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Dual hesitant Pythagorean fuzzy Bonferroni mean operators in multi-attribute decision making

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In this paper, we investigate the multiple attribute decision making problems based on the Bonferroni mean operators with dual Pythagorean hesitant fuzzy information. Firstly, we introduce the concept and basic operations of the dual hesitant Pythagorean fuzzy sets, which is a new extension of Pythagorean fuzzy sets. Then, motivated by the idea of Bonferroni mean operators, we have developed some Bonferroni mean aggregation operators for aggregating dual hesitant Pythagorean fuzzy information. The prominent characteristic of these proposed operators are studied. Then, we have utilized these operators to develop some approaches to solve the dual hesitant Pythagorean fuzzy multiple attribute decision making problems. Finally, a practical example for supplier selection in supply chain management is given to verify the developed approach and to demonstrate its practicality and effectiveness.

Key words: multiple attribute decision making (MADM), dual hesitant Pythagorean fuzzy sets, dual hesitant Pythagorean fuzzy Bonferroni mean (DHPFBM) operator, dual hesitant Pythagorean fuzzy geometric Bonferroni mean(DHPFGBM) operator, supplier selection, supply chain management

1. Introduction

Atanassov [1] introduced the concept of intuitionistic fuzzy set (IFS) characterized by a membership function and a non-membership function, which is a generalization of the concept of fuzzy set [2] whose basic component is only a membership function. Xu [3] developed some aggregation operators, such as the intuitionistic fuzzy weighted averaging operator, intuitionistic fuzzy ordered weighted averaging operator and intuitionistic fuzzy hybrid aggregation operator for aggregating intuitionistic fuzzy values and established various properties of these operators [4] developed some new geometric aggregation operators, such as the intuitionistic fuzzy weighted geo-

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metric (IFWG) operator, the intuitionistic fuzzy ordered weighted geometric (IFOWG) operator and the intuitionistic fuzzy hybrid geometric (IFHG) operator, which extend the WG and OWG operators to accommodate the environment in which the given arguments are intuitionistic fuzzy sets which are characterized by a membership function and a non-membership function. Li, Gao and Wei [5] extended the Hamy mean (HM) operator, the Dombi Hamy mean (DHM) operator, the Dombi dual Hamy mean (DDHM), with the intuitionistic fuzzy numbers (IFNs) to propose the intuitionistic fuzzy Dombi Hamy mean (IFDHM) operator, intuitionistic fuzzy weighted Dombi Hamy mean (IFWDHM) operator, intuitionistic fuzzy Dombi dual Hamy mean (IFDDHM) operator, and intuitionistic fuzzy weighted Dombi dual Hamy mean (IFWDDHM) operator. Xu and Yager [6] developed an intuitionistic fuzzy Bonferroni Mean (IFBM) and discuss its variety of special cases. Su, Xia, Chen and Wang [7] proposed a new aggregation operator called induced generalized intuitionistic fuzzy ordered weighted averaging (IG-IFOWA) operator. Agarwal, Hanmandlu and Biswas [8] defined a probabilistic and decision attitude aggregation operator for intuitionistic fuzzy environment. More recently, Pythagorean fuzzy set (PFS) [9, 10] has emerged as an effective tool for depicting uncertainty of the MADM problems. The PFS is also characterized by the membership degree and the non-membership degree, whose sum of squares is less than or equal to 1, the PFS is more general than the IFS. In some cases, the PFS can solve the problems that the IFS cannot, for example, if a DM gives the membership degree and the non-membership degree as 0.8 and 0.6, respectively, then it is only valid for the PFS. In other words, all the intuitionistic fuzzy degrees are a part of the Pythagorean fuzzy degrees, which indicates that the PFS is more powerful to handle the uncertain problems. Zhang and Xu [11] provided the detailed mathematical expression for PFS and introduced the concept of Pythagorean fuzzy number (PFN) and developed a Pythagorean fuzzy TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) for handling the MCDM problem within PFNs. Garg [12] proposed a novel correlation coefficient and weighted correlation coefficient formulation to measure the relationship between two PFSs. Ma and Xu [13] defined some novel Pythagorean fuzzy weighted geometric/averaging operators for Pythagorean fuzzy information, which can neutrally treat the membership degree and the nonmembership degree, and investigate the relationships among these operators and those existing ones. Peng and Yang [14] defined the Choquet integral operator for Pythagorean fuzzy aggregation operators, such as Pythagorean fuzzy Choquet integral average (PFCIA) operator and Pythagorean fuzzy Choquet integral geometric (PFCIG) operator and proposed two approaches to multiple attribute group decision making with attributes involving dependent and independent by the PFCIA operator and multi-attributive border approximation area comparison (MABAC) in Pythagorean fuzzy environment. Ren, Xu and Gou [15] extended the TODIM (an acronym in Portuguese for Interactive Multi-criteria Decision Making) approach [16–20] to solve the multi-criteria decision making (MCDM) problems with Pythagorean fuzzy information. Zhang [21] developed a closeness index-based Pythagorean fuzzy QUALIFLEX method to address hierarchical multicriteria decision making problems within Pythagorean fuzzy environment based on PFNs and IVPFNs. Liang, Xu and Darko [22] developed the Pythagorean fuzzy geometric Bonferroni mean and weighted Pythagorean fuzzy geometric Bonferroni mean (WPFGBM) operators describing the interrelationship between arguments and some special properties of them are also investigated. Wei [23] utilized arithmetic and geometric operations to develop some Pythagorean fuzzy interaction aggregation operators: Pythagorean fuzzy interaction weighted average (PFIWA) operator, Pythagorean fuzzy interaction weighted geometric (PFIWG) operator, Pythagorean fuzzy interaction ordered weighted average (PFIOWA) operator, Pythagorean fuzzy interaction ordered weighted geometric (PFIOWG) operator, Pythagorean fuzzy interaction hybrid average (PFIHA) operator and Pythagorean fuzzy interaction hybrid geometric (PFIHG) operator. Bolturk [24] developed the Pythagorean fuzzy extension of CODAS method. Li, Wei and Lu [25] extended the Hamy mean (HM) operator and dual Hamy mean (DHM) operator [25–28] with Pythagorean fuzzy numbers (PFNs) to propose Pythagorean fuzzy Hamy mean (PFHM) operator, weighted Pythagorean fuzzy Hamy mean (WPFHM) operator, Pythagorean fuzzy dual Hamy mean (PFDHM) operator, weighted Pythagorean fuzzy dual Hamy mean (WPFDHM) operator. Wei and Lu [29] extended Maclaurin symmetric mean (MSM) operator to Pythagorean fuzzy environment to propose the Pythagorean fuzzy Maclaurin symmetric mean and Pythagorean fuzzy weighted Maclaurin symmetric mean operators. Wei and Lu [30] utilized power aggregation operators [31–33] to develop some Pythagorean fuzzy power aggregation operators: Pythagorean fuzzy power average operator, Pythagorean fuzzy power geometric operator, Pythagorean fuzzy power weighted average operator, Pythagorean fuzzy power weighted geometric operator, Pythagorean fuzzy power ordered weighted average operator, Pythagorean fuzzy power ordered weighted geometric operator, Pythagorean fuzzy power hybrid average operator, and Pythagorean fuzzy power hybrid geometric operator. Wei and Wei [34] presented 10 similarity measures between Pythagorean fuzzy sets (PFSs) based on the cosine function by considering the degree of membership, degree of nonmembership and degree of hesitation in PFSs. Wei [35] utilized Hamacher operations and power aggregation operators to develop some Pythagorean fuzzy Hamacher power aggregation operators. Nie, Tian, Wang and Hu [36] investigated an effective means to aggregate uncertain information and then employ it into settling multiple criteria decision making (MCDM) problems within the Pythagorean fuzzy circumstances. Peng [37] presented an algorithm for solving multi-criteria decision making (MCDM) problem based on Weighted Distance Based Approximation (WDBA). Based on the traditional VIKOR (Vise Kriterijumska Optimizacija I Kompromisno Resenje) method [38]of MCDM, Liang, Zhang, Xu and Jamaldeen [39] provided a new perspective of a compromised solution, which can handle the decision maker's psychological behavior by inducing TODIM (a Portuguese acronym meaning Interactive Multi-Criteria Decision Making). Khan, Khan, Shahzad and Abdullah [40] presented the notion of Pythagorean cubic fuzzy sets in which the membership degree and non-membership degree are cubic fuzzy numbers which hold the conditions that the square sum of its membership degree is less than or equal to 1.

Wei and Lu [41] proposed the concept and basic operations of the dual hesitant Pythagorean fuzzy sets (DHPFSs), which are a new extension of PFS [42–47] and have developed some Hamacher aggregation operators for aggregating dual hesitant Pythagorean fuzzy information. It's very evident that the DHPFSs consist of two parts, that is, the membership hesitancy function and the non-membership hesitancy function, supporting a more exemplary and flexible access to assign values for each element in the domain, and we have to handle two kinds of hesitancy in this situation. For example, in a MADM problem, some decision makers consider as possible values for the membership degree of x into the set A a few different values 0.4, 0.5, and 0.6, and for the non-membership degrees 0.1, 0.2 and 0.3 replacing just one number or a tuple. Utilizing DHPFSs can take much more information into account, the more values we obtain from the decision makers, the greater epistemic certainty we have, and thus, compared to the existing sets, DHPFSs can be regarded as a more comprehensive set, which supports a more flexible approach when the decision makers provide their judgments.

