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The Poincaré theorem in linear circuit synthesis

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Abstract. The paper deals with linear circuits synthesis with periodic parameters. It was proved that the time-varying voltages and currents of inner branches of such circuits can be calculated using linear recursive equations with periodic coefficients if signals on port are given. The stability theorem of periodic solution was formulated. Hereby described the synthesis problems appear when compensation of power supply systems is considered.

Key words: periodically time-varying networks, synthesis, optimization, stability.

1. Introduction

The problem of electrical circuit synthesis which accomplish the assumed voltage-current distribution appears frequently in the electrotechnics, electronic and first of all in the power-electronic domain. It is the power-electronic that the problem of matching source to load appears in and it is called the compensation task. It consists in such a modification of the receiver circuit as to meet the optimal condition of the source signal. The exact optimal criteria result from the minimization of some evaluation functionals under the condition of the prescribed value of the provided power P. These varied functionals and their solutions are presented in the paper [1].

The circuits connected to the existing power network to assure the optimal signal distribution are called the compensatory circuits. Figure 1 shows: a) the circuit without compensation, b) the circuit with two-terminal shunt compensations, c) the serious compensation, d) two-port compensatory network.

The advantage of the two terminal compensation is its simplicity but such a circuit does not assure appropriate power balance. We will prove it in the shunt compensation example (commonly used). The two terminal compensator is usually lossy because from the power balance in Fig. 1b results:

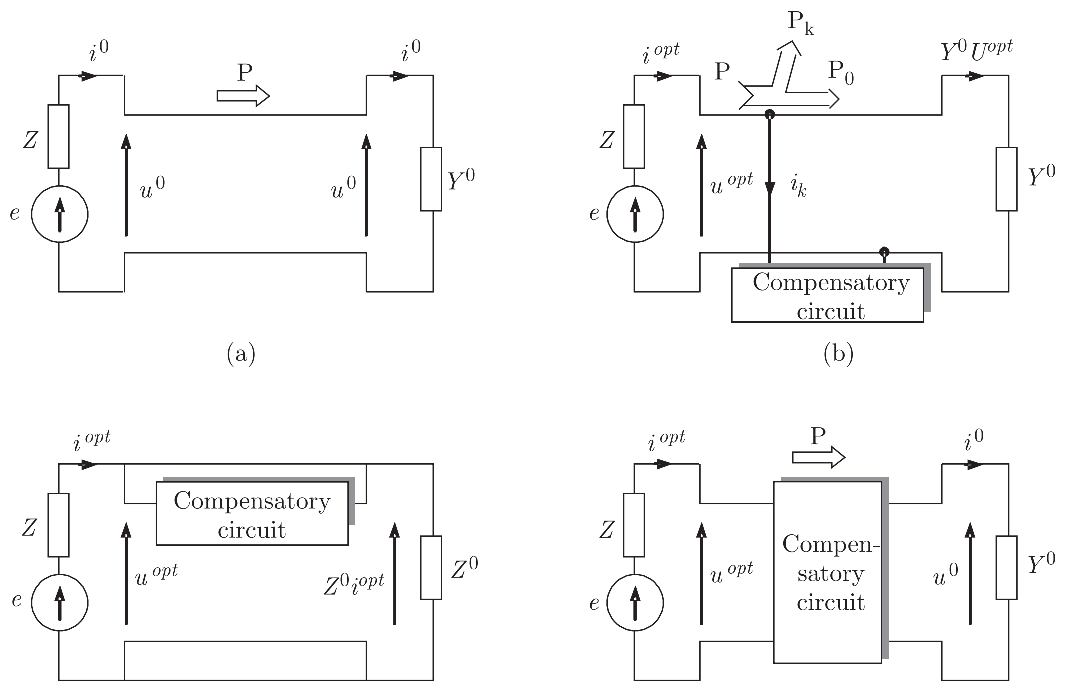

Fig. 1. Simple compensatory circuits: circuit without the compensator (a), shunt compensation (b), serious compensation (c), lossless two port compensation (d)

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$$\begin{aligned}(\boldsymbol{u}^{opt}, \boldsymbol{i}^{K}) &= (\boldsymbol{u}^{opt}, \boldsymbol{i}^{opt}) - (\boldsymbol{Y}^{0}\boldsymbol{u}^{opt}, \boldsymbol{u}^{opt}) \\ &= P - (\boldsymbol{Y}^{0}\boldsymbol{u}^{opt}, u^{opt})\end{aligned} \tag{1}$$

but the power condition says

$$P - (\boldsymbol{Y}^{0}\boldsymbol{u}^{0}, \boldsymbol{u}^{0}) = 0 \tag{2}$$

then the active power of the compensator

$$(\boldsymbol{u}^{opt}, \boldsymbol{i}^{K}) \neq 0$$

where: (,) stands for the dot product of signals, and $\boldsymbol{Y}^0$ is the admittance operator of the load.

From (1) and (2) results that under the prescribed power P condition, the circuit does not assure the assumed power P_0 of the load. On the other hand when we prescribe P_0 we arrive at the casual value of P. However, when assuming the lossless compensatory circuit we get an equation $P = P_0$ but the value of P is incontrollable.

The ideal compensatory circuit is the four-terminal network which can assure to the load the conditions before the compensation, and to the source its own optimal conditions (Fig. 1a and 1d). Such a four terminal compensatory circuit can be lossless (passive) because:

$$(\boldsymbol{u}^{o}, \boldsymbol{i}^{o}) - (\boldsymbol{u}_{opt}, \boldsymbol{i}_{opt}) = 0 \tag{3}$$

Nevertheless, the two-terminal compensatory networks are worth paying attention to. Because of their its simplicity they are convenient to construct complex networks.

2. Time varying compensatory branch

While analyzing in detail the shrunk compensatory networks [2] the most advantageous seems to be the use of a two-terminal network consisting of the linear operator $\boldsymbol{Z}^k$ and the voltage source $\boldsymbol{e}^k$ (both controlled by the optimal current signal).

$$\boldsymbol{e}^{k} = \boldsymbol{e}^{k}(\boldsymbol{i}^{opt}) \tag{4}$$

$$\boldsymbol{Z}^{k} = \boldsymbol{Z}^{k}(\boldsymbol{i}^{opt}). \tag{5}$$

One can formulate two equivalent conceptions of the compensatory branch synthesis:

a) without the controlled source but with the time-varying linear operator (5),
b) with the controlled source (4) and the stationary linear two-terminal network $\boldsymbol{Z}^k$, often very simple

The equivalent circuits representing the foregoing conceptions are shown below.

According to the conception (a) we needed to construct the simple linear time-varying GC branch as shown in Fig. 3.

The shrunk compensatory branch has to produce the optimal voltage signal on its terminals:

$$u(t) = u^{opt}(t) = e(t) - Zi^{opt}(t)$$

and the optimal compensatory current (see Figs. 1b,2,3)

$$i(t) = i^{opt}(t) - Y^{0}u^{opt}(t)$$

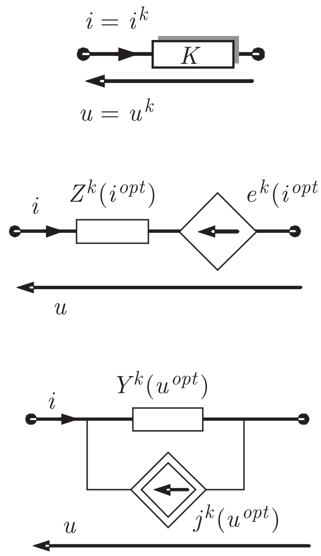


Fig. 2. Compensatory branches representing equivalent conceptions

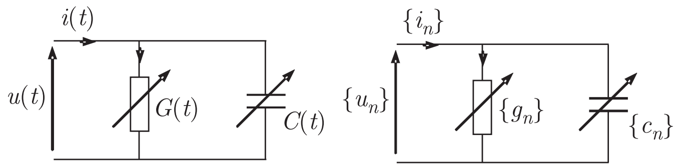


Fig. 3. Time-varying GC compensatory branch and its discrete counterpart

The differential equation describing the time-varying GC branch shown in Fig. 3 has a form of (using the time derivative of charge):

$$G_R(t)u(t) + \frac{d}{dt}[C(t)u(t)] = i(t)$$

but it can be rewritten in the following way

$$G(t)u(t) + C(t)\frac{du}{dt} = i(t) \tag{6}$$

After the discretisation of the Eq. (6) with the sampling time τ we achieve the difference equation

$$u_n g_n + \Delta u_n c_n = i_n \tag{7}$$

where the instant values are: $u_n = u(n\tau)$, $\Delta u_n = u_n - u_{n-1}$, $i_n = i(n\tau)$, $g_n = G(n\tau)$, $c_n = \frac{1}{\tau}C(n\tau)$.

