these studies, the average errors:

$$\varepsilon_{tr}^{(1)-(4)} = \frac{1}{m} \sum_{k=1}^{m} \left| f_1^{(1)-(4)}(k) - f(k) \right|, \tag{41}$$

were determined on the basis of $m \in \mathbf{N} \backslash \{0\}$ results of $f_1^{(1)-(4)}(k)$ within a fixed measurement time.

4.1. Simulation results

During the first simulation studies, it was assumed that signal samples were obtained by A–D conversion of the signal, which was sampled with a frequency f_s known with an inaccuracy $\varepsilon_{fs} \in \mathbf{R}_+$. It was also assumed that such a signal occurred in the presence of noise whose source was a measuring channel.

Let us assume that the estimators $f_1^{(1)-(4)}$ are calculated based on the quantized samples:

$$y_q[n] = Q(y[n]) = Q(x[n] + n[n]), \qquad n = 0..M-1, \tag{42}$$

of the sum $y(t) = x(t) + n(t)$ of a sinusoidal signal $x(t)$ and an additive Gaussian noise $n(t)$.

The samples:

$$x[n] = A\sin\left(2\pi f \frac{1}{f_s\left(1+\dfrac{\delta_{f_s}}{100}\right)} n + \varphi\right) \tag{43}$$

of the signal $x(t)$ occur in the presence of the samples $n[n]$ of the Gaussian noise $n(t)$ which is characterized by the standard deviation $\sigma_n \in \mathbf{R}_+$. If by:

$$\mathrm{SNR} = 10\log\left(\frac{0.5A^2}{\sigma_n^2}\right), \tag{44}$$

we put a signal-to-noise ratio expressed in decibels, then:

$$\sigma_n = \frac{1}{\sqrt{10^{\frac{\mathrm{SNR}}{10}}}} \frac{A}{\sqrt{2}}. \tag{45}$$

The quantization operation:

$$Q(\pm z) = q \cdot \mathrm{round}\left(\pm\frac{z}{q} \pm \frac{1}{2}\right) \tag{46}$$

was performed on the samples $y[n]$ of the signal $y(t)$ in an ideal round-off A–D converter with a step size:

$$q = \frac{2A}{2^B}, \tag{47}$$

where $B \in \mathbf{N}\backslash\{0,1\}$ is a resolution of the A-D converter, whereas round$(\cdot)$ is a function rounding off a real number to the nearest integer [20].

Assuming $A = 5$ V, $f = 4$ kHz, and $\varphi = 0$, the effect of noise and A–D processing parameters on the maximum errors $\varepsilon^{(1)-(4)}$ has been examined (Figs. 1–4). The errors $\varepsilon^{(1)-(4)}$ were determined on the basis of the errors $\varepsilon_f^{(1)-(4)}(k)$ out of $K = 1\,000$ repetitions of the experiment, for $N = 1$ and Δ_M assuming uniform values from the interval (3) with a step size $(2/M)/100$.

In the first place, the effect of the number of samples M and of Gaussian noise with the standard deviation σ_n on the errors $\varepsilon^{(1)-(4)}$ was examined (Fig. 1). It ought to be noted that if $\sigma_n > 0$, and $M \geq 4$, then the error $\varepsilon^{(2)}$ is less than the errors $\varepsilon^{(3)}$ and $\varepsilon^{(4)}$, and generally less than the error $\varepsilon^{(1)}$. If, however, $\sigma_n > 0$ and $M > 5$, then also the error $\varepsilon^{(1)}$ is less than the errors $\varepsilon^{(3)}$ and $\varepsilon^{(4)}$.

It should be taken into account that a noise level has been expressed with SNR. The SNR levels were selected due to the values of this quantity obtained in real measuring conditions and due to the research results presented in the literature. In this paper and in the papers [21, 22]