All the above-mentioned information aggregating operators and measures are based on the assumption that input arguments are independent and hence, in sometimes, these input arguments may be unable to justify the decision maker goals. On the other hand, in our real-life situation, it may be possible that there are interactions among the different attributes in a MADM process. To address such type of issues, Bonferroni mean (BM) operator [48] and geometric Bonferroni mean (GBM) operator [49], has prominent characteristics to capture the interrelationship among the multi-input arguments. In the past few years, the BM and GBM have received more and more attentions, many important results both in theory and application are developed [50–58]. Therefore, by considering the advantages of the DHPFSs and the BM, GBM operator during the information fusion process, the present study enhanced these works in that direction. DHPFSs has been used to handle the uncertainties in the data in the form of DHPFSs while BM and GBM operator is used to considering the interrelationships between the different attributes. As far as we are aware, there are no researches conducted under this direction and hence it is meaningful to pay any attention to it. Thus, in this paper we shall propose some Bonferroni mean aggregation operators for fusing the dual hesitant Pythagorean fuzzy information. Further, some of their desirable properties have also been analyzed. Finally, based on these operators, a decision-making approach has been presented under DHPFS environment and illustrate with a numerical example to validate the approach through some comparative study with the existing approaches.

In order to do so, the rest of the paper is organized as follows. Some basic concepts on PFS and DHPFSs have been introduced in the next section. Section 3, presented the BM operators under DHPFS environment namely, DHPFBM and DHPFGBM along with their certain properties. In Section 4, we presented the dual hesitant Pythagorean fuzzy generalized Bonferroni mean (DHPFGBM) operator and dual hesitant Pythagorean fuzzy generalized geometric Bonferroni mean (DHPFGGBM) operator along with their certain properties. In Section 5, we presented dual hesitant Pythagorean fuzzy dual Bonferroni mean (DHPFDBM) operator and dual hesitant Pythagorean fuzzy dual geometric Bonferroni mean (DHPFDGBM) operator along with their certain properties. In Section 6, based on these operators, we shall present some methods for MADM problems with DHPFNs. In Section 7, we present a numerical example for supplier selection in supply chain management with DHPFNs in order to illustrate the method proposed in this paper and we gave a comparative analysis with existing models. Section 8 concludes the paper with some remarks.

2. Preliminaries

2.1. Pythagorean fuzzy set

The basic concepts of PFSs [9, 10] are briefly reviewed in this section.

Definition 1 [9, 10] *Let X be a fix set. A PFS is an object having the form*

$$P = \{\{\langle x, (\mu_P(x),\ \nu_P(x))\rangle\ |x \in X\},\tag{1}$$

where the function $\mu_P :\ X \to [0, 1]$ *defines the degree of membership and the function* $\nu_P :\ X \to [0, 1]$ *defines the degree of non-membership of the element* $x \in X$ *to P, respectively, and, for every* $x \in X$*, it holds that*

$$\left(\mu_p(x)\right)^2 + \left(\nu_p(x)\right)^2 \leqslant 1.\tag{2}$$

Definition 2 [11] *Let* $\widetilde{a}_1 = (\mu_1, \nu_1)$*,* $\widetilde{a}_2 = (\mu_2, \nu_2)$*, and* $\widetilde{a} = (\mu, \nu)$ *be three Pythagorean fuzzy numbers, and some basic operations on them are defined as* *follows:*

$$(1)\ \ \widetilde{a}_1 \oplus \widetilde{a}_2 = \left(\sqrt{(\mu_1)^2 + (\mu_2)^2 - (\mu_1)^2 (\mu_2)^2}, \nu_1\nu_2 \right);$$

$$(2)\ \ \widetilde{a}_1 \otimes \widetilde{a}_2 = \left(\mu_1\mu_2, \sqrt{(\nu_1)^2 + (\nu_2)^2 - (\nu_1)^2 (\nu_2)^2} \right);$$

$$(3)\ \ \pi\widetilde{a} = \left(\sqrt{1-(1-\mu^2)^\pi}, \nu^\pi \right), \quad \pi > 0;$$

$$(4)\ \ (\widetilde{a})^\pi = \left(\mu^\pi, \sqrt{1-(1-\nu^2)^\pi} \right), \quad \pi > 0;$$

$$(5)\ \ \widetilde{a}^c = (\nu, \mu).$$

2.2. Dual hesitant Pythagorean fuzzy set

In this section, Wei & Lu [41] proposed the concept of the dual hesitant Pythagorean fuzzy sets (DHPFSs), which is a new extension of PFS [10, 32, 59] and dual hesitant fuzzy set [60].

Definition 3 [41] *Let X be a fixed set, then a dual hesitant Pythagorean fuzzy set (DHPFS) on X is described as:*

$$D = (\langle x, h_P(x), g_P(x)\rangle \,|x \in X). \tag{3}$$

In which $h_P(x)$ *and* $g_P(x)$ *are two sets of some values in* $[0, 1]$, *denoting the possible membership degrees and non-membership degrees of the element* $x \in X$ *to the set D respectively, with the conditions:*

$$\gamma^2 + \eta^2 \leqslant 1, \tag{4}$$

where $\gamma \in h_P(x)$, $\eta \in g_P(x)$, *for all* $x \in X$. *For convenience, the pair* $\widetilde{d}(x) = (h_P(x), g_P(x))$ *is called a dual hesitant Pythagorean fuzzy number (DHPFN) denoted by* $\widetilde{d} = (h, g)$, *with the conditions:* $\gamma \in h$, $\eta \in g$, $0 \leqslant \gamma, \eta \leqslant 1$, $0 \leqslant \gamma^2 + \eta^2 \leqslant 1$.

To compare the DHPFNs, in the following, Wei & Lu [41] gave the following comparison laws:

Definition 4 *[41] Let* $d = (h, g)$ *be a DHPFNs,*
$s(d) = \frac{1}{2}\left(1 + \frac{1}{\#h}\sum_{\gamma\in h}\gamma^2 - \frac{1}{\#g}\sum_{\eta\in g}\eta^2\right)$ *the score function of d, and*
$p(d) = \frac{1}{\#h}\sum_{\gamma\in h}\gamma^2 + \frac{1}{\#g}\sum_{\eta\in g}\eta^2$ *the accuracy function of d, where #h*

and #g are the numbers of the elements in h and g respectively, then, let $d_i = (h_i, g_i)$ $(i = 1, 2)$ *be any two DHPFNs, we have the following comparison laws:*

- *If* $s(d_1) > s(d_2)$, *then* d_1 *is superior to* d_2, *denoted by* $d_1 \succ d_2$;
- *If* $s(d_1) = s(d_2)$, *then*
  1. *If* $p(d_1) = p(d_2)$, *then* d_1 *is equivalent to* d_2, *denoted by* $d_1 \sim d_2$;
  2. *If* $p(d_1) > p(d_2)$, *then* d_1 *is superior to* d_2, *denoted by* $d_1 \succ d_2$.

Then, Wei & Lu [41] defined some new operations on the DHPFNs d, d_1 *and* d_2:

1. $d^{\pi} = \bigcup_{\gamma \in h, \eta \in g} \left\{ \{\gamma^{\pi}\}, \left\{ \sqrt{1 - (1 - \eta^2)^{\pi}} \right\} \right\}, \quad \pi > 0;$

2. $\pi d = \bigcup_{\gamma \in h, \eta \in g} \left\{ \left\{ \sqrt{1 - (1 - \gamma^2)^{\pi}} \right\}, \{\eta^{\pi}\} \right\}, \quad \pi > 0;$

3. $d_1 \oplus d_2 = \bigcup_{\substack{\gamma_1 \in h_1, \gamma_2 \in h_2, \\ \eta_1 \in g_1, \eta_2 \in g_2}} \left\{ \left\{ \sqrt{(\gamma_1)^2 + (\gamma_2)^2 - (\gamma_1)^2 (\gamma_2)^2} \right\}, \{\eta_1 \eta_2\} \right\};$

4. $d_1 \otimes d_2 = \bigcup_{\substack{\gamma_1 \in h_1, \gamma_2 \in h_2, \\ \eta_1 \in g_1, \eta_2 \in g_2}} \left\{ \{\gamma_1 \gamma_2\}, \left\{ \sqrt{(\eta_1)^2 + (\eta_2)^2 - (\eta_1)^2 (\eta_2)^2} \right\} \right\}.$

2.3. Bonferroni mean

Definition 5 [48] *Let* $p, q > 0$ *and* a_i $(i = 1, 2, \cdots, n)$ *be a collection of nonnegative crisp numbers. The Bonferroni mean (BM) operator is defined as follows:*

$$BM^{p,q}(a_1, a_2, \cdots, a_n) = \left(\sum_{i,j=1}^{n} a_i^p a_j^q \right)^{1/(p+q)}. \tag{5}$$

Definition 6 [49] *Let* $p, q > 0$ *and* a_i $(i = 1, 2, \cdots, n)$ *be a collection of nonnegative crisp numbers, the Bonferroni mean (BM) operator is defined as follows:*

$$GBM^{p,q}(a_1, a_2, \cdots, a_n) = \frac{1}{p+q} \prod_{i,j=1}^{n} \left(pa_i + qa_j \right). \tag{6}$$

2.4. GWBM operator and GWGBM operator

Zhu, Xu and Xia [49] defined the generalized BM (GBM) operator and generalized geometric BM (GGBM) operator.