The determination of the sought discrete $\{g_n\}$ and $\{c_n\}$ sequences from the Eq. (7) is equivocal. As to achieve an unique solution we introduce an additional condition:

$$()^2 + (\Delta c_n)^2 \to \min \tag{8}$$

(for any $n \in$ Integer) where:

$$\Delta g_n = g_n - g_{n-1}, \quad \Delta c_n = c_n - c_{n-1}.$$

The formula (8) minimize the parameter changes along the time, making the practical realization of $\{g + n\}$ and $\{c_n\}$ easier.

The conditions (7) and (8) make together the optimization task under 'smooth conductance and capacity changes' condition:

$$\begin{aligned} (\Delta g_n)^2 + (\Delta c_n)^2 &\to \min \\ u_n g_n + \Delta u_n c_n &= i_n \end{aligned} \tag{9}$$

The solution of (9) is a pair of values $[g_n, c_n]$ calculated from the previous ones $[g_{n-1}, c_{n-1}]$. Thus, as a matter of fact, we seek the recurrent transformation $[g_{n-1}, c_{n-1}] \to [g_n, c_n]$ where the instant values u_n, Δu_n and i_n are known.

Applying the Lagrange multiplier method we arrive at the unbounded minimum task:

$$f_\lambda(g_n, c_n) = (\Delta g_n)^2 + (\Delta c_n)^2 + \lambda(u_n g_n + \Delta u_n c_n) \to \min \tag{10}$$

Calculating the difference of (10) we get the necessary minimum condition:

$$\begin{aligned} \delta f_\lambda(g_n, c_n) &= f_\lambda(g_n + \delta g_n, c_n + \delta c_n) - f_\lambda(g_n, c_n) \\ &= (2\Delta g_n + \lambda u_n)\delta g_n + (2\Delta c_n + \lambda \Delta u_n)\delta c_n \\ &+ (\delta g_n)^2 + (\delta c_n)^2 > 0 \end{aligned} \tag{11}$$

for any δg_n, δc_n and $n \in$ Integer.

Inequality (11) is met when:

$$\begin{aligned} 2\Delta g_n + \lambda u_n &= 0 \\ 2\Delta c_n + \lambda \Delta u_n &= 0. \end{aligned} \tag{12}$$

Connecting both (12) and (7) to make the set of linear equations depending on g_n, c_n, λ we get:

$$\begin{aligned} u_n g_n + \Delta u_n c_n &= i_n \\ 2g_n + u_n \lambda &= 2g_{n-1} \\ 2c_n + \Delta u_n \lambda &= 2c_{n-1}. \end{aligned} \tag{13}$$

Solving the above we get the transformation:

$$[g_{n-1}, c_{n-1}] \to [g_n, c_n]$$

which can be written in a matrix form

$$\begin{aligned} \begin{bmatrix} g_n \\ c_n \end{bmatrix} = \frac{1}{(u_n)^2 + (\Delta u_n)^2} \Bigg(& \begin{bmatrix} (\Delta u_n)^2 & -u_n \Delta u_n \\ -u_n \Delta u_n & (u_n)^2 \end{bmatrix} \\ & \times \begin{bmatrix} g_{n-1} \\ c_{n-1} \end{bmatrix} + \begin{bmatrix} u_n i_n \\ \Delta u_n i_n \end{bmatrix} \Bigg) \end{aligned} \tag{14}$$

or in a shortened form

$$\boldsymbol{x}_n = \boldsymbol{A}_n \boldsymbol{x}_{n-1} + \boldsymbol{\alpha}_n \tag{15}$$

where

$$\begin{aligned} \boldsymbol{A}_n &= \frac{1}{(u_n)^2 + (\Delta u_n)^2} \begin{bmatrix} (\Delta u_n)^2 & -u_n \Delta u_n \\ -u_n \Delta u_n & (u_n)^2 \end{bmatrix} \\ \boldsymbol{\alpha}_n &= \frac{1}{(u_n)^2 + (\Delta u_n)^2} \begin{bmatrix} u_n i_n \\ \Delta u_n i_n \end{bmatrix} \\ \boldsymbol{x}_n &= \begin{bmatrix} g_n \\ c_n \end{bmatrix}. \end{aligned}$$

The recursive formula in the classical form allows us to evolve the g, c parameters to reach the solution when the strong inequality

$$(u_n)^2 + (\Delta u_n)^2 > 0 \tag{16}$$

is met. Thus, the prescribed voltage signal and its derivative must not simultaneously be zero.

3. Multi-dimensional problem

Some generalization of the minimization procedure described in 2 is the time-varying coefficients synthesis

$$x^0(t), x^1(t), \ldots, x^{M-1}(t)$$

of the linear differential equation

$$x^0(t)u(t) + x^1(t)u^{(1)}(t) + \ldots + x^{M-1}(t)u^{(M-1)}(t) = s(t) \tag{17}$$

where: $u(t)$, $u^{(1)}(t)$, ... , $u^{(M-1)}(t)$ – a prescribed signal with its derivatives, $s(t)$ – a single prescribed signal. Such a problem appears in a complex multi-element linear circuit.

It is more convenient to change the differential into the difference equation (see previous chapter)

$$b_n^0 x_n^0 + b_n^1 x_n^1 + \ldots + b_n^{M-1} x_n^{M-1} = s_n \tag{18}$$

where: $b_n^k = (\Delta^k u)_n = (\Delta^{k-1} u)_n - (\Delta^{k-1} u)_{n-1}$ – stands for k-th derivative of the prescribed signal $\{u_n\}$:

$$u_n = \frac{1}{\tau} u(n\tau)$$

Also, this time the unique determination of the time-varying vector

$$\boldsymbol{x}_n = \left[x_n^0, x_n^1, \ldots, x_n^{M-1}\right]^T$$

using only (18) is impossible. We require an additional minimum condition which together with (18) makes a constrained minimum condition.

$$\begin{aligned} (\boldsymbol{b}_n, \boldsymbol{x}_n) &= s_n \\ (\Delta \boldsymbol{x}_n, \Delta \boldsymbol{x}_n) &\to \min \end{aligned} \tag{19}$$

where $\boldsymbol{x}_n = \left[x_n^0, x_n^1, \ldots, x_n^{M-1}\right]^T$ – the vector of the sought coefficient, $\Delta \boldsymbol{x}_n = \boldsymbol{x}_n - \boldsymbol{x}_{n-1}$, T – stands for transposition.