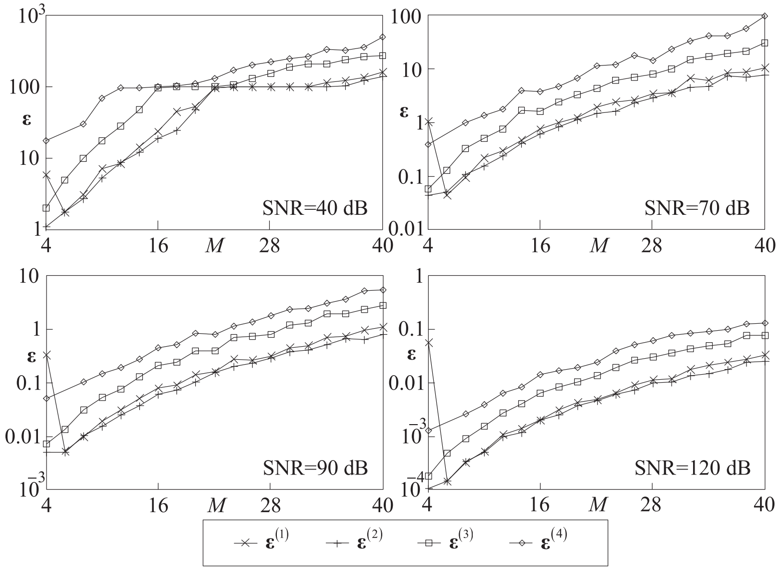

Fig. 1. Errors $\varepsilon^{(1)-(4)}$ as a function of *M* for a fixed SNR ($B = \infty$, $\varepsilon_{fs} = 0$).

it was shown that SNR values are within a range of [65 dB; 75 dB] for the function generator used in the measurements. Vizireanu stated in [16] that the estimator $f_1^{(3)}$ can be useful if SNR > 45 dB. Alegria and others showed in [23] that if SNR > 70 dB, then $\varepsilon^{(4)} < 2\%$. Our research results confirmed the occurrence of large estimation errors for small SNR values. For example, if SNR = 35 dB and $M = 10$, then $\varepsilon^{(1)} = 14\%$, $\varepsilon^{(2)} = 9.2\%$, $\varepsilon^{(3)} = 33\%$ and $\varepsilon^{(4)} = 99\%$, but if SNR = 10 dB, then $\varepsilon^{(1)} = 100\%$, $\varepsilon^{(2)} = 247\%$, $\varepsilon^{(3)} = 218\%$ and $\varepsilon^{(4)} = 393\%$. Therefore, the paper presents the values of errors for SNR from 40 dB to 120 dB. The maximum level of SNR equal to 120 dB was assumed in order to disclose the estimators' properties that could be studied when the noise influence was negligible. In practice, a level of SNR greater than 100 dB characterizes signals from precision sources, such as calibrators.

It should also be noted that the estimators $f_1^{(1)-(4)}$ required calculation of the arc cos function. The domain of this function is a set of all real numbers from an interval $[-1, 1]$. Alegria and others suggested in [23] that if the arguments of the arc cos function did not belong to its domain, then the obtained results should be rejected. Next, the frequency estimation process should be repeated for new data until the correct calculation results are obtained. Therefore in our work, if the arc cos function arguments were not within a range of its domain, then the calculation results of $f_1^{(1)-(4)}$ were rejected. The results of $f_1^{(1),(2)}$ were also rejected if the conditions (12) and (24) had not been satisfied. It was found that the results of $f_1^{(1),(2)}$ had not been rejected if $M \leq 40$ and SNR > 55 dB. However, the exclusion of the results of $f_1^{(3)}$ and $f_1^{(4)}$ estimators had no place for $M \leq 40$, as well as for SNR > 65 dB and SNR > 70 dB, respectively.

In the next step, the simulations consisting in checking the effect of *B* on the errors $\varepsilon^{(1)-(4)}$ (Fig. 2) were carried out. It can be noted that the quantizing operation has an effect on the increase in the value of the errors $\varepsilon^{(1)-(4)}$. It does not, however, change the results of error comparison. Besides the cases when $M < 6$, the errors $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ are significantly less than the errors $\varepsilon^{(3)}$ and $\varepsilon^{(4)}$.