Definition 7 [49] *Let $p,q,r > 0$ and a_i $(i = 1,2,\cdots,n)$ be a collection of nonnegative crisp numbers. The generalized BM (GBM) operator is defined as follows:*

$$GBM^{p,q,r}(a_1,a_2,\cdots,a_n) = \left(\sum_{i,j,k=1}^{n} \frac{1}{n^3} a_i^p a_j^q a_k^r\right)^{1/(p+q+r)}. \tag{7}$$

Definition 8 [49] *Let $p,q,r > 0$ and a_i $(i = 1,2,\cdots,n)$ be a collection of nonnegative crisp numbers. If*

$$GGBM^{p,q,r}(a_1,a_2,\cdots,a_n) = \frac{1}{p+q+r}\prod_{i,j,k=1}^{n}\left(pa_i + qa_j + ra_k\right)^{\frac{1}{n^3}}. \tag{8}$$

Then $GGBM^{p,q,r}$ is called the generalized geometric BM (GGBM) operator.

2.5. DGWBM operator and DGWGBM operator

Definition 9 [61] *Let a_i $(i = 1,2,\cdots,n)$ be a collection of nonnegative crisp numbers. If*

$$DGBM^{K}(a_1,a_2,\cdots,a_n) = \left(\sum_{i_1,i_2,\ldots,i_n=1}^{n}\left(\prod_{j=1}^{n}\frac{1}{n}a_{i_j}^{k_j}\right)\right)^{1/\sum_{j=1}^{n}k_j}, \tag{9}$$

where $K = (k_1,k_2,\cdots k_n)^T$ is parameter vector with $k_i \geqslant 0$ $(i = 1,2,\cdots,n)$.

Definition 10 [61] *Let a_i $(i = 1,2,3,\cdots,n)$ be a collection of nonnegative crisp numbers. If*

$$DGGBM^{K}(a_1,a_2,\cdots,a_n) = \frac{1}{\sum\limits_{j=1}^{n}k_j}\left(\prod_{i_1,i_2,\cdots,i_n=1}^{n}\left(\sum_{j=1}^{n}\left(k_j p_{i_j}\right)\right)^{\prod_{j=1}^{n}\frac{1}{n}}\right), \tag{10}$$

where $K = (k_1,k_2,\cdots k_n)^T$ is parameter vector with $k_i \geqslant 0$ $(i = 1,2,3,\cdots,n)$.

3. Dual hesitant Pythagorean fuzzy Bonferroni mean operators

This section we fuse dual hesitant Pythagorean fuzzy set with Bonferroni mean operator and proposed the dual hesitant Pythagorean fuzzy Bonferroni mean (DHPFBM) operator and dual hesitant Pythagorean fuzzy geometric Bonferroni mean (DHPFGBM) operator.

3.1. DHPFBM operator

Definition 11 *Let $t, r > 0$, $d_j = \left(h_j, g_j\right)$ $(j = 1, 2, \cdots, n)$ be a set of DHPFN in which $h_P(x)$ and $g_P(x)$ are two sets of some values in $[0, 1]$, then the dual hesitant Pythagorean fuzzy Bonferroni mean (DHPFBM) operator is defined as*

$$DHPFBM^{t,r}\left(d_1, d_2, \cdots, d_n\right) = \left(\bigoplus_{i,j=1}^{n} \frac{1}{n^2}\left(d_i^t \otimes d_j^r\right)\right)^{1/(t+r)}. \tag{11}$$

Theorem 1 *Let $t, r > 0$ and $d_j = \left(h_j, g_j\right)$ $(j = 1, 2, \cdots, n)$ be a collection of DHPFNs in which $\gamma_j \in h_j$, $\eta_j \in g_j$, then their aggregated value by using the DHPFBM operator is also a DHPFN, and*

$$DHPFBM^{t,r}\left(d_1, d_2, \cdots, d_n\right) = \left(\bigoplus_{i,j=1}^{n} \frac{1}{n^2}\left(d_i^t \otimes d_j^r\right)\right)^{1(t+r)}$$

$$= \bigcup_{\substack{\gamma \in h,\\ \eta \in g}} \left\{\left\{\left(\sqrt{1 - \prod_{i,j=1}^{n}\left(1 - \gamma_i^{2t}\gamma_j^{2r}\right)^{\frac{1}{n^2}}}\right)^{1/(t+r)}\right\},\right.$$
$$\left.\left\{\sqrt{1 - \left(1 - \prod_{i,j=1}^{n}\left(1 - \left(1 - \eta_i^2\right)^t\left(1 - \eta_j^2\right)^r\right)^{\frac{1}{n^2}}\right)^{1/(t+r)}}\right\}\right\}. \tag{12}$$

Proof. According the definition 4, we can get

$$d_i^t = \bigcup_{\substack{\gamma_i \in h_i,\\ \eta_i \in g_i}} \left\{\left\{\gamma_i^t\right\}, \left\{\sqrt{1 - \left(1 - \eta_i^2\right)^t}\right\}\right\}, \tag{13}$$

$$d_j^r = \bigcup_{\substack{\gamma_j \in h_j,\\ \eta_j \in g_j}} \left\{\left\{\gamma_j^r\right\}, \left\{\sqrt{1 - \left(1 - \eta_j^2\right)^r}\right\}\right\}, \tag{14}$$

$$d_i^t \otimes d_j^r = \bigcup_{\substack{\gamma_j \in h_j, \\ \eta_j \in g_j}} \left\{ \left\{ \gamma_i^t \gamma_j^r \right\}, \left\{ \sqrt{1 - \left(1 - \eta_i^2\right)^t \left(1 - \eta_j^2\right)^r} \right\} \right\}. \tag{15}$$

Thereafter,

$$\frac{1}{n^2}\left(d_i^t \otimes d_j^r\right) = \bigcup_{\substack{\gamma_j \in h_j, \\ \eta_j \in g_j}} \left\{ \left\{ \sqrt{1 - \left(1 - \gamma_i^{2t} \gamma_j^{2r}\right)^{\frac{1}{n^2}}} \right\}, \left\{ \left(\sqrt{1 - \left(1 - \eta_i^2\right)^t \left(1 - \eta_j^2\right)^r} \right)^{\frac{1}{n^2}} \right\} \right\}. \tag{16}$$

Furthermore,

$$\bigoplus_{i,j=1}^{n} \frac{1}{n^2}\left(d_i^t \otimes d_j^r\right) = \bigcup_{\substack{\gamma_j \in h_j, \\ \eta_j \in g_j}} \left\{ \left\{ \sqrt{1 - \prod_{i,j=1}^{n} \left(1 - \gamma_i^{2t} \gamma_j^{2r}\right)^{\frac{1}{n^2}}} \right\}, \left\{ \left(\sqrt{\prod_{i,j=1}^{n} \left(1 - \left(1 - \eta_i^2\right)^t \left(1 - \eta_j^2\right)^r\right)} \right)^{\frac{1}{n^2}} \right\} \right\}. \tag{17}$$

Therefore,

$$\mathrm{DHPFBM}^{t,r}\left(d_1, d_2, \cdots, d_n\right) = \left(\bigoplus_{i,j=1}^{n} \frac{1}{n^2}\left(d_i^t \otimes d_j^r\right) \right)^{1(t+r)} = \bigcup_{\substack{\gamma \in h, \\ \eta \in g}} \left\{ \left\{ \left(\sqrt{1 - \prod_{i,j=1}^{n} \left(1 - \gamma_i^{2t} \gamma_j^{2r}\right)^{\frac{1}{n^2}}} \right)^{1/(t+r)} \right\}, \left\{ \sqrt{1 - \left(1 - \prod_{i,j=1}^{n} \left(1 - \left(1 - \eta_i^2\right)^t \left(1 - \eta_j^2\right)^r\right)^{\frac{1}{n^2}}\right)^{1/(t+r)}} \right\} \right\}. \tag{18}$$

Thereafter, we can get

$$0 \leqslant \left(\sqrt{1-\prod_{i,j=1}^{n}\left(1-\gamma_i^{2t}\gamma_j^{2r}\right)^{\frac{1}{n^2}}}\right)^{1/(t+r)} \leqslant 1, \tag{19}$$

$$0 \leqslant \sqrt{1-\left(1-\prod_{i,j=1}^{n}\left(1-\left(1-\eta_i^2\right)^t\left(1-\eta_j^2\right)^r\right)^{\frac{1}{n^2}}\right)^{1/(t+r)}} \leqslant 1. \tag{20}$$

And we know $\gamma^2+\eta^2 \leqslant 1$, so

$$\begin{aligned}&\left(\sqrt{1-\prod_{i,j=1}^{n}\left(1-\gamma_i^{2t}\gamma_j^{2r}\right)^{\frac{1}{n^2}}}\right)^{1/(t+r)}\\ &\quad\leqslant \sqrt{\left(1-\prod_{i,j=1}^{n}\left(1-\left(1-\eta_i^2\right)^t\left(1-\eta_j^2\right)^r\right)^{\frac{1}{n^2}}\right)^{1/(t+r)}}.\end{aligned} \tag{21}$$

Therefore,

$$\begin{aligned}&\left(\left(\sqrt{1-\prod_{i,j=1}^{n}\left(1-\gamma_i^{2t}\gamma_j^{2r}\right)^{\frac{1}{n^2}}}\right)^{1/(t+r)}\right)^2+\\ &\quad\left(\sqrt{1-\left(1-\prod_{i,j=1}^{n}\left(1-\left(1-\eta_i^2\right)^t\left(1-\eta_j^2\right)^r\right)^{\frac{1}{n^2}}\right)^{1/(t+r)}}\right)^2\\ &\quad\leqslant\left(1-\prod_{i,j=1}^{n}\left(1-\left(1-\eta_i^2\right)^t\left(1-\eta_j^2\right)^r\right)^{\frac{1}{n^2}}\right)^{1/(t+r)}+1\\ &\quad-\left(1-\prod_{i,j=1}^{n}\left(1-\left(1-\eta_i^2\right)^t\left(1-\eta_j^2\right)^r\right)^{\frac{1}{n^2}}\right)^{1/(t+r)}=1.\end{aligned} \tag{22}$$

So, we complete the proof.