Task (19) can be formulated in a convenient form using a matrix notation:

$$\begin{aligned} \boldsymbol{b}_n^T \boldsymbol{x}_n &= s_n \\ (\Delta \boldsymbol{x}_n)^T \Delta \boldsymbol{x}_n &\to \min \end{aligned} \tag{20}$$

Further, we can extend the foregoing problem adding more constrains conditions, by adding more linear difference equations:

$$\begin{aligned} b_n^{0,0} x_n^0 + b_n^{1,0} x_n^1 + \ldots + b_n^{M-1,0} x_n^{M-1} &= s_n^0 \\ b_n^{0,1} x_n^0 + b_n^{1,1} x_n^1 + \ldots + b_n^{M-1,1} x_n^{M-1} &= s_n^0 \\ \ldots\ldots\ldots \\ b_n^{0,P-1} x_n^0 + b_n^{1,p-1} x_n^1 + \ldots + b_n^{M-1,p-1} x_n^{M-1} &= s_n^{p-1} \end{aligned} \tag{21}$$

under the condition that $P < M$. Then we formulate the following minimum tasks

$$\begin{aligned} \boldsymbol{B}_n^T \boldsymbol{x}_n &= s_n \\ (\Delta \boldsymbol{x}_n)^T \Delta \boldsymbol{x}_n &\to \min \end{aligned} \tag{22}$$

where:
$[\boldsymbol{B}_n^T]^{ij} = b_n^{i,j}$ – the prescribed time-varying matrix;
$\boldsymbol{s}_n = \left[s_n^0, s_n^1, \ldots, s_n^{P-1}\right]^T$ – the prescribed signals vector.

The solution of (22) consists in finding the values $\boldsymbol{x}_n$ by the use of the previous one $\boldsymbol{x}_{n-1}$. A suitable functional for the task (22) has a form:

$$f_\lambda(\boldsymbol{x}_n) = (\Delta \boldsymbol{x}_n)^T \Delta \boldsymbol{x}_n + \boldsymbol{x}_n^T \boldsymbol{B}_n \boldsymbol{\lambda} \tag{23}$$

where $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \ldots, \lambda_p]^T$ – the vector of the Lagrange factors.

Taking into consideration that

$$\boldsymbol{\delta} \Delta \boldsymbol{x}_n = \boldsymbol{\delta x}_n$$

We get the condition for the variation of (23)

$$\begin{aligned} \delta f_\lambda(\boldsymbol{x}_n) &= f_\lambda(\boldsymbol{x}_n + \delta \boldsymbol{x}_n) - f_\lambda(\boldsymbol{x}_n) \\ &= (\delta \boldsymbol{x}_n)^T \Delta \boldsymbol{x}_n + (\Delta \boldsymbol{x}_n)^T \delta \boldsymbol{x}_n \\ &+ (\delta \boldsymbol{x}_n)^T \boldsymbol{B}_n \boldsymbol{\lambda} + (\delta \boldsymbol{x}_n)^T \delta \boldsymbol{x}_n = (\delta \boldsymbol{x}_n)^T \\ &\times (2\boldsymbol{x}_n - 2\boldsymbol{x}_{n-1} + \boldsymbol{B}_n \boldsymbol{\lambda}) + (\delta \boldsymbol{x}_n)^T \delta \boldsymbol{x}_n. \end{aligned} \tag{24}$$

The variation (24) is positive for any $\delta \boldsymbol{x}_n$ if only

$$2\boldsymbol{x}_n - 2\boldsymbol{x}_{n-1} + \boldsymbol{B}_n \boldsymbol{\lambda} = 0$$

thus

$$\boldsymbol{x}_n = \boldsymbol{x}_{n-1} - 0.5 \boldsymbol{B}_n \boldsymbol{\lambda}. \tag{25}$$

Substituting (25) to the first equation of (22) we get

$$\boldsymbol{B}_n^T \boldsymbol{x}_{n-1} - 0.5 \boldsymbol{B}_n^T \boldsymbol{B}_n \boldsymbol{\lambda} = \boldsymbol{s}_n$$

Therefore, we get the set of linear equations for λ coefficients:

$$\boldsymbol{B}_n^T \boldsymbol{B}_n \boldsymbol{\lambda} = 2\boldsymbol{B}_n^T \boldsymbol{x}_{n-1} - 2\boldsymbol{s}_n \tag{26}$$

If $\boldsymbol{B}_n^T \boldsymbol{B}_n$ is not singular for any n then there exists $\boldsymbol{\lambda}$

$$\boldsymbol{\lambda} = 2(\boldsymbol{B}_n^T \boldsymbol{B}_n)^{-1} \boldsymbol{B}_n^T \boldsymbol{x}_{n-1} - 2(\boldsymbol{B}_n^T \boldsymbol{B}_n)^{-1} \boldsymbol{s}_n \tag{27}$$

Substituting it to (25) we get a solution of a minimum task (22)

$$\boldsymbol{x}_n = \left[1 - \boldsymbol{B}_n (\boldsymbol{B}_n^T \boldsymbol{B}_n)^{-1} \boldsymbol{B}_n^T\right] \boldsymbol{x}_{n-1} + \boldsymbol{B}_n (\boldsymbol{B}_n^T \boldsymbol{B}_n)^{-1} \boldsymbol{s}_n \tag{28}$$

which can be rewritten in a classical form (15) where:

$$\boldsymbol{A}_n = 1 - \boldsymbol{B}_n (\boldsymbol{B}_n^T \boldsymbol{B}_n)^{-1} \boldsymbol{B}_n^T \tag{29}$$

$$\boldsymbol{\alpha}_n = \boldsymbol{B}_n (\boldsymbol{B}_n^T \boldsymbol{B}_n)^{-1} \boldsymbol{s}_n. \tag{30}$$

In a particular case of the minimum task (19) or (22) with the single constraint the recursive formula takes the form of:

$$\boldsymbol{x}_n = \left[1 - \frac{\boldsymbol{b}_n \boldsymbol{b}_n^T}{\boldsymbol{b}_n^T \boldsymbol{b}_n}\right] \boldsymbol{x}_{n-1} + \frac{\boldsymbol{b}_n}{\boldsymbol{b}_n^T \boldsymbol{b}_n} s_n. \tag{31}$$

This is a general form of (15) where:

$$\boldsymbol{b}_n = \begin{bmatrix} u_n \\ \Delta u_n \end{bmatrix}; \quad \boldsymbol{x}_n = \begin{bmatrix} g_n \\ c_n \end{bmatrix}.$$

The above solution exists if for any n the norm of bn meets the strong inequality condition

$$||\boldsymbol{b}_n|| = \sqrt{\boldsymbol{b}_n^T \boldsymbol{b}_n} \neq 0 \tag{32}$$

It is clear that (16) is a particular case of (32).

4. Time-varying circuits – a stability problem

In a real situation the input signals often change periodically. Thus the matrix $\boldsymbol{A}_n$ and $\boldsymbol{\alpha}_n$ (Eq. 15) are periodic too, with the period N:

$$\boldsymbol{A}_{n+N} = \boldsymbol{A}_n$$

$$\boldsymbol{\alpha}_{n+1} = \boldsymbol{\alpha}_n$$

The solution of the recursive Eq. (15) with the initial condition $\boldsymbol{x}_0$ has the form of:

$$\boldsymbol{x}_n = \boldsymbol{Y}_n \boldsymbol{x}_0 + \boldsymbol{y}_n \tag{33}$$

where $\boldsymbol{Y}_n, \boldsymbol{y}_n$, are called a resolved matrix and a vector.

Substituting the general solution (33) in (15) we get:

$$\boldsymbol{x}_n = \boldsymbol{A}_n (\boldsymbol{Y}_{n-1} \boldsymbol{x}_0 + \boldsymbol{y}_{n-1}) + \boldsymbol{\alpha}_n \tag{34}$$

where the recursive formulas for a resolved matrix and a vector have a from:

$$\boldsymbol{Y}_n = \boldsymbol{A}_n \boldsymbol{Y}_{n-1} \quad \boldsymbol{Y}_0 = 1 \tag{35}$$

$$\boldsymbol{y}_n = \boldsymbol{A}_n \boldsymbol{y}_{n-1} + \boldsymbol{\alpha}_n \quad \boldsymbol{y}_0 = 0 \tag{36}$$

for $n = 1, 2, \ldots N$.