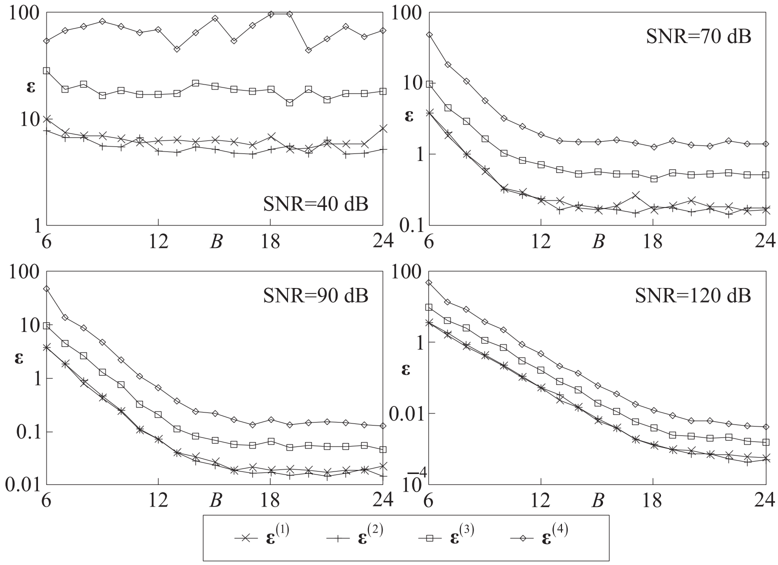

Fig. 2. Errors $\varepsilon^{(1)-(4)}$ as a function of B for fixed M and SNR values ($M = 10$, $\varepsilon_{fs} = 0$).

The simulations consisting in checking the effect of ε_{fs} on the errors $\varepsilon^{(1)-(4)}$ have also been carried out (Fig. 3). In the simulation research, it was assumed that the error $\varepsilon_{fs} < 1\%$, which is often the case in practice. From the obtained research results it follows that the inaccuracy ε_{fs} of a frequency f_s influences the error of the frequency f measurement. The obtained results show that if in the measurement theoretically no other sources of error besides ε_{fs} occur, then $\varepsilon_{fs} = \varepsilon^{(1)-(4)}$.

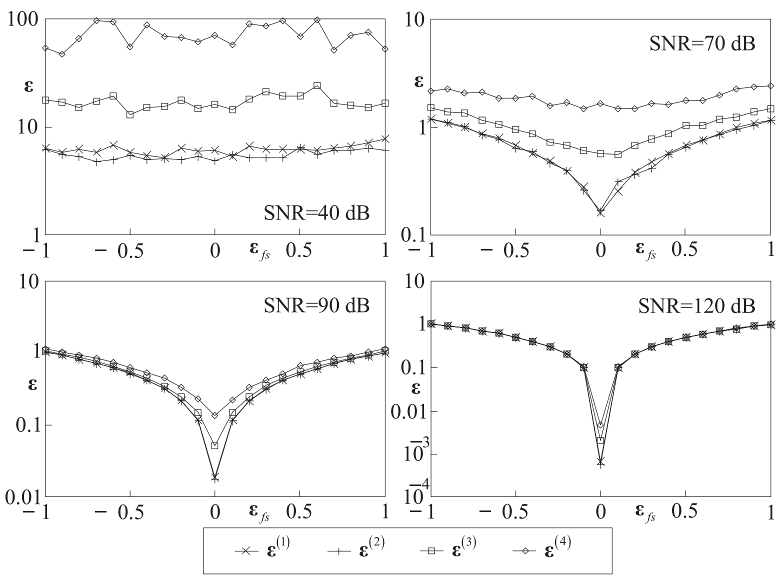

Fig. 3. Errors $\varepsilon^{(1)-(4)}$ as a function of ε_{fs} for fixed M and SNR values ($M = 10$, $B = \infty$).

Hence, it follows that it is possible to obtain high accuracies of the frequency f measurements by using a sampling device whose sampling frequency f_s is also known with a high accuracy. If, however, in a measurement channel additional signal disturbances occur, then the inaccuracy of the frequency f measurement will increase. In practice, the influence of disturbances can still be minimized by means of carrying out multiple measurements and averaging the obtained results.

Unlike $f_1^{(4)}$, the estimators $f_1^{(1)-(3)}$ were calculated when the signal $x(t)$ did not contain a constant component $A_0 \in \mathbf{R}$. However, in real measuring situations, we often deal with sinusoidal signals containing A_0. In order to verify the influence of A_0 on the errors $\varepsilon^{(1)-(4)}$, the estimators were calculated assuming that the sinusoidal component $x(t)$ of the signal $y(t)$ contains A_0. The simulation results had shown that with an increase of A_0, the values of errors $\varepsilon^{(1)-(3)}$ increased. Moreover, if $A_0 \approx 0$, then $\varepsilon^{(1),(2)} < \varepsilon^{(3)} < \varepsilon^{(4)}$ (Fig. 4). In the assumed range of A_0, the error $\varepsilon^{(4)}$ is approximately constant and depends to a large extent on an SNR level.