Example 1. Let $a_1 = \{(0.4, 0.2), (0.5, 0.1)\}$, $a_2 = \{(0.5, 0.6), (0.3, 0.7)\}$, $a_3 = \{(0.4, 0.3)\}$ be three DHPFNs, and $t = r = 2$, the aggregation result as follows:

$$\begin{aligned}
&\text{DHPFBM}^{t,r}(a_1, a_2, a_3)\\
&= \text{DHPFBM}^{t,r}\{(0.4, 0.2), (0.5, 0.1)\}, \{(0.5, 0.6), (0.3, 0.7)\}, \{(0.4, 0.3)\}\\
&= \left\{\left(\left(\sqrt{\begin{array}{l} 1- \\ \left(1-0.4^{2\times 2}\times 0.4^{2\times 2}\right)^{1/(3\times 3)}\times \\ \left(1-0.4^{2\times 2}\times 0.5^{2\times 2}\right)^{1/(3\times 3)}\times \\ \left(1-0.4^{2\times 2}\times 0.4^{2\times 2}\right)^{1/(3\times 3)}\times \\ \left(1-0.5^{2\times 2}\times 0.4^{2\times 2}\right)^{1/(3\times 3)}\times \\ \left(1-0.5^{2\times 2}\times 0.5^{2\times 2}\right)^{1/(3\times 3)}\times \\ \left(1-0.5^{2\times 2}\times 0.4^{2\times 2}\right)^{1/(3\times 3)}\times \\ \left(1-0.4^{2\times 2}\times 0.4^{2\times 2}\right)^{1/(3\times 3)}\times \\ \left(1-0.4^{2\times 2}\times 0.5^{2\times 2}\right)^{1/(3\times 3)}\times \\ \left(1-0.4^{2\times 2}\times 0.4^{2\times 2}\right)^{1/(3\times 3)} \end{array}}\right)^{\frac{1}{2+2}}, 1-\sqrt{\left(\begin{array}{l} 1- \\ \left(1-0.96^2\times 0.96^2\right)^{1/(3\times 3)}\times \\ \left(1-0.96^2\times 0.64^2\right)^{1/(3\times 3)}\times \\ \left(1-0.96^2\times 0.91^2\right)^{1/(3\times 3)}\times \\ \left(1-0.64^2\times 0.96^2\right)^{1/(3\times 3)}\times \\ \left(1-0.64^2\times 0.64^2\right)^{1/(3\times 3)}\times \\ \left(1-0.64^2\times 0.91^2\right)^{1/(3\times 3)}\times \\ \left(1-0.91^2\times 0.96^2\right)^{1/(3\times 3)}\times \\ \left(1-0.91^2\times 0.64^2\right)^{1/(3\times 3)}\times \\ \left(1-0.91^2\times 0.91^2\right)^{1/(3\times 3)} \end{array}\right)^{\frac{1}{2+2}}}\right), \cdots\right\}\\
&= \{(0.4412, 0.3548), (0.3750, 0.3767), (0.4734, 0.3129), (0.4232, 0.3320)\}.
\end{aligned}$$

In the next, we introduce three kinds of property of DHPFBM.

Property 1 (Idempotency), let $t, r > 0$ and $d_i = (h_i, g_i)$ $(i = 1, 2, 3, \cdots, n)$ be two sets of DHPFNs, if d_i $(i = 1, 2, \cdots, n)$ are equal, that is $d_i = d = (h, g)$, then

$$\text{DHPFBM}^{t,r}(d_1, d_2, \cdots, d_n) = d. \tag{23}$$

Proof.

$$\begin{aligned}
\text{DHPFBM}^{t,r}(d_1, d_2, \cdots, d_n) &= \left(\bigoplus_{i,j=1}^{n} \frac{1}{n^2}\left(d^t \otimes d^r\right)\right)^{1/(t+r)}\\
&= d\left(\bigoplus_{i,j=1}^{n} \frac{1}{n^2}\right)^{1(t+r)}\\
&= d.
\end{aligned} \tag{24}$$

Property 2 (Monotonicity), let $d_j = (h_{d_j}, g_{d_j})$ and $b_j = (h_{b_j}, g_{b_j})$ $(j = 1, 2, 3, \cdots, n)$ be two sets of DHPFNs, if $\forall(\gamma_{d_j})^2 \leqslant \forall(\gamma_{b_j})^2$, $\gamma_{d_j} \in h_{d_j}$, $\gamma_{b_j} \in h_{b_j}$ and

$\forall(\eta_{d_j})^2 \geqslant \forall(\eta_{b_j})^2$, $\eta_{d_j} \in g_{d_j}$, $\eta_{b_j} \in g_{b_j}$ then

$$\mathrm{DHPFBM}^{t,r}(d_1, d_2, \cdots, d_n) \leqslant \mathrm{DHPFBM}^{t,r}(b_1, b_2, \cdots, b_n). \tag{25}$$

Proof. We also can obtain

$$\gamma_{d_i}^{2t}\gamma_{d_j}^{2r} \leqslant \gamma_{b_i}^{2t}\gamma_{b_j}^{2r}, \tag{26}$$

$$\prod_{i,j=1}^{n}\left(1-\gamma_{d_i}^{2t}\gamma_{d_j}^{2r}\right) \geqslant \prod_{i,j=1}^{n}\left(1-\gamma_{b_i}^{2t}\gamma_{b_j}^{2r}\right), \tag{27}$$

$$1-\prod_{i,j=1}^{n}\left(1-\gamma_{d_i}^{2t}\gamma_{d_j}^{2r}\right)^{\frac{1}{n^2}} \leqslant 1-\prod_{i,j=1}^{n}\left(1-\gamma_{b_i}^{2t}\gamma_{b_j}^{2r}\right)^{\frac{1}{n^2}}. \tag{28}$$

Therefore:

$$\left(\sqrt{1-\prod_{i,j=1}^{n}\left(1-\gamma_{d_i}^{2t}\gamma_{d_j}^{2r}\right)^{\frac{1}{n^2}}}\right)^{1/(t+r)} \leqslant \left(\sqrt{1-\prod_{i,j=1}^{n}\left(1-\gamma_{b_i}^{2t}\gamma_{b_j}^{2r}\right)^{\frac{1}{n^2}}}\right)^{1/(t+r)}. \tag{29}$$

Thus:

$$\left(\left(\sqrt{1-\prod_{i,j=1}^{n}\left(1-\gamma_{d_i}^{2t}\gamma_{d_j}^{2r}\right)^{\frac{1}{n^2}}}\right)^{1/(t+r)}\right)^2 \leqslant \left(\left(\sqrt{1-\prod_{i,j=1}^{n}\left(1-\gamma_{b_i}^{2t}\gamma_{b_j}^{2r}\right)^{\frac{1}{n^2}}}\right)^{1/(t+r)}\right)^2, \tag{30}$$

which means $\gamma_d^2 \leqslant \gamma_b^2$. Similarly, we can obtain $\eta_d^2 \geqslant \eta_b^2$.

If $\forall\gamma_{d_j}^2 < \forall\gamma_{b_j}^2$ and $\forall\eta_{d_j}^2 > \forall\eta_{b_j}^2$ then

$$\mathrm{DHPFBM}^{t,r}(d_1, d_2, \cdots, d_n) < \mathrm{DHPFBM}^{t,r}(b_1, b_2, \cdots, b_n);$$

If $\forall\gamma_{d_j}^2 < \forall\gamma_{b_j}^2$ and $\forall\eta_{d_j}^2 = \forall\eta_{b_j}^2$ then

$$\mathrm{DHPFBM}^{t,r}(d_1, d_2, \cdots, d_n) < \mathrm{DHPFBM}^{t,r}(b_1, b_2, \cdots, b_n);$$

If $\forall\gamma_{d_j}^2 = \forall\gamma_{b_j}^2$ and $\forall\eta_{d_j}^2 > \forall\eta_{b_j}^2$ then

$$\mathrm{DHPFBM}^{t,r}(d_1, d_2, \cdots, d_n) < \mathrm{DHPFBM}^{t,r}(b_1, b_2, \cdots, b_n);$$

If $\forall\gamma_{d_j}^2 = \forall\gamma_{b_j}^2$ and $\forall\eta_{d_j}^2 = \forall\eta_{b_j}^2$ then

$$\mathrm{DHPFBM}^{t,r}(d_1, d_2, \cdots, d_n) = \mathrm{DHPFBM}^{t,r}(b_1, b_2, \cdots, b_n).$$

Therefore, the proof of property 2 is completed.