For the sake of periodicity of $\boldsymbol{A}_n$ and $\boldsymbol{\alpha}_n$ the recursive formulas (35) and (36) are executed only N times. Then we get:

$$\begin{aligned} (1, 0) \to (\boldsymbol{Y}_1, \boldsymbol{y}_1) \to (\boldsymbol{Y}_2, \boldsymbol{y}_2) \ldots \to (\boldsymbol{Y}_{N-1}, \boldsymbol{y}_{N-1}) \\ \to (\boldsymbol{Y}_N, \boldsymbol{y}_N) = (\boldsymbol{Y}, \boldsymbol{y}) \end{aligned}$$

We call the function:

$$\boldsymbol{x} \to \boldsymbol{Y}_x + \boldsymbol{y} \tag{37}$$

the Poincaré mapping [2,3].

Therefore the Poincaré mapping for x is the ending point of the trajectory (15) starting from $\boldsymbol{x} = \boldsymbol{x}_0$ and continuing over one period from 1 to N (Fig. 4).

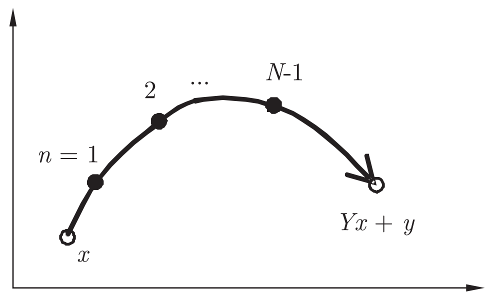


Fig. 4. Poincaré mapping trajectory

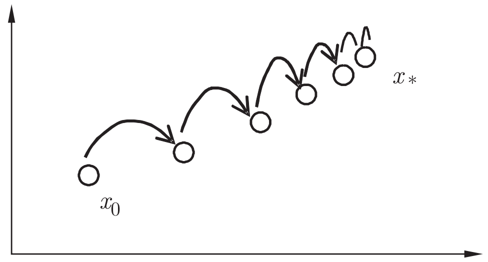


Fig. 5. Cyclic Poincaré mapping

The solution of (15) becomes N-periodic for n approaching infinity if the cyclic Poincaré mapping

$$\boldsymbol{x}_n = \boldsymbol{Y}\boldsymbol{x}_{n-1} + \boldsymbol{y} \tag{38}$$

drives to the steady point (Fig. 5.)

$$\boldsymbol{x}_* = \boldsymbol{Y}\boldsymbol{x}_* + \boldsymbol{y} \tag{39}$$

Thus the steady point can be calculated from

$$(1 - \boldsymbol{Y})\boldsymbol{x}_* = \boldsymbol{y} \tag{40}$$

and if $(1 - \boldsymbol{Y})$ is not singular

$$\boldsymbol{x}_* = (1 - \boldsymbol{Y})^{-1}\boldsymbol{y}. \tag{41}$$

On the other side the cyclic Poincaré mapping gives the series

$$\boldsymbol{x}_n - \boldsymbol{Y}^n\boldsymbol{x}_0 = (1 + \boldsymbol{Y} + \boldsymbol{Y}^1 + \boldsymbol{Y}^2 + \ldots + \boldsymbol{Y}^{n-1})\boldsymbol{y}. \tag{42}$$

From (42) results that the convergence

$$\boldsymbol{x}_n \to \boldsymbol{x}_* \tag{43}$$

takes place if for any $\boldsymbol{x}_0$ and $n \to \infty$

$$\boldsymbol{Y}^n\boldsymbol{x}_0 \to 0 \tag{44}$$

or if, and only if,

$$(1 + \boldsymbol{Y} + \boldsymbol{Y}^1 + \boldsymbol{Y}^2 + \ldots + \boldsymbol{Y}^{n-1}) \to (1 - \boldsymbol{Y})^{-1}$$
$$\text{for} \quad n \to \infty$$

what results from the following formula

$$\begin{aligned}
&(1 + \boldsymbol{Y} + \boldsymbol{Y}^1 + \boldsymbol{Y}^2 + \ldots + \boldsymbol{Y}^{n-1})(1 - \boldsymbol{Y}) \\
&= (1 + \boldsymbol{Y} + \boldsymbol{Y}^1 + \boldsymbol{Y}^2 + \ldots + \boldsymbol{Y}^{n-1}) \\
&- (\boldsymbol{Y} + \boldsymbol{Y}^1 + \boldsymbol{Y}^2 + \ldots + \boldsymbol{Y}^{n-1} + \boldsymbol{Y}^n) \\
&= (1 - \boldsymbol{Y}^n) \to 1
\end{aligned} \tag{45}$$

The $\boldsymbol{Y}^n$ matrix is the solution of the recursive equation

$$\boldsymbol{x}_n = \boldsymbol{Y}\boldsymbol{x}_{n-1} \tag{46}$$

which means that

$$\boldsymbol{x}_n = \boldsymbol{Y}^n\boldsymbol{x}_0. \tag{47}$$

The solution of (46) has a form

$$\boldsymbol{x}_n = z^n\boldsymbol{a}$$

and is convergent for $|z| < 1$ i.e. $\boldsymbol{x}_n \to 0$ and $\boldsymbol{Y}^n \to 0$ for any $\boldsymbol{a}$ and $\boldsymbol{x}_0$.

Substituting above to (46) we get

$$z^n\boldsymbol{a} = \boldsymbol{Y}z^{n-1}\boldsymbol{a}$$

or

$$(1z - \boldsymbol{Y})\boldsymbol{a} = 0$$

Therefore (46) converges only if all eigenvalues of $\boldsymbol{Y}$ (from here $\boldsymbol{Y}$ is called the Poincaré matrix) meets the condition of $|z| < 1$. Therefore the following theorem is true.

The linear set of Eqs. (15) with an N-periodical $\boldsymbol{A}_n$ matrix and $\boldsymbol{\alpha}_n$ vector has a stable solution if all eigenvalues of the Poincaré matrix $z : |1z - \boldsymbol{Y}| = 0$ include the unit circle i.e.

$$\bigwedge_{z:|z|>0} |1z - \boldsymbol{Y}| \neq 0 \tag{48}$$

Example 1. For various couples of the periodic signals $[\{u_n\}, \{i_n\}]$ the convergence process of $[\{g_n\}, \{c_n\}]$ was studied according to the formula (14). In the picture 6 the individual Poincaré points are marked. There are also written the eigenvalues of the Poincaré matrix.

Example 2. In this example the synthesis problem of the time-varying parameters will be formulated. The parameters are: conductance and capacity of the structure shown in Fig. 7.

In the circuit shown in Fig. 7 the pairs

$$\{u_n^1, i_n^1\}, \{u_n^2, i_n^2\}, \{u_n^3, i_n^3\},$$

stand for samples of the prescribed signals on ports and

$$\boldsymbol{x}_n = \left[x_n^1, x_n^2, x_n^3, x_n^4, x_n^5, x_n^6, x_n^7\right]^T$$

the vector of the conductance-capacity elements of inner branches of the three port network is the sought.