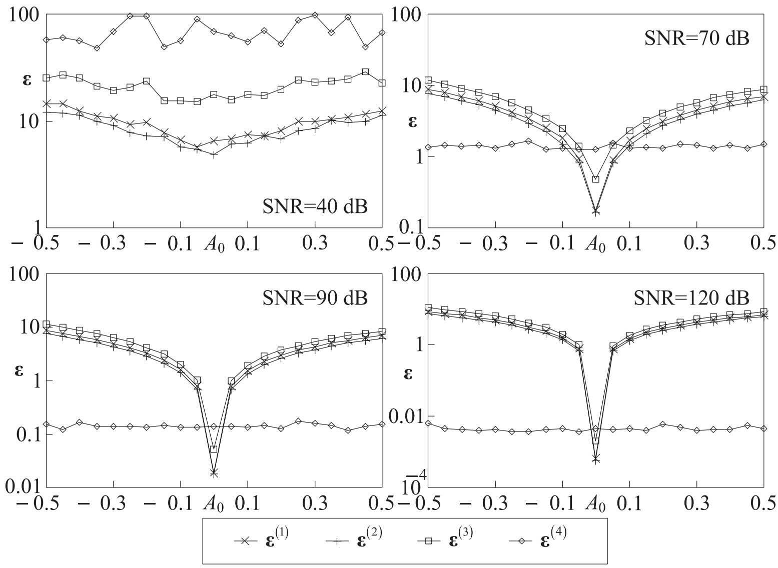

Fig. 4. Errors $\varepsilon^{(1)-(4)}$ as a function of A_0 for fixed M and SNR values ($M = 10$, $B = \infty$, $\varepsilon_{fs} = 0$).

Another study concerned the frequency tracking of a sinusoidal signal and of a sinusoidal signal with a modulated frequency, by means of $f_1^{(1)-(4)}(k)$ (Figs. 5, 6). It was possible because the estimators $f_1^{(1)-(4)}(k)$ could be calculated from a sequence of samples $x[k-1]$, $x[k]$, $x[k+1]$, $x[k+2]$ of the signal $x(t)$. A test was performed consisting of tracking a frequency $f(k) = f$ of the signal $y(t) = x(t) + n(t)$ and a modulated frequency:

$$f(k) = \frac{k_f}{f_s} k + f_0^{(m)} \tag{48}$$

of the signal:

$$y^{(m)}(t) = A\cos\left(2\pi\left(\frac{k_f}{2}t + f_0^{(m)}\right)t + \varphi\right) + n(t). \tag{49}$$

The chirp modulation has been applied, where:

$$k_f = \frac{f_1^{(m)} - f_0^{(m)}}{T^{(m)}}, \tag{50}$$

$f_0^{(m)}$, $f_1^{(m)}$ and $T^{(m)}$ are a rate of frequency change, beginning and ending modulation frequencies and a duration of modulation, respectively [18]. Both signals $y(t)$ and $y^{(m)}(t)$ contained a Gaussian noise $n(t)$. The following measurement conditions were assumed: $f_s = 4$ kHz, $f = 400$ Hz, $N = 100$, $f_0^{(m)} = 0$, $f_1^{(m)} = 1$ kHz, $T^{(m)} = 1$ s, $\Delta_M = 1$, $B = \infty$, $\varepsilon_{fs} = 0$.