Property 3 (Boundedness), let $t, r > 0$ and $d_j = \left(h_{d_j}, g_{d_j}\right)$ $(j = 1, 2, 3, \cdots, n)$ be a collection of DHPFNs. If $d^+ = \cup_{\gamma_j \in h_{d_j}, \eta_j \in g_{d_j}} \{\{\max_i(\gamma_i)\}, \{\min_i(\eta_i)\}\}$ and $d^- = \cup_{\gamma_j \in h_{d_j}, \eta_j \in g_{d_j}} \{\{\min_i(\gamma_i)\}, \{\max_i(\eta_i)\}\}$, then

$$d^- \leqslant \mathrm{DHPFBM}^{t,r}(d_1, d_2, \cdots, d_n) \leqslant d^+. \tag{31}$$

Proof. From property 1 we can obtain

$$\mathrm{DHPFBM}^{t,r}(d^+, d^+, \cdots, d^+) = d^+, \quad \mathrm{DHPFBM}^{t,r}(d^-, d^-, \cdots, d^-) = d^-. \tag{32}$$

So, from property 2 we can obtain:

$$\begin{aligned} d^- = \mathrm{DHPFBM}^{t,r}(d^-, d^-, \cdots, d^-) \leqslant \\ \mathrm{DHPFBM}^{t,r}(d_1, d_2, \cdots, d_n) \leqslant \\ \mathrm{DHPFBM}^{t,r}(d^+, d^+, \cdots, d^+) = d^+. \end{aligned} \tag{33}$$

3.2. DHPFGBM operator

We extend GBM to DHPFN and introduced the Dual hesitant Pythagorean fuzzy Geometric Bonferroni mean (DHPFGBM) continue.

Definition 12 *Let $t, r > 0$, $d_j = \left(h_j, g_j\right)$ $(j = 1, 2, \cdots, n)$ be a set of DHPFN in which $h_P(x)$ and $g_P(x)$ are two sets of some values in $[0, 1]$, then the dual hesitant Pythagorean Fuzzy geometric Bonferroni mean (DHPFGBM) operator is defined as*

$$DHPFGBM^{t,r}(d_1, d_2, \cdots, d_n) = \frac{1}{t+r} \bigotimes_{i,j=1}^{n} \left(td_i \oplus rd_j\right)^{\frac{1}{n^2}}. \tag{34}$$

Theorem 2 *Let $t, r > 0$ and $d_j = \left(h_j, g_j\right)$ $(j = 1, 2, \cdots, n)$ be a collection of DHPFNs in which $\gamma_j \in h_j$, $\eta_j \in g_j$, then their aggregated value by using the DHPFGBM operator is also a DHPFN, and*

$$\begin{aligned} &DHPFGBM^{t,r}(d_1, d_2, \cdots, d_n) \\ &= \bigcup_{\gamma \in h, \eta \in g} \left\{ \left\{ \left(\sqrt{1 - \left(1 - \prod_{i,j=1}^{n} \left(1 - \left(1 - \gamma_i^2\right)^t \left(1 - \gamma_j^2\right)^r\right)^{\frac{1}{n^2}}\right)} \right)^{1/(t+r)} \right\}, \right. \\ &\qquad \left. \left\{ \left(\sqrt{1 - \prod_{i,j=1}^{n} \left(1 - \eta_i^{2t} \eta_j^{2r}\right)^{\frac{1}{n^2}}} \right)^{1/(t+r)} \right\} \right\}. \end{aligned} \tag{35}$$

Proof.

$$td_i = \bigcup_{\gamma_i \in h_i, \eta_i \in g_i} \left\{ \left\{ \sqrt{1 - \left(1 - \gamma_i^2\right)^t} \right\}, \left\{ \eta_i^t \right\} \right\}, \tag{36}$$

$$rd_j = \bigcup_{\gamma_j \in h_j, \eta_j \in g_j} \left\{ \left\{ \sqrt{1 - \left(1 - \gamma_j^2\right)^r} \right\}, \left\{ \eta_j^r \right\} \right\}. \tag{37}$$

Thereafter,

$$td_i \oplus rd_j = \bigcup_{\gamma \in h, \eta \in g} \left\{ \left\{ \sqrt{1 - \left(1 - \gamma_i^2\right)^t \left(1 - \gamma_j^2\right)^r} \right\}, \left\{ \eta_i^t \eta_j^r \right\} \right\}, \tag{38}$$

$$\left(td_i \oplus rd_j\right)^{\frac{1}{n^2}} = \bigcup_{\substack{\gamma \in h, \\ \eta \in g}} \left\{ \left\{ \left(\sqrt{1 - \left(1 - \gamma_i^2\right)^t \left(1 - \gamma_j^2\right)^r} \right)^{\frac{1}{n^2}} \right\}, \left\{ \sqrt{1 - \left(1 - \eta_i^{2t} \eta_j^{2r}\right)^{\frac{1}{n^2}}} \right\} \right\}. \tag{39}$$

Therefore,

$$\bigotimes_{i,j=1}^{n} \left(td_i \oplus rd_j\right)^{\frac{1}{n^2}} = \bigcup_{\gamma \in h, \eta \in g} \left\{ \left\{ \left(\sqrt{\prod_{i,j=1}^{n} \left(1 - \left(1 - \gamma_i^2\right)^t \left(1 - \gamma_j^2\right)^r\right)} \right)^{\frac{1}{n^2}} \right\}, \left\{ \sqrt{1 - \prod_{i,j=1}^{n} \left(1 - \gamma_i^{2t} \gamma_j^{2r}\right)^{\frac{1}{n^2}}} \right\} \right\}. \tag{40}$$

Thus

$$\mathrm{DHPFGBM}^{t,r}\left(d_1, d_2, \cdots, d_n\right) = \bigcup_{\gamma \in h, \eta \in g} \left\{ \left\{ \left(\sqrt{1 - \left(1 - \prod_{i,j=1}^{n} \left(1 - \left(1 - \gamma_i^2\right)^t \left(1 - \gamma_j^2\right)^r\right)^{\frac{1}{n^2}}\right)} \right)^{1/(t+r)} \right\}, \left\{ \left(\sqrt{1 - \prod_{i,j=1}^{n} \left(1 - \eta_i^{2t} \eta_j^{2r}\right)^{\frac{1}{n^2}}} \right)^{1/(t+r)} \right\} \right\}. \tag{41}$$

Thereafter:

$$0 \leqslant \sqrt{1-\left(1-\prod_{i,j=1}^{n}\left(1-\left(1-\gamma_i^2\right)^t\left(1-\gamma_j^2\right)^r\right)^{\frac{1}{n^2}}\right)^{1/(t+r)}} \leqslant 1, \tag{42}$$

$$0 \leqslant \left(\sqrt{1-\prod_{i,j=1}^{n}\left(1-\eta_i^{2t}\eta_j^{2r}\right)^{\frac{1}{n^2}}}\right)^{1/(t+r)} \leqslant 1. \tag{43}$$

Because of $\gamma^2 + \eta^2 \leqslant 1$,

$$\begin{aligned}&\left(\sqrt{1-\prod_{i,j=1}^{n}\left(1-\eta_i^{2t}\eta_j^{2r}\right)^{\frac{1}{n^2}}}\right)^{1/(t+r)}\\ &\leqslant \sqrt{\left(1-\prod_{i,j=1}^{n}\left(1-\left(1-\gamma_i^2\right)^t\left(1-\gamma_j^2\right)^r\right)^{\frac{1}{n^2}}\right)^{1/(t+r)}}.\end{aligned} \tag{44}$$

Therefore,

$$\begin{aligned}&\left(\sqrt{1-\left(1-\prod_{i,j=1}^{n}\left(1-\left(1-\gamma_i^2\right)^t\left(1-\gamma_j^2\right)^r\right)^{\frac{1}{n^2}}\right)^{1/(t+r)}}\right)^2 +\\ &\left(\left(\sqrt{1-\prod_{i,j=1}^{n}\left(1-\eta_i^{2t}\eta_j^{2r}\right)^{\frac{1}{n^2}}}\right)^{1/(t+r)}\right)^2 \leqslant\\ &1-\left(1-\prod_{i,j=1}^{n}\left(1-\left(1-\gamma_i^2\right)^t\left(1-\gamma_j^2\right)^r\right)^{\frac{1}{n^2}}\right)^{1/(t+r)}\\ &+\left(1-\prod_{i,j=1}^{n}\left(1-\left(1-\gamma_i^2\right)^t\left(1-\gamma_j^2\right)^r\right)^{\frac{1}{n^2}}\right)^{1/(t+r)} = 1.\end{aligned} \tag{45}$$

Thereby completing the proof.