According to Fig. 7 we can formulate the following current equations:

$$\begin{aligned}
u_n^1 x_n^1 + (\mathit{\Delta} u_n^1 - \mathit{\Delta} u_n^2)x_n^6 + (u_n^1 - u_n^3)x_n^7 &= i_n^1 \\
u_n^2 x_n^2 + (\mathit{\Delta} u_n^2 - \mathit{\Delta} u_n^1)x_n^6 + (u_n^2 - u_n^3)x_n^5 &= i_n^2 \\
u_n^3 x_n^3 + \mathit{\Delta} u_n^3 x_n^4 + (u_n^3 - u_n^2)u_n^5 + (u_n^3 - u_n^1)x_n^7 &= i_n^3
\end{aligned}$$

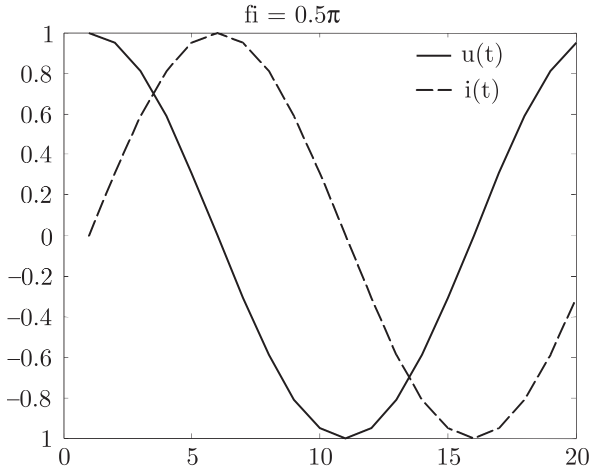
they can be rewritten in a standard form which has appeared before in (22)

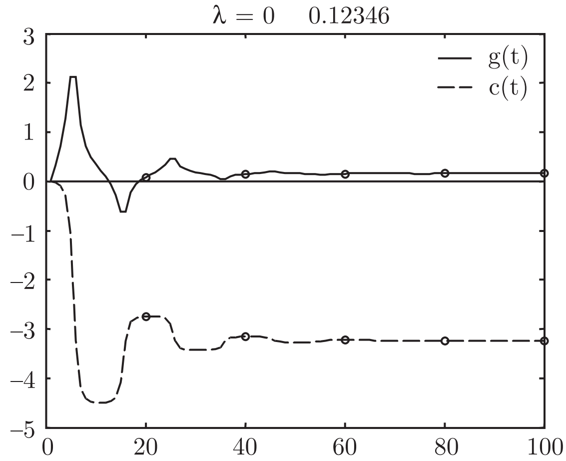
$$\boldsymbol{B}_n^T\boldsymbol{x}_n = \boldsymbol{s}_n$$

i.e.:

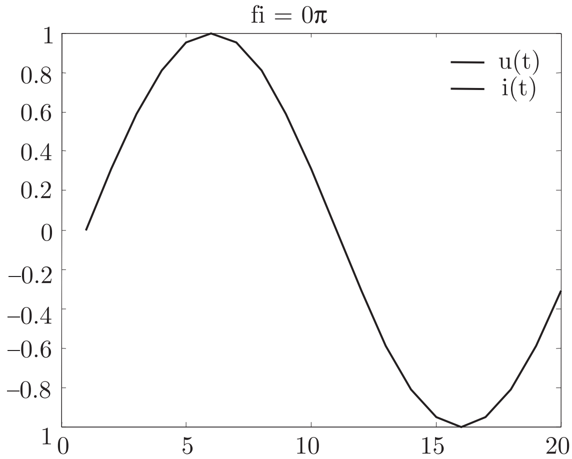
$$\begin{array}{|c|c|c|c|c|c|c|}
\hline
u_n^1 & & & & & \mathit{\Delta}(u_n^1 - u_n^2) & u_n^1 - u_n^3 \\
\hline
 & u_n^2 & & & u_n^2 - u_n^3 & \mathit{\Delta}(u_n^2 - u_n^1) & \\
\hline
 & & u_n^3 & \mathit{\Delta} u_n^1 & u_n^3 - u_n^2 & & u_n^3 - u_n^1 \\
\hline
\end{array} \times$$

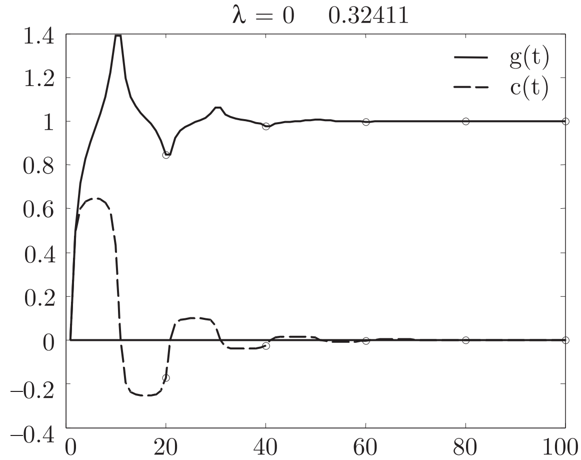
$$\times \begin{array}{|c|}
\hline
x_n^1 \\ \hline
x_n^2 \\ \hline
x_n^3 \\ \hline
x_n^4 \\ \hline
x_n^5 \\ \hline
x_n^6 \\ \hline
x_n^7 \\ \hline
\end{array} = \begin{array}{|c|}
\hline
i_n^1 \\ \hline
i_n^2 \\ \hline
i_n^3 \\ \hline
\end{array}$$


fi = 0.5π
u(t)
i(t)
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
1
0
5
10
15
20

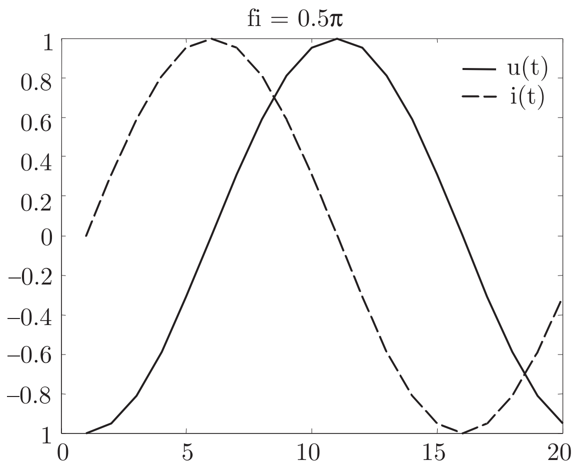

λ = 0 0.12346
g(t)
c(t)
3
2
1
0
−1
−2
−3
−4
−5
0
20
40
60
80
100

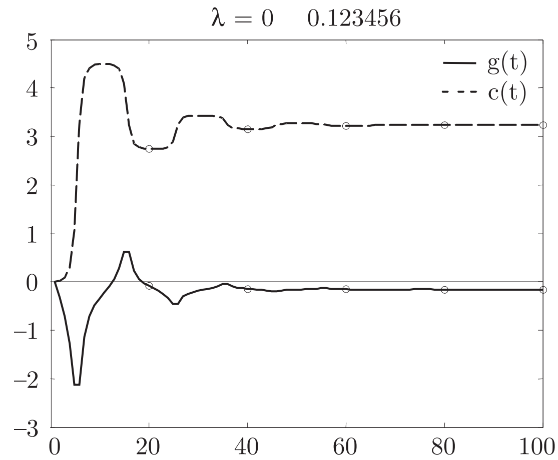
(a)


fi = 0π
u(t)
i(t)
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
1
0
5
10
15
20

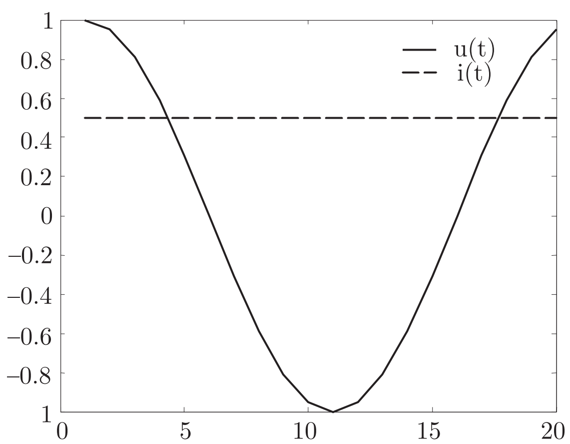

λ = 0 0.32411
g(t)
c(t)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
0
20
40
60
80
100