Unlike in the previous simulation studies, the initial phases of traced sinusoidal signals are not constant and are not equal to 0 rad. For this reason, the estimators $f_1^{(1)-(4)}(k)$ for proper operation (elimination of excessive estimation errors) require additional conditions: $|y[k]| > \Theta$, $|y[k+1]| > \Theta$ and $|y[k] - y[k+1]| > \Theta$, where $\Theta \in \mathbf{R}_+$. Vizireanu used a similar approach in the paper [16]. If during the trace of $f(k)$ the conditions for calculating $f_1^{(1)-(4)}(k)$ have not been satisfied, the current results of the estimators $f_1^{(1)-(4)}(k)$ are rejected and replaced with the previous ones, obtained on the basis of $f_1^{(1)-(4)}(k-1)$. Examples of results of tracking a frequency $f(k)$ for the signals $y(t)$ and $y^{(m)}(t)$ for SNR $= 70$ dB are shown in Figs. 5 and 6. On the basis of (41) the errors $\varepsilon_{tr}^{(1)-(4)}$ of tracking $f(k)$ were calculated. The values of errors $\varepsilon_{tr}^{(1)-(4)}$ for selected values of SNR and Θ are shown in Table 1. When calculating $\varepsilon_{tr}^{(1)-(4)}$ it was assumed that $m = N \cdot f_s/f - 3$ for $y(t)$ and $m = T^{(m)} \cdot f_s - 3$ for $y^{(m)}(t)$. The obtained results (Table 1) shows that the value of Θ affects the values of errors $\varepsilon_{tr}^{(1)-(4)}$. It was observed that increasing the value of Θ from 0 to around $A/2$ improves the accuracy of frequency estimation for all algorithms. Our research has also showed that if $\Theta > A/2$, then the estimation accuracy is reduced. In addition, in the case of tracking $f(k)$ of the signal $y(t)$, $\varepsilon_{tr}^{(1),(2)} < \varepsilon_{tr}^{(3),(4)}$. However, in the case of tracking $f(k)$ of the signal $y^{(m)}(t)$, $\varepsilon_{tr}^{(1),(2)} < \varepsilon_{tr}^{(3),(4)}$ for 40 dB $<$ SNR $<$ 70 dB.

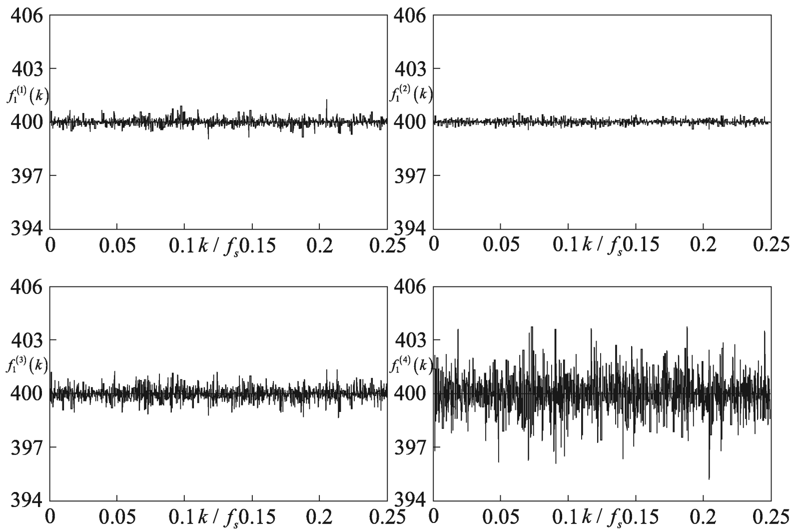

Fig. 5. Tracking $f(k)$ of the signal $y(t)$ with $f_1^{(1)-(4)}(k)$ (SNR $= 70$ dB, $\Theta = 0.1$ V).

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Table 1. Errors $\varepsilon_{tr}^{(1)-(4)}$ of tracking $f(k)$ of signals $y(t)$ and $y^{(m)}(t)$.

SNR [dB]	Θ [V]	$y(t)$				$y^{(m)}(t)$			
		$\varepsilon_{tr}^{(1)}$ [Hz]	$\varepsilon_{tr}^{(2)}$ [Hz]	$\varepsilon_{tr}^{(3)}$ [Hz]	$\varepsilon_{tr}^{(4)}$ [Hz]	$\varepsilon_{tr}^{(1)}$ [Hz]	$\varepsilon_{tr}^{(2)}$ [Hz]	$\varepsilon_{tr}^{(3)}$ [Hz]	$\varepsilon_{tr}^{(4)}$ [Hz]
40	0	6.6	82	75	127	13	16	27	100
70		0.20	81	80	109	1.3	2.1	2.1	17
90		0.021	87	78	103	1.2	1.4	1.5	7.3
120		$6.5\cdot10^{-4}$	81	72	107	1.2	1.3	1.3	17
40	10^{-14}	6.4	84	81	132	12	14	25	96
70		0.20	73	78	112	1.4	2.0	2.2	16
90		0.021	80	83	113	1.3	2.2	1.3	6.4
120		$6.1\cdot10^{-4}$	76	73	100	1.2	1.9	1.3	6.7
40	0.1	5.5	3.9	9.7	47	6.7	14	22	56
70		0.17	0.12	0.30	0.92	1.1	1.1	1.4	6.0
90		0.017	0.011	0.030	0.088	0.87	0.90	0.79	1.8
120		$5.0\cdot10^{-4}$	$3.6\cdot10^{-4}$	$9.5\cdot10^{-4}$	$2.9\cdot10^{-3}$	0.88	0.91	0.79	1.7
40	$A/2$	5.5	3.7	9.6	23	9.1	7.0	14	58
70		0.17	0.13	0.31	1.9	0.57	0.67	0.69	54
90		0.017	0.011	0.028	1.3	0.35	0.55	0.29	53
120		$4.9\cdot10^{-4}$	$3.7\cdot10^{-4}$	$9.0\cdot10^{-4}$	1.3	0.34	0.54	0.27	53