Example 2. Let $a_1 = \{(0.4, 0.2), (0.5, 0.1)\}$, $a_2 = \{(0.5, 0.6), (0.3, 0.7)\}$, $a_3 = \{(0.4, 0.3)\}$ be three DHPFNs, and $t = r = 2$, the aggregation result as follows:

$$\begin{aligned}
&\text{DHPFGBM}^{t,r}(a_1, a_2, a_3)\\
&= \text{DHPFGBM}^{t,r}\{(0.4, 0.2), (0.5, 0.1)\}, \{(0.5, 0.6), (0.3, 0.7)\}, \{(0.4, 0.3)\}\\
&= \left\{ \left(\left(\sqrt{\begin{pmatrix} 1- \\ \left(1-0.84^2\times 0.84^2\right)^{1/(3\times 3)} \times \\ \left(1-0.84^2\times 0.75^2\right)^{1/(3\times 3)} \times \\ \left(1-0.84^2\times 0.84^2\right)^{1/(3\times 3)} \times \\ \left(1-0.75^2\times 0.84^2\right)^{1/(3\times 3)} \times \\ \left(1-0.75^2\times 0.75^2\right)^{1/(3\times 3)} \times \\ \left(1-0.75^2\times 0.84^2\right)^{1/(3\times 3)} \times \\ \left(1-0.84^2\times 0.84^2\right)^{1/(3\times 3)} \times \\ \left(1-0.84^2\times 0.75^2\right)^{1/(3\times 3)} \times \\ \left(1-0.84^2\times 0.84^2\right)^{1/(3\times 3)} \end{pmatrix}^{\frac{1}{2+2}}}, \sqrt{1-\begin{pmatrix} 1- \\ \left(1-0.2^{2\times 2}\times 0.2^{2\times 2}\right)^{1/(3\times 3)} \times \\ \left(1-0.2^{2\times 2}\times 0.6^{2\times 2}\right)^{1/(3\times 3)} \times \\ \left(1-0.2^{2\times 2}\times 0.4^{2\times 2}\right)^{1/(3\times 3)} \times \\ \left(1-0.6^{2\times 2}\times 0.2^{2\times 2}\right)^{1/(3\times 3)} \times \\ \left(1-0.6^{2\times 2}\times 0.6^{2\times 2}\right)^{1/(3\times 3)} \times \\ \left(1-0.6^{2\times 2}\times 0.4^{2\times 2}\right)^{1/(3\times 3)} \times \\ \left(1-0.4^{2\times 2}\times 0.2^{2\times 2}\right)^{1/(3\times 3)} \times \\ \left(1-0.4^{2\times 2}\times 0.6^{2\times 2}\right)^{1/(3\times 3)} \times \\ \left(1-0.4^{2\times 2}\times 0.4^{2\times 2}\right)^{1/(3\times 3)} \end{pmatrix}^{\frac{1}{2+2}}} \right) \right), \cdots \right\}\\
&= \{(0.4328, 0.4646), (0.3665, 0.5388), (0.4662, 0.4633), (0.3986, 0.5380)\}.
\end{aligned}$$

Similar to DHPFBM, the DHPFGBM has the same properties. The proofs of these properties are similar to that of the properties of DHPFGBM. Accordingly, the proofs are omitted to save space.

Property 4 (Idempotency), let $t, r > 0$ and $d_i = (h_i, g_i)$ $(i = 1, 2, 3, \cdots, n)$ be two sets of DHPFNs, If d_i $(i = 1, 2, \cdots, n)$ are equal, that is $d_i = d = (h, g)$, then

$$\text{DHPFGBM}^{t,r}(d_1, d_2, \cdots, d_n) = d. \tag{46}$$

Property 5 (Monotonicity), let $d_j = \left(h_{d_j}, g_{d_j}\right)$ and $b_j = \left(h_{b_j}, g_{b_j}\right)$ $(j = 1, 2, 3, \cdots, n)$ be two sets of DHPFNs, If $\forall \left(\gamma_{d_j}\right)^2 \leqslant \forall \left(\gamma_{b_j}\right)^2, \gamma_{d_j} \in h_{d_j}, \gamma_{b_j} \in h_{b_j}$ and $\forall \left(\eta_{d_j}\right)^2 \geqslant \forall \left(\eta_{b_j}\right)^2, \eta_{d_j} \in g_{d_j}, \eta_{b_j} \in g_{b_j}$ then

$$\text{DHPFGBM}^{t,r}(d_1, d_2, \cdots, d_n) \leqslant \text{DHPFGBM}^{t,r}(d_1, d_2, \cdots, d_n). \tag{47}$$

Property 6 (Boundedness), let $t, r > 0$ and $d_j = \left(h_{d_j}, g_{d_j}\right)$ $(j = 1, 2, 3, \cdots, n)$ be a collection of DHPFNs. If

$d^+ = \bigcup_{\gamma_j \in h_{d_j}, \eta_j \in g_{d_j}} \{\{\max_i(\gamma_i)\}, \{\min_i(\eta_i)\}\}$ and

$$d^- = \bigcup_{\gamma_j \in h_{d_j}, \eta_j \in g_{d_j}} \{\{\min_i (\gamma_i)\}, \{\max_i (\eta_i)\}\}, \text{ then}$$

$$d^- \leqslant \text{DHPFGBM}^{t,r} (d_1, d_2, \cdots d_n) \leqslant d^+. \tag{48}$$

4. Dual hesitant Pythagorean fuzzy generalized Bonferroni mean operators

In this section, we combine dual hesitant Pythagorean fuzzy set with Bonferroni mean operators to propose the dual hesitant Pythagorean fuzzy generalized Bonferroni mean (DHPFGBM) operator and dual hesitant Pythagorean fuzzy generalized geometric Bonferroni mean (DHPFGGBM) operator.

4.1. DHPFGBM operator

Definition 13 *Let $t, r > 0$, $d_j = \left(h_j, g_j\right)$ $(j = 1, 2, \cdots, n)$ be a set of DHPFN in which $h_P(x)$ and $g_P(x)$ are two sets of some values in $[0, 1]$, if*

$$DHPFGBM^{s,t,r} (d_1, d_2, \cdots, d_n) = \left(\bigoplus_{i,j,k=1}^{n} \frac{1}{n^3} d_i^{\alpha} \otimes d_j^{\beta} \otimes d_k^{\gamma} \right)^{1/(\alpha+\beta+\gamma)} . \tag{49}$$

Then $DHPFGBM^{s,t,r}$ is called the dual hesitant Pythagorean fuzzy generalized Bonferroni mean (DHPFGBM) operator.

Theorem 3 *Let $s, t, r > 0$ and $d_j = \left(h_j, g_j\right)$ $(j = 1, 2, \cdots, n)$ be a collection of DHPFNs. The aggregated value by DHPFGBM is also a DHPFN and*

$$\begin{aligned} & DHPFGBM^{s,t,r} (d_1, d_2, \cdots, d_n) \\ & = \left(\bigoplus_{i,j,k=1}^{n} \frac{1}{n^3} d_i^{s} \otimes d_j^{t} \otimes d_k^{r} \right)^{1/(\alpha+\beta+\gamma)} \\ & = \bigcup_{\gamma \in h, \eta \in g} \left\{ \left\{ \left(\sqrt{1 - \prod_{i,j,k=1}^{n} \left(1 - \gamma_i^{2s} \gamma_j^{2t} \gamma_k^{2r}\right)^{\frac{1}{n^3}}} \right)^{1/(s+t+r)} \right\}, \right. \\ & \left. \left\{ \sqrt{1 - \left(1 - \prod_{i,j,k=1}^{n} \left(1 - \left(1 - \eta_i^2\right)^s \left(1 - \eta_j^2\right)^t \left(1 - \eta_k^2\right)^r\right)^{\frac{1}{n^3}} \right)^{1/(s+t+r)}} \right\} \right\}. \end{aligned} \tag{50}$$

Proof. According to Definition 4, we can obtain

$$d_i^s = \bigcup_{\gamma_i \in h_i, \eta_i \in g_i} \left\{ \left\{ \gamma_i^s \right\}, \left\{ \sqrt{1 - \left(1 - \eta_i^2\right)^s} \right\} \right\}, \tag{51}$$

$$d_j^t = \bigcup_{\gamma_j \in h_j, \eta_j \in g_j} \left\{ \left\{ \gamma_j^t \right\}, \left\{ \sqrt{1 - \left(1 - \eta_j^2\right)^t} \right\} \right\}, \tag{52}$$

$$d_k^r = \bigcup_{\gamma_k \in h_k, \eta_k \in g_k} \left\{ \left\{ \gamma_k^r \right\}, \left\{ \sqrt{1 - \left(1 - \eta_k^2\right)^r} \right\} \right\}. \tag{53}$$

Thus,

$$d_i^s \otimes d_j^t \otimes d_k^r = \bigcup_{\gamma \in h, \eta \in g} \left\{ \left\{ \gamma_i^s \gamma_j^t \gamma_k^r \right\}, \left\{ \sqrt{1 - \left(1 - \eta_i^2\right)^s \left(1 - \eta_j^2\right)^t \left(1 - \eta_k^2\right)^r} \right\} \right\}. \tag{54}$$

Thereafter,

$$\frac{1}{n^3}\left(d_i^s \otimes d_j^t \otimes d_k^r\right) = \bigcup_{\substack{\gamma \in h, \\ \eta \in g}} \left\{ \left\{ \sqrt{1 - \left(1 - \gamma_i^{2s} \gamma_j^{2t} \gamma_k^{2r}\right)^{\frac{1}{n^3}}} \right\}, \left\{ \left(\sqrt{1 - \left(1 - \eta_i^2\right)^s \left(1 - \eta_j^2\right)^t \left(1 - \eta_k^2\right)^r} \right)^{\frac{1}{n^3}} \right\} \right\}. \tag{55}$$