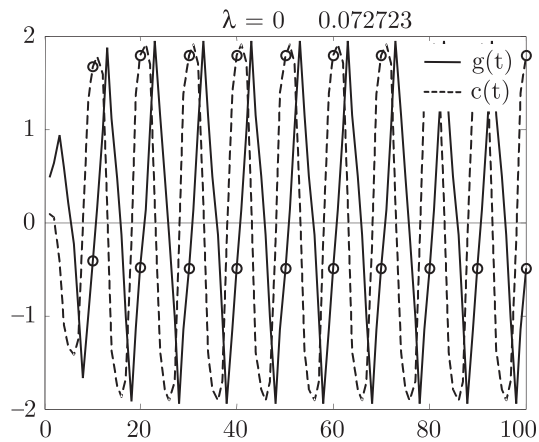
(b)


fi = 0.5π
u(t)
i(t)
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
1
0
5
10
15
20


λ = 0 0.123456
g(t)
c(t)
5
4
3
2
1
0
−1
−2
−3
0
20
40
60
80
100

(c)


u(t)
i(t)
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
1
0
5
10
15
20


λ = 0 0.072723
g(t)
c(t)
2
1
0
−1
−2
0
20
40
60
80
100

(d)

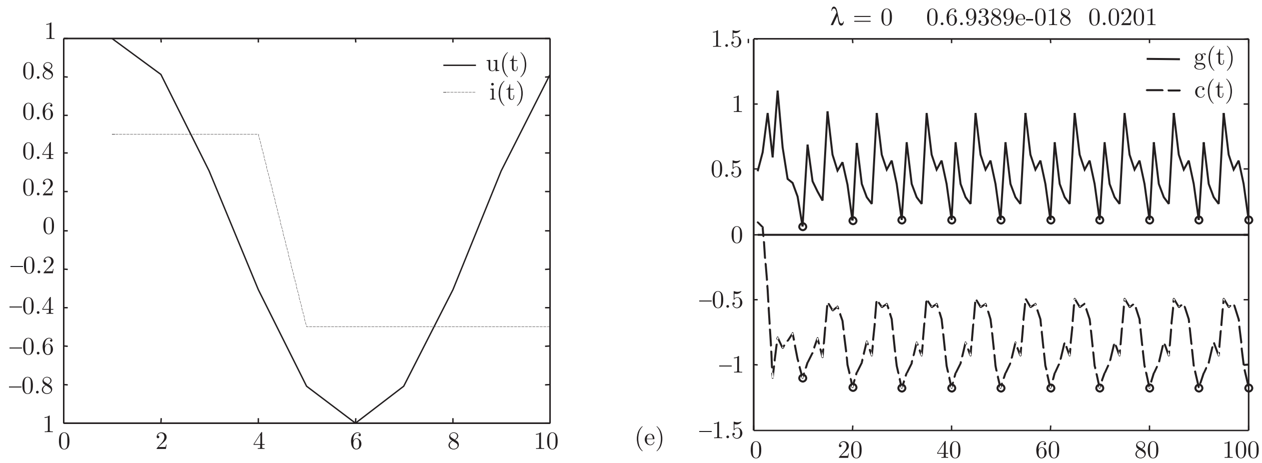


Fig. 6. Time diagram of $\{g_n\}$, $\{c_n\}$ of the time-varying parallel GC branch, o – marks N-period. After short time parameters $\{g_n\}$, $\{c_n\}$ become periodic

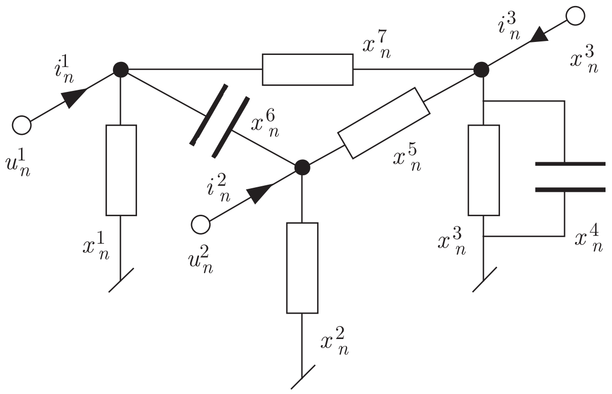


Fig. 7. Three-port network with conductance-capacity elements

In the example described above the problem dimensions are: $P = 3$ (number of conditions); $M = 7$ (number of the sought elements).

The matrix $\boldsymbol{B}_n^T\boldsymbol{B}_n$ which decides about the existence of the Eq. (28) has the following form

$a+b+c$	b	c
b	$b+d+e$	e
c	e	$c+e+f+g$

where:

$$a = (u_n^1)^2; b = \left[\Delta(u_n^1 - u_n^2)\right]2; c = (u_n^1 - u_n^3)^2;$$
$$d = (u_n^2)^2; e = (u_n^2 - u_n^3)^2; f = (u_n^3)^2; g = (\Delta u_n^3)^2;$$

The determinant of the above matrix is equal

$$(a+b+c)[d(e+f+g)+e(f+g)] + (a+b)c(d+e) \\ + ab(c+e+f+g) + bc(3e+f+g)$$

and all its elements are positive, so the determinant will usually be nonzero. Therefore, the set of recursive Eq. (28) exists.

5. Additional conditions

The next generalization of (22) is the minimum task with the additional elements of power condition.

$$\begin{aligned} \Delta\boldsymbol{x}_n^T\Delta\boldsymbol{x}_n &\to \min \\ \boldsymbol{B}_n^T\boldsymbol{x}_n &= \boldsymbol{s}_n \\ \boldsymbol{x}_n^T\boldsymbol{Q}_n\boldsymbol{x}_n &= q_n \end{aligned} \tag{49}$$

$\boldsymbol{Q}_n$ – a positive symmetric matrix for any n
q_n – a prescribed signal.

Introducing the vector of Lagrange multiplier $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \ldots, \lambda_p,]^T$ an appropriate functional can be written as

$$f_{\lambda_0,\lambda}(\boldsymbol{x}_n) = (\Delta\boldsymbol{x}_n)^T\Delta\boldsymbol{x}_n + \lambda_0\boldsymbol{x}_n^T\boldsymbol{Q}_n\boldsymbol{x} - 2\boldsymbol{\lambda}^T\boldsymbol{B}_n\boldsymbol{x}_n \tag{50}$$

and its variation is equal

$$\begin{aligned} \delta f_{\lambda_0,\lambda}(\boldsymbol{x}_n) &= f_{\lambda_0,\lambda}(\boldsymbol{x}_n + \delta\boldsymbol{x}_n) - f_{\lambda_0,\lambda}(\boldsymbol{x}_n) \\ &= 2((\Delta\boldsymbol{x}_n)^T + \lambda_0\boldsymbol{x}_n^T\boldsymbol{Q}_n - \boldsymbol{\lambda}^T\boldsymbol{B}_n^T)\delta\boldsymbol{x}_n \\ &+ (\delta\boldsymbol{x}_n)^T(1 + \lambda_0\boldsymbol{Q}_n)\delta\boldsymbol{x}_n \end{aligned}$$

which gives the optimizing equation:

$$(1 + \lambda_0\boldsymbol{Q}_n)\boldsymbol{x}_n = x_{n-1} + \boldsymbol{B}_n\boldsymbol{\lambda}. \tag{51}$$

However, for the indeterminate Lagrange coefficients we must formulate an additional set of the differential equations:

$$\begin{aligned} \frac{d\lambda_0}{dt} &= \boldsymbol{x}_n^T\boldsymbol{Q}_n\boldsymbol{x}_n - q_n \\ \frac{d\boldsymbol{\lambda}}{dt} &= \boldsymbol{s}_n - \boldsymbol{B}_n^T\boldsymbol{x}_n \end{aligned} \tag{52}$$

The stable singular point of (52) i.e.

$$\frac{d\lambda_0}{dt} \to 0, \qquad \frac{d\boldsymbol{\lambda}}{dt} \to 0$$

and Eq. (51) make the solution of (49). The existence of such a singular point has been proved in [2]. Thus, the recursive Eq. (51) with (52) determine the trajectory of the element's vector $\{\boldsymbol{x}_n\}$.