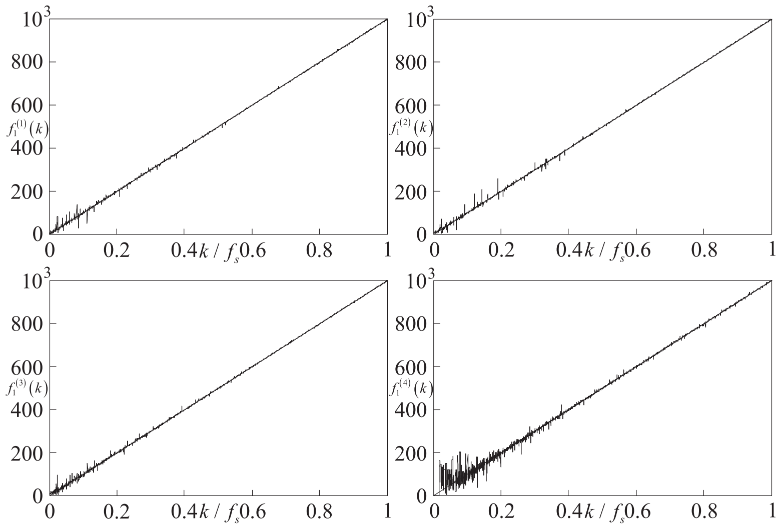

Fig. 6. Tracking $f(k)$ of the signal $y^{(m)}(t)$ with $f_1^{(1)-(4)}(k)$ (SNR = 70 dB, $\Theta = 0.1$ V).

On the basis of the simulation results, the following conclusions can be drawn:
- if $M \geq 4$, then $\varepsilon^{(2)}$ is less than $\varepsilon^{(3)}$ and $\varepsilon^{(4)}$, and generally less than $\varepsilon^{(1)}$;
- if $M > 5$, then $\varepsilon^{(1)}$ is less than $\varepsilon^{(3)}$ and $\varepsilon^{(4)}$;
- if $M = 4$, then $\varepsilon^{(1)}$ is greater than $\varepsilon^{(3)}$, and – in the case of SNR > 40 dB – also generally greater than $\varepsilon^{(4)}$;

– if $M = 5$, then $\varepsilon^{(1)}$ is generally greater than $\varepsilon^{(3)}$, and always less than $\varepsilon^{(4)}$;
– if SNR < 40 dB, then $\varepsilon^{(1)-(4)}$ assume values above 1%;
– $f_1^{(1),(2)}$ enables to track a sinusoidal signal frequency with an accuracy greater than that obtained for $f_1^{(3),(4)}$;
– if SNR < 70 dB, then $f_1^{(1),(2)}$ enables to track a sinusoidal signal modulated frequency with an accuracy greater than that obtained for $f_1^{(3),(4)}$.

4.2. Experimental research results

Taking into account the simulation results, a measurement experiment consisting in the acquisition of samples of a sinusoidal voltage with an amplitude $A = 5$ V, a frequency f and an initial phase $\varphi = 0$ was planned and carried out. The voltage signal was generated by an Agilent 33220A function generator. The signal was sampled using a National Instruments PCI-6025E data acquisition card. The samples were acquired for selected values of frequency f and sampling frequency f_s, with the assumption that $f_s \gg f$, to enable unambiguous identification of the frequency on the basis of the collected signal samples. The measurement resolution B, and the error ε_{fs} of the sampling frequency f_s amounted, respectively, to $B = 12$, and $\varepsilon_{fs} = 0.01\%$, and followed from the measurement board specification. During the sampling, a clock signal from a function generator was used to trigger a measurement in such a way that the first voltage sample collected within one sample series had the phase corresponding to a zero initial phase of the generated signal.