Furthermore,

$$\bigoplus_{i,j,k=1}^{n} \frac{1}{n^3}\left(d_i^s \otimes d_j^t \otimes d_k^r\right) = \bigcup_{\substack{\gamma \in h, \\ \eta \in g}} \left\{ \left\{ \sqrt{1 - \prod_{i,j,k=1}^{n} \left(1 - \gamma_i^{2s} \gamma_j^{2t} \gamma_k^{2r}\right)^{\frac{1}{n^3}}} \right\}, \left\{ \left(\sqrt{\prod_{i,j,k=1}^{n} \left(1 - \left(1 - \eta_i^2\right)^s \left(1 - \eta_j^2\right)^t \left(1 - \eta_k^2\right)^r\right)} \right)^{\frac{1}{n^3}} \right\} \right\}. \tag{56}$$

Therefore,

$$\begin{aligned}
&\mathrm{DHPFGBM}^{s,t,r}\left(d_1, d_2, \cdots, d_n\right) \\
&= \left(\bigoplus_{i,j,k=1}^{n} \frac{1}{n^3} d_i^s \otimes d_j^t \otimes d_k^r\right)^{1/(\alpha+\beta+\gamma)} \\
&= \bigcup_{\substack{\gamma\in h,\\ \eta\in g}} \left\{\left\{\left(\sqrt{1-\prod_{i,j,k=1}^{n}\left(1-\gamma_i^{2s}\gamma_j^{2t}\gamma_k^{2r}\right)^{\frac{1}{n^3}}}\right)^{1/(s+t+r)}\right\},\right. \\
&\left.\left\{\sqrt{1-\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\eta_i^2\right)^s\left(1-\eta_j^2\right)^t\left(1-\eta_k^2\right)^r\right)^{\frac{1}{n^3}}\right)^{1/(s+t+r)}}\right\}\right\}.
\end{aligned} \tag{57}$$

Hence, (50) is maintained. Thereafter:

$$0 \leqslant \left(\sqrt{1-\prod_{i,j,k=1}^{n}\left(1-\gamma_i^{2s}\gamma_j^{2t}\gamma_k^{2r}\right)^{\frac{1}{n^3}}}\right)^{1/(s+t+r)} \leqslant 1, \tag{58}$$

$$0 \leqslant \sqrt{1-\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\eta_i^2\right)^s\left(1-\eta_j^2\right)^t\left(1-\eta_k^2\right)^r\right)^{\frac{1}{n^3}}\right)^{1/(s+t+r)}} \leqslant 1. \tag{59}$$

Because $\gamma^2 + \eta^2 \leqslant 1$,

$$\begin{aligned}
&\left(\sqrt{1-\prod_{i,j,k=1}^{n}\left(1-\gamma_i^{2s}\gamma_j^{2t}\gamma_k^{2r}\right)^{\frac{1}{n^3}}}\right)^{1/(s+t+r)} \leqslant \\
&\sqrt{\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\eta_i^2\right)^s\left(1-\eta_j^2\right)^t\left(1-\eta_k^2\right)^r\right)^{\frac{1}{n^3}}\right)^{1/(s+t+r)}}.
\end{aligned} \tag{60}$$

Therefore,

$$\begin{aligned}&\left(\left(\sqrt{1-\prod_{i,j,k=1}^{n}\left(1-\gamma_i^{2s}\gamma_j^{2t}\gamma_k^{2r}\right)^{\frac{1}{n^3}}}\right)^{1/(s+t+r)}\right)^2 + \\ &\left(\sqrt{1-\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\eta_i^2\right)^s\left(1-\eta_j^2\right)^t\left(1-\eta_k^2\right)^r\right)^{\frac{1}{n^3}}\right)^{1/(s+t+r)}}\right)^2 \\ &\leqslant \left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\eta_i^2\right)^s\left(1-\eta_j^2\right)^t\left(1-\eta_k^2\right)^r\right)^{\frac{1}{n^3}}\right)^{1/(s+t+r)} + 1 \\ &-\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\eta_i^2\right)^s\left(1-\eta_j^2\right)^t\left(1-\eta_k^2\right)^r\right)^{\frac{1}{n^3}}\right)^{1/(s+t+r)} = 1.\end{aligned} \tag{61}$$

Thereby completing the proof.

Moreover, DHPFGBM has the following properties.

Property 7 (Idempotency), if d_i $(i = 1, 2, \cdots, n)$ are equal, that is $d_i = d = (h, g)$, then

$$\mathrm{DHPFGBM}^{s,t,r}\left(d_1, d_2, \cdots, d_n\right) = d. \tag{62}$$

Proof.

$$\begin{aligned}&\mathrm{DHPFGBM}^{s,t,r}\left(d_1, d_2, \cdots, d_n\right) \\ &= \left(\bigoplus_{i,j,k=1}^{n}\frac{1}{n^3}\left(d_i^s \otimes d_j^t \otimes d_k^r\right)\right)^{1/(s+t+r)} \\ &= d\left(\bigoplus_{i,j,k=1}^{n}\frac{1}{n^3}\right)^{1/(s+t+r)} \\ &= d.\end{aligned} \tag{63}$$

Property 8 (Monotonicity), let $d_j = \left(h_{d_j}, g_{d_j}\right)$ and $b_j = \left(h_{b_j}, g_{b_j}\right)$ $(j = 1, 2, 3, \cdots, n)$ be two sets of DHPFNs, If $\forall\left(\gamma_{d_j}\right)^2 \leqslant \forall\left(\gamma_{b_j}\right)^2$, $\gamma_{d_j} \in h_{d_j}$, $\gamma_{b_j} \in h_{b_j}$ and $\forall\left(\eta_{d_j}\right)^2 \geqslant \forall\left(\eta_{b_j}\right)^2$, $\eta_{d_j} \in g_{d_j}$, $\eta_{b_j} \in g_{b_j}$ then

$$\mathrm{DHPFGBM}^{s,t,r}\left(d_1, d_2, \cdots, d_n\right) \leqslant \mathrm{DHPFGBM}^{s,t,r}\left(b_1, b_2, \cdots, b_n\right). \tag{64}$$

Proof. We can obtain

$$\gamma_{d_i}^{2s}\gamma_{d_j}^{2t}\gamma_{d_k}^{2r} \leqslant \gamma_{b_i}^{2s}\gamma_{b_j}^{2t}\gamma_{b_k}^{2r}, \tag{65}$$

$$\prod_{i,j,k=1}^{n}\left(1-\gamma_{d_i}^{2s}\gamma_{d_j}^{2t}\gamma_{d_k}^{2r}\right) \geqslant \prod_{i,j,k=1}^{n}\left(1-\gamma_{b_i}^{2s}\gamma_{b_j}^{2t}\gamma_{b_k}^{2r}\right), \tag{66}$$

$$1-\prod_{i,j,k=1}^{n}\left(1-\gamma_{d_i}^{2s}\gamma_{d_j}^{2t}\gamma_{d_k}^{2r}\right)^{\frac{1}{n^3}} \leqslant 1-\prod_{i,j,k=1}^{n}\left(1-\gamma_{b_i}^{2s}\gamma_{b_j}^{2t}\gamma_{b_k}^{2r}\right)^{\frac{1}{n^3}}. \tag{67}$$

Therefore:

$$\begin{aligned}&\left(\sqrt{1-\prod_{i,j,k=1}^{n}\left(1-\gamma_{d_i}^{2s}\gamma_{d_j}^{2t}\gamma_{d_k}^{2r}\right)^{\frac{1}{n^3}}}\right)^{1/(s+t+r)} \leqslant \\ &\leqslant\left(\sqrt{1-\prod_{i,j,k=1}^{n}\left(1-\gamma_{b_i}^{2s}\gamma_{b_j}^{2t}\gamma_{b_k}^{2r}\right)^{\frac{1}{n^3}}}\right)^{1/(s+t+r)}.\end{aligned} \tag{68}$$

Thus,

$$\begin{aligned}&\left(\left(\sqrt{1-\prod_{i,j,k=1}^{n}\left(1-\gamma_{d_i}^{2s}\gamma_{d_j}^{2t}\gamma_{d_k}^{2r}\right)^{\frac{1}{n^3}}}\right)^{1/(s+t+r)}\right)^2 \leqslant \\ &\leqslant\left(\left(\sqrt{1-\prod_{i,j,k=1}^{n}\left(1-\gamma_{b_i}^{2s}\gamma_{b_j}^{2t}\gamma_{b_k}^{2r}\right)^{\frac{1}{n^3}}}\right)^{1/(s+t+r)}\right)^2,\end{aligned} \tag{69}$$

which means $\forall\gamma_d^2 \leqslant \forall\gamma_b^2$. Similarly, we can obtain $\forall\eta_d^2 \geqslant \forall\eta_b^2$.