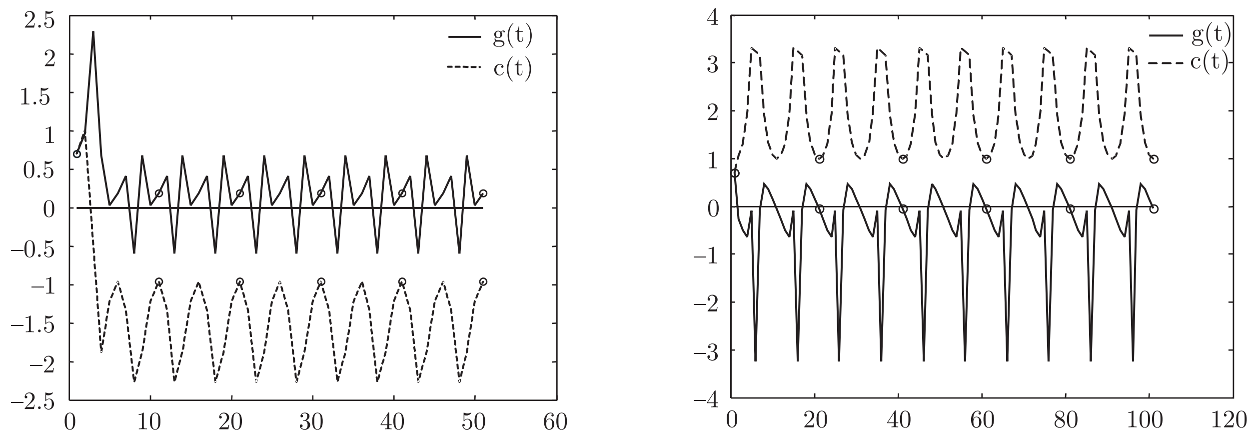


Fig. 8. The time-varying parameters g, c for the signals u and i from Figs. 6a and 6c calculated according to formula (61), o – marks N-period

Example 3. In the elementary circuit shown in fig 3. we put the extended minimum task

$$\begin{gathered}\varDelta \boldsymbol{x}_n^T \varDelta \boldsymbol{x}_n \rightarrow \min \\ \boldsymbol{b}_n^T \boldsymbol{x}_n = \boldsymbol{i}_n \\ \boldsymbol{x}_n^T \boldsymbol{x}_n = q_n\end{gathered} \tag{53}$$

$$\boldsymbol{b}_n = \begin{bmatrix} u_n \\ \varDelta u_n \end{bmatrix}; \qquad \boldsymbol{x}_n = \begin{bmatrix} g_n \\ c_n \end{bmatrix}$$

in which the condition of limiting the values of the circuit elements was added to the original Eq. (9).

The optimizing Eq. (51) now takes a form:

$$x_n = \frac{1}{1+\lambda_0}(\boldsymbol{x}_{n-1} + \lambda_1 \boldsymbol{b}_n) \tag{54}$$

where the lamdas are sought from the additional differential equations:

$$\begin{gathered}\frac{d\lambda_0}{dt} = \frac{1}{(1+\lambda_0)^2}\left(\boldsymbol{x}_{n-1}^T + \lambda_1 \boldsymbol{b}_n^T\right)(\boldsymbol{x}_{n-1} + \lambda_1 \boldsymbol{b}_n) - q_n \\ \frac{d\lambda_1}{dt} = i_n - \frac{1}{1+\lambda_0}\boldsymbol{b}_n^T(\boldsymbol{x}_{n-1} + \lambda_1 \boldsymbol{b}_n).\end{gathered} \tag{55}$$

Substituting the following $1+\lambda_0 = \mu$, $\lambda_1 = \lambda$ we get the equation of singular point (55) in a shortened form

$$\begin{gathered}\boldsymbol{x}_{n-1}^T \boldsymbol{x}_{n-1} + 2\lambda \boldsymbol{b}_n^T \boldsymbol{x}_{n-1} + \lambda^2 \boldsymbol{b}_n^T \boldsymbol{b}_n = q_n \mu^2 \\ \lambda \boldsymbol{b}_n^T \boldsymbol{b}_n = i_n \mu - \boldsymbol{b}_n^T \boldsymbol{x}_{n-1}\end{gathered} \tag{56}$$

The solution of (56) has a fairly simple form

$$\mu_n = \sqrt{\frac{(\boldsymbol{x}_{n-1}^T \boldsymbol{x}_{n-1})(\boldsymbol{b}_n^T \boldsymbol{b}_n) - (\boldsymbol{b}_n^T \boldsymbol{x}_{n-1})^2}{q_n(\boldsymbol{b}_n^T \boldsymbol{b}_n) - i_n^2}} \tag{57}$$

$$\lambda_n = \frac{i_n \mu_n - \boldsymbol{b}_n^T \boldsymbol{x}_{n-1}}{\boldsymbol{b}_n^T \boldsymbol{b}_n}. \tag{58}$$

The recursive Eq. (54) has now an explicit form in its right side without the additional differential equations:

$$\boldsymbol{x}_n = \frac{1}{\mu_n}(\boldsymbol{x}_{n-1} + \lambda_n \boldsymbol{b}_n) \tag{59}$$

the functions μ_n, λ_n are defined according to (57 i 58).

From the analysis of the Eq. (57) results that the condition to the real μ_n exists has a form

$$q_n \boldsymbol{b}_n^T \boldsymbol{b}_n - i_n^2 > 0$$

or

$$q_n > \frac{(i_n)^2}{(u_n)^2 + (\varDelta u_n)^2}. \tag{60}$$

Received inequality (60) is the relation between three signals involved in this task.

Substituting some real signals to the formulas (57, 58 and 59) the recursive Eq. (59) for the sequences $\{g_n\}$, $\{c_n\}$ takes a form

$$\begin{gathered}g_n = \frac{1}{\mu_n}(g_{n-1} + \lambda_n u_n) \\ c_n = \frac{1}{\mu_n}(c_{n-1} + \lambda_n \varDelta u_n)\end{gathered} \tag{61}$$

where

$$\mu_n = \sqrt{\frac{[(g_{n-1})^2 + (c_{n-1})^2][(u_n)^2 + (\Delta u_n)^2] - (u_n g_{n-1} + \Delta u_n c_{n-1})^2}{q_n[(u_n)^2 + (\Delta u_n)^2] - (i_n)^2}},$$

$$\lambda_n = \frac{i_n \mu_n - (u_n g_{n-1} + \Delta u_n c_{n-1})}{(u_n)^2 + (\Delta u_n)^2}.$$

The resultant recursive formula is now nonlinear. The numerical example of the above formula is shown in Fig. 8.

6. Conclusion and some realization of time-varying elements

The present article considered mainly the synthesis of the time-varying branch $G(t)C(t)$ and the circuits constructed with it. Such circuits are able to maintain the assumed signals on ports. It was proved that the parameters $g(t)$, $c(t)$ are sought with some recursive formulas. In a particular case such formulas have periodic coefficients. To solve them the algorithm (35–36) was introduced. It was called the periodic iteration scheme [2] similarly to the case of differential equation [4]. Now we will make some remarks on the technical realization of the time-varying branches $G(t)C(t)$. Figure 9 shows some realization of the element $\{g_n\}$ based on a high-low switch conception. The switch period is τ.

As to the negative time-varying conductance realization make possible we must use the negative constant one. Figure 9 depicts it. Analogically we can realize $\{c_n\}$ element. Another conception is presented in Fig. 10 [5].