As a result of the performed measurements, four sets (files) containing $K = 100$ measurement series comprising the samples $y_q[n]$ of the sinusoidal voltage signal were obtained. For each measurement series, the errors $\varepsilon_f^{(1)-(4)}(k)$ were calculated, assuming for the sake of the calculations that the true frequency was the frequency f set on the generator (inaccuracy of the frequency f given by the generator manufacturer was 0.002%). Then, the maximum errors $\varepsilon^{(1)-(4)}$ were determined based on K results of the errors $\varepsilon_f^{(1)-(4)}(k)$. The obtained errors $\varepsilon^{(1)-(4)}$ are presented in Table 2. Table 2 also shows the numbers of samples $M' \in \mathbf{N}\backslash\{0..2\}$ determined for each of the sample sets and falling within the first period of the measured signal. Additionally, Table 2 contains intervals of the SNR values determined on the basis of samples for each measurement series.

Table 2. Results of the errors $\varepsilon^{(1)-(4)}$ and the intervals of SNR for Files 1–4.

	File 1	**File 2**	**File 3**	**File 4**
f [kHz]	4	3.3	0.0623	0.112
f_s [kHz]	200	200	20	20
M'	51	61	321	179
$\varepsilon^{(1)}$ [%]	13	19	490	170
$\varepsilon^{(2)}$ [%]	10	15	440	210
$\varepsilon^{(3)}$ [%]	24	34	820	200
$\varepsilon^{(4)}$ [%]	50	76	1300	570
SNR [dB]	[69.62, 74.12]	[67.09, 73.47]	[69.06, 70.64]	[69.18, 70.77]

Due to large values of the errors $\varepsilon^{(1)-(4)}$, a procedure of reducing the number of samples M' was run in order to obtain more accurate results of the frequency f estimation. This procedure consisted in selecting in each file a finite number of sample subsets out of the sets of basic samples. The selection of samples was imposed by the result of dividing the frequency f_s by $D \in \mathbf{N} \backslash \{0\}$. The division of f_s was stopped as soon as $M' < 4$. For the obtained subsets the errors $\varepsilon^{(1)-(4)}$ were calculated. Tables 3–6 present selected results of the errors $\varepsilon^{(1)-(4)}$. Plots of the errors $\varepsilon^{(1)-(4)}$ as functions of $f_s/(D \cdot f)$ (Fig. 7) are also presented.

Table 3. Results of the errors $\varepsilon^{(1)-(4)}$ for File 1.

D	**4**	**6**	**8**	**10**	**12**	**16**
f [kHz]	4					
f_s/D [kHz]	50	33.(3)	25	20	16.(6)	12.5
M'	13	9	7	6	5	4
$\varepsilon^{(1)}$ [%]	0.20	0.055	0.026	0.028	0.13	0.011
$\varepsilon^{(2)}$ [%]	0.25	0.057	0.025	0.032	0.027	0.0090
$\varepsilon^{(3)}$ [%]	0.60	0.19	0.085	0.055	0.026	0.037
$\varepsilon^{(4)}$ [%]	1.9	0.77	2.0	0.20	0.047	0.0093

Table 4. Results of the errors $\varepsilon^{(1)-(4)}$ for File 2.

D	**3**	**6**	**9**	**12**	**15**	**18**
f [kHz]	3.3					
f_s/D [kHz]	66.(6)	33.(3)	22.(2)	16.(6)	13.(3)	11.(1)
M'	21	11	7	6	5	4
$\varepsilon^{(1)}$ [%]	0.89	0.14	0.033	0.029	0.31	0.024
$\varepsilon^{(2)}$ [%]	0.81	0.13	0.033	0.024	0.028	0.014
$\varepsilon^{(3)}$ [%]	2.2	0.32	0.14	0.045	0.027	0.035
$\varepsilon^{(4)}$ [%]	4.1	0.95	1.2	0.26	0.045	0.015

Table 5. Results of the errors $\varepsilon^{(1)-(4)}$ for File 3.