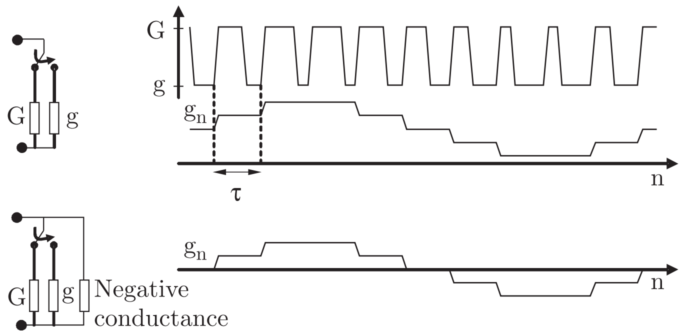


Fig. 9. Binary controlled time-varying conductance

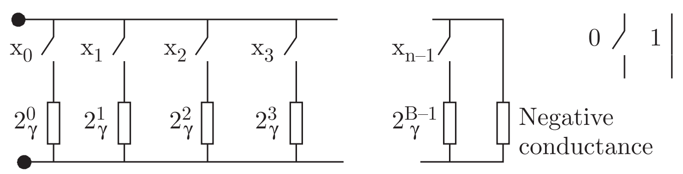


Fig. 10. Realization of $\{g_n\}$ based on a high-low switch concept

According to this conception the function g is realized of value – in binary mode – equal

$$g = (X_0 2^0 + X_1 2^1 + X_2 2^2 + \ldots + X_0 2^{B-1})\gamma$$

This conductance is driven by 0-1 switching vector (B-bits)

$$X = (X_0, X_1, X_2, \ldots X_{B-1}), \quad X_i \in \{0, 1\}$$

In that way the control in the range $\left[0, (2^B - 1)\gamma\right]$ with the γ quantization error is possible. As to make negative values possible we must add the shrunk constant negative conductance (Fig. 10). Such circuit makes the realization of the conductance in symmetric range $\left[-(2^B - 1)\gamma, (2^B - 1)\gamma\right]$ possible. Analogically we can realize the $\{c_n\}$ element.

As a matter of fact the realization of $G(t)C(t)$ branch is based on a driving algorithm and a transformation:

$$\begin{bmatrix} u_n \\ i_n \end{bmatrix} \rightarrow \begin{bmatrix} g_n \\ c_n \end{bmatrix}$$

with the use of periodic iteration algorithm. This transformation can be realized diversely e.g. with the use of a controlled voltage source.

In this way we arrive at a general compensation theorem in which conductance is substituted by an equivalent source. This principle is depicted in Fig. 11.

The controlled voltage source can be realized in diverse ways. One is the two voltage level PWM modulation shown in Fig. 12 [2]. The branch with PWM source can imitate many other branches.

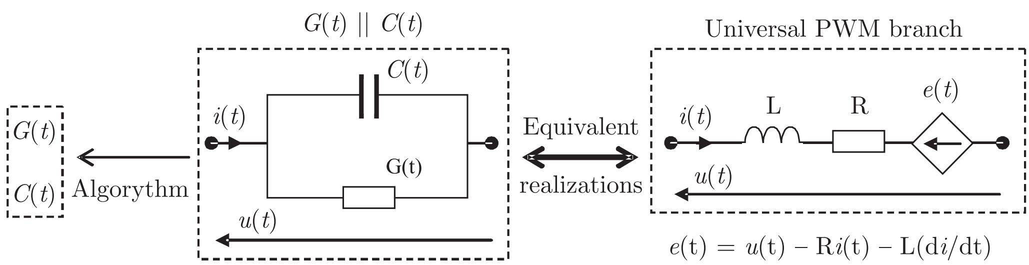


Fig. 11. General compensation theorem

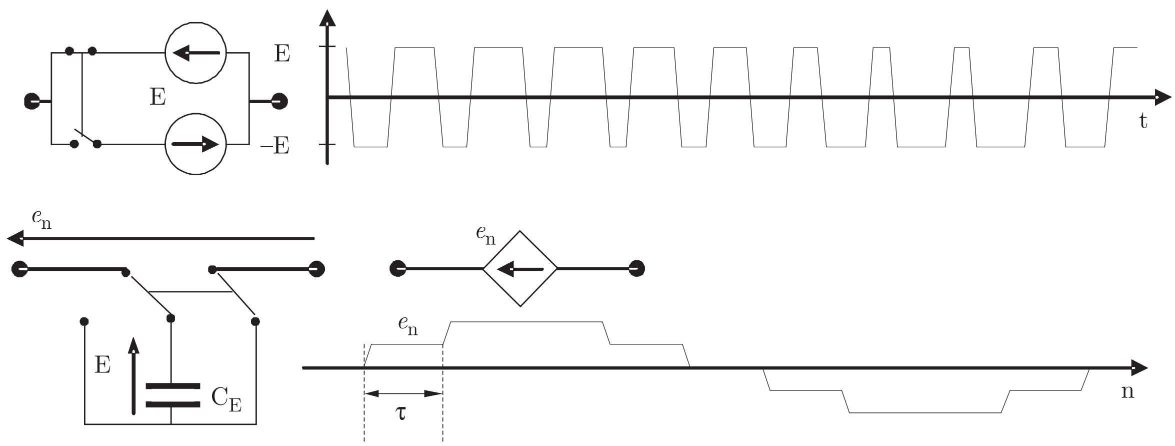
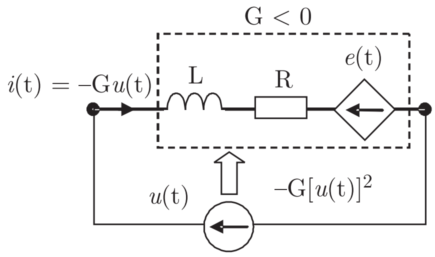


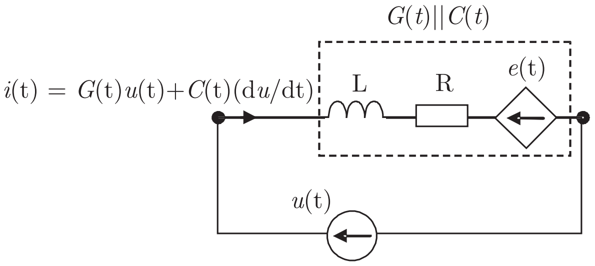
Fig. 12. Realization of voltage controlled source with two voltage level modulation



$$u(t) \to e(t) = u(t) - Ri(t) - L(di/dt) = (1 + RG)u(t) + LG(du/dt)$$

$$\{u_n\} \to u_n = (1 + RG)u_n + (1/\tau)LG(u_n - u_{n-1})$$

Fig. 13. Realization of negative conductance using the PWM voltage source



$$u(t) \to e(t) = [1 + RG(t)]u(t) + RC(t)(du/dt) - L(d/dt)[G(t)u(t)] - L(d/dt)[C(t)(du/dt)]$$

$$u \to e = (1 + RG - RCD - LDG - LDCD)u$$

Fig. 14. Realization of time-varying $G(t)C(t)$ branch using the PWM voltage source

In Figs. 13 and 14 the negative conductance and $G(t)C(t)$ branch realization are shown, with the voltage controlled source used. Also the driving rules of the voltage source in an analog and a digital form are written there. For the digital version the following symbols are introduced:

$$\boldsymbol{e} = \underset{n}{col}[e_n]; \quad \boldsymbol{u} = \underset{n}{col}[u_n]$$

$$\boldsymbol{G} = \underset{n}{diag}[g_n]; \ \boldsymbol{C} = \underset{n}{diag}[c_n]; \ \boldsymbol{D} - \text{difference matrix.}$$

These are the 'intelligent' systems which respond to the input signal $\boldsymbol{u}$ and control the source $\boldsymbol{e}$ according to the appropriate algorithm. Such branches as well as time-varying GC branches, can realize the optimal voltage-current distribution on the complex power networks. This issue is described in more detail in [6].

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