D	**15**	**29**	**43**	**57**	**71**	**85**
f [kHz]	0.0623					
f_s/D [kHz]	1.(3)	0.689655	0.465116	0.350877	0.28169	0.235294
M'	22	12	8	6	5	4
$\varepsilon^{(1)}$ [%]	0.81	0.14	0.049	0.021	0.038	0.045
$\varepsilon^{(2)}$ [%]	0.70	0.15	0.049	0.024	0.032	0.017
$\varepsilon^{(3)}$ [%]	2.4	0.34	0.13	0.079	0.041	0.017
$\varepsilon^{(4)}$ [%]	5.6	1.2	0.83	0.92	0.092	0.024

The conclusions following from the experiment results are similar to those following from the simulation results. Except for the cases when $f_s f \leq 5$, the errors $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ are significantly less than the errors $\varepsilon^{(3)}$ and $\varepsilon^{(4)}$.

Table 6. Results of the errors $\varepsilon^{(1)-(4)}$ for File 4.

D	9	17	25	33	41	49
f [kHz]	0.112					
f_s/D [kHz]	2.(2)	1.1764706	0.8	0.(60)	0.4878049	0.4081633
M'	20	11	8	6	5	4
$\varepsilon^{(1)}$ [%]	1.3	0.11	0.034	0.034	0.074	0.050
$\varepsilon^{(2)}$ [%]	0.65	0.11	0.039	0.029	0.036	0.017
$\varepsilon^{(3)}$ [%]	3.4	0.30	0.11	0.050	0.035	0.020
$\varepsilon^{(4)}$ [%]	6.0	1.2	0.63	0.37	0.054	0.024

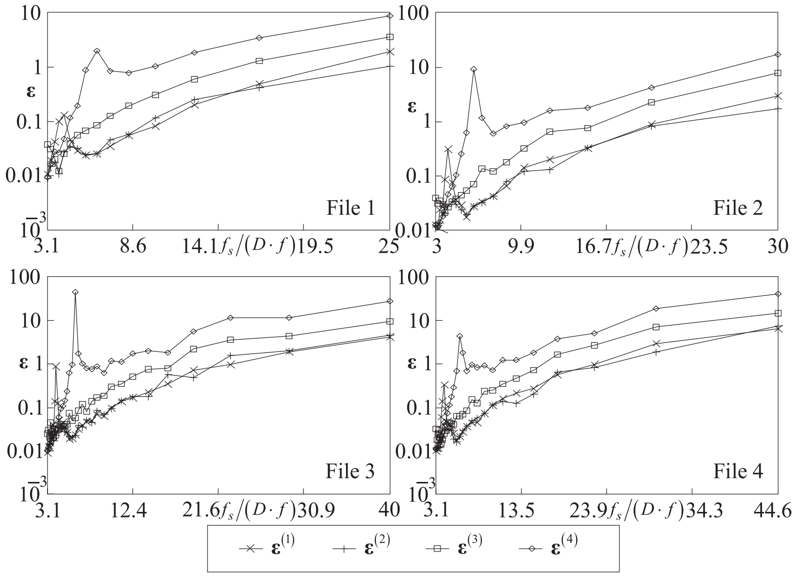

Fig. 7. Errors $\varepsilon^{(1)-(4)}$ as functions of $f_s/(D\cdot f)$.

5. Conclusion

In this paper, two new four-point sinusoidal signal frequency estimators are presented. The invented estimators have been compared with the Vizireanu estimators. The simulation and experimental research results have shown that the estimators enable frequency estimation and frequency tracking with greater accuracy. It ought to be recognized as advantageous that the invented estimators are very effective regarding the calculation speed and do not require the acquisition of signal samples from the whole period. The estimators suffer from a serious shortcoming though. As it is in the case of the Vizireanu estimators, an increase in the number of samples and in the disturbance level results in an increase in the error values of the estimators. Usually, during signal parameter estimation by other methods, an increase in the amount of data leads to an increase in parameter estimation accuracy. It is often accompanied by an increase in the number of calculation operations and in lengthening of the calculation time. An increase in the cost associated with the acquisition and data processing, as well as with archiving the obtained results, can also be expected. Hence, in the measurement systems in which the calculation cost and time consumption are of importance, using the estimators invented by the authors of this paper is highly recommended.

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