If $\forall\gamma_{d_j}^2 < \forall\gamma_{b_j}^2$ and $\forall\eta_{d_j}^2 > \forall\eta_{b_j}^2$ then

$$\text{DHPFGBM}^{s,t,r}\left(d_1, d_2, \cdots, d_n\right) < \text{DHPFGBM}^{s,t,r}\left(b_1, b_2, \cdots, b_n\right);$$

If$\forall\gamma_{d_j}^2 < \forall\gamma_{b_j}^2$ and $\forall\eta_{d_j}^2 = \forall\eta_{b_j}^2$ then

$$\text{DHPFGBM}^{s,t,r}\left(d_1, d_2, \cdots, d_n\right) < \text{DHPFGBM}^{s,t,r}\left(b_1, b_2, \cdots, b_n\right);$$

If $\forall\gamma_{d_j}^2 = \forall\gamma_{b_j}^2$ and $\forall\eta_{d_j}^2 > \forall\eta_{b_j}^2$ then

$$\text{DHPFGBM}^{s,t,r}\left(d_1, d_2, \cdots, d_n\right) < \text{DHPFGBM}^{s,t,r}\left(b_1, b_2, \cdots, b_n\right);$$

If $\forall\gamma_{d_j}^2 = \forall\gamma_{b_j}^2$ and $\forall\eta_{d_j}^2 = \forall\eta_{b_j}^2$ then

$$\text{DHPFGBM}^{s,t,r}(d_1, d_2, \cdots, d_n) = \text{DHPFGBM}^{s,t,r}(b_1, b_2, \cdots, b_n).$$

Therefore, the proof of Property 8 is completed.

Property 9 (Boundedness), let $t, r > 0$ and $d_j = \left(h_{d_j}, g_{d_j}\right)$ $(j = 1, 2, 3, \cdots, n)$ be a collection of DHPFNs. If $d^+ = \cup_{\gamma_j \in h_{d_j}, \eta_j \in g_{d_j}} \{\{\max_i(\gamma_i)\}, \{\min_i(\eta_i)\}\}$ and $d^- = \cup_{\gamma_j \in h_{d_j}, \eta_j \in g_{d_j}} \{\{\min_i(\gamma_i)\}, \{\max_i(\eta_i)\}\}$, then

$$d^- \leqslant \text{DHPFGBM}^{s,t,r}(d_1, d_2, \cdots, d_n) \leqslant d^+. \tag{70}$$

Proof. From Property 7 we can obtain

$$\begin{aligned} &\text{DHPFGBM}^{s,t,r}(d^+, d^+, \cdots, d^+) \\ &\quad = d^+ \text{DHPFGBM}^{s,t,r}(d^-, d^-, \cdots, d^-) = d^-. \end{aligned} \tag{71}$$

From Property 8, we can obtain

$$\begin{aligned} d^- = {} & \text{DHPFGBM}^{s,t,r}(d^-, d^-, \cdots, d^-) \leqslant \\ & \text{DHPFGBM}^{s,t,r}(d_1, d_2, \cdots, d_n) \leqslant \\ & \text{DHPFGBM}^{s,t,r}(d^+, d^+, \cdots, d^+) = p^+. \end{aligned} \tag{72}$$

Therefore,

$$d^- \leqslant \text{DHPFGBM}^{s,t,r}(d_1, d_2, \cdots, d_n) \leqslant d^+. \tag{73}$$

4.2. DHPFGGBM operator

Thereafter, we extend GGBM operator [24] to DHPFN and propose the dual hesitant Pythagorean fuzzy generalized geometric Bonferroni mean (DHPFG-GBM) operator.

Definition 14 *Let $s, t, r > 0$, $d_j = \left(h_j, g_j\right)$ $(j = 1, 2, \cdots, n)$ be a set of DHPFN in which $h_P(x)$ and $g_P(x)$ are two sets of some values in* $[0, 1]$. *If*

$$DHPFGGBM^{s,t,r}(d_1, d_2, \cdots, d_n) = \frac{1}{s+t+r} \bigotimes_{i,j,k=1}^{n} \left(sd_i \oplus td_j \oplus rd_k\right)^{\frac{1}{n^3}}. \tag{74}$$

then $\text{DHPFGGBM}^{s,t,r}$ is called DHPFGGBM operator.

Theorem 4 *Let $s, t, r > 0$ and $d_j = \left(h_j, g_j\right)$ $(j = 1, 2, \cdots, n)$ be a collection of DHPFNs. The aggregated value by DHPFGGBM is also a DHPFN and*

$$\begin{aligned}
&DHPFGGBM^{s,t,r}\left(d_1, d_2, \cdots, d_n\right) \\
&= \frac{1}{s+t+r} \bigotimes_{i,j,k=1}^{n} \left(sd_i \oplus td_j \oplus rd_k\right)^{\frac{1}{n^3}} \\
&= \bigcup_{\substack{\gamma \in h,\\ \eta \in g}} \\
&\left\{\left\{\left(\sqrt{1-\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\gamma_i^2\right)^s\left(1-\gamma_j^2\right)^t\left(1-\gamma_k^2\right)^r\right)^{\frac{1}{n^3}}\right)}\right)^{1/(s+t+r)}\right\},\right. \\
&\left.\left\{\left(\sqrt{1-\prod_{i,j,k=1}^{n}\left(1-\eta_i^{2\alpha}\eta_j^{2\beta}\eta_k^{2\gamma}\right)^{\frac{1}{n^3}}}\right)^{1/(s+t+r)}\right\}\right\}.
\end{aligned} \tag{75}$$

Proof. Through Definition 4, we can obtain

$$sd_i = \bigcup_{\gamma_i \in h_i, \eta_i \in g_i} \left\{\left\{\sqrt{1-\left(1-\gamma_i^2\right)^s}\right\}, \left\{\eta_i^s\right\}\right\}, \tag{76}$$

$$td_j = \bigcup_{\gamma_j \in h_j, \eta_j \in g_j} \left\{\left\{\sqrt{1-\left(1-\gamma_j^2\right)^t}\right\}, \left\{\eta_j^t\right\}\right\}, \tag{77}$$

$$rd_k = \bigcup_{\gamma_k \in h_k, \eta_k \in g_k} \left\{\left\{\sqrt{1-\left(1-\gamma_k^2\right)^r}\right\}, \left\{\eta_k^r\right\}\right\}. \tag{78}$$

Thereafter,

$$\begin{aligned}
&sd_i \oplus td_j \oplus rd_k \\
&= \bigcup_{\substack{\gamma \in h,\\ \eta \in g}} \left\{\left\{\sqrt{1-\left(1-\gamma_i^2\right)^s\left(1-\gamma_j^2\right)^t\left(1-\gamma_k^2\right)^r}\right\}, \left\{\eta_i^s\eta_j^t\eta_k^r\right\}\right\}.
\end{aligned} \tag{79}$$

Thereafter,

$$\begin{aligned}&\left(sd_i \oplus td_j \oplus rd_k\right)^{\frac{1}{n^3}}\\ &= \bigcup_{\gamma\in h,\eta\in g}\left\{\left\{\left(\sqrt{1-\left(1-\gamma_i^2\right)^s\left(1-\gamma_j^2\right)^t\left(1-\gamma_k^2\right)^r}\right)^{\frac{1}{n^3}}\right\},\right.\\ &\qquad\left.\left\{\sqrt{1-\left(1-\eta_i^{2s}\eta_j^{2t}\eta_k^{2r}\right)^{\frac{1}{n^3}}}\right\}\right\}.\end{aligned} \tag{80}$$

Therefore,

$$\begin{aligned}&\bigotimes_{i,j,k=1}^{n}\left(sd_i \oplus td_j \oplus rd_k\right)^{\frac{1}{n^3}}\\ &= \bigcup_{\gamma\in h,\eta\in g}\left\{\left\{\left(\sqrt{\prod_{i,j,k=1}^{n}\left(1-\left(1-\gamma_i^2\right)^s\left(1-\gamma_j^2\right)^t\left(1-\gamma_k^2\right)^r\right)}\right)^{\frac{1}{n^3}}\right\},\right.\\ &\qquad\left.\left\{\sqrt{1-\prod_{i,j,k=1}^{n}\left(1-\eta_i^{2s}\eta_j^{2t}\eta_k^{2r}\right)^{\frac{1}{n^3}}}\right\}\right\}.\end{aligned} \tag{81}$$

Thus

$$\begin{aligned}&\mathrm{DHPFGGBM}^{s,t,r}\left(d_1, d_2, \cdots, d_n\right)\\ &= \frac{1}{s+t+r}\bigotimes_{i,j,k=1}^{n}\left(sd_i \oplus td_j \oplus rd_k\right)^{\frac{1}{n^3}}\\ &= \bigcup_{\substack{\gamma\in h,\\ \eta\in g}}\left\{\left(\sqrt{1-\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\gamma_i^2\right)^s\left(1-\gamma_j^2\right)^t\left(1-\gamma_k^2\right)^r\right)^{\frac{1}{n^3}}\right)}\right)^{1/(s+t+r)},\right.\\ &\qquad\left.\left\{\left(\sqrt{1-\prod_{i,j,k=1}^{n}\left(1-\eta_i^{2\alpha}\eta_j^{2\beta}\eta_k^{2\gamma}\right)^{\frac{1}{n^3}}}\right)^{1/(s+t+r)}\right\}\right\}.\end{aligned} \tag{82}$$

Hence, (75) is maintained. Thereafter:

$$0 \leqslant \sqrt{1-\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\gamma_i^2\right)^s\left(1-\gamma_j^2\right)^t\left(1-\gamma_k^2\right)^r\right)^{\frac{1}{n^3}}\right)}^{1/(s+t+r)} \leqslant 1, \tag{83}$$

$$0 \leqslant \left(\sqrt{1-\prod_{i,j,k=1}^{n}\left(1-\eta_i^{2s}\eta_j^{2t}\eta_k^{2r}\right)^{\frac{1}{n^3}}}\right)^{1/(s+t+r)} \leqslant 1. \tag{84}$$

Because $\gamma^2+\eta^2 \leqslant 1$,

$$\left(\sqrt{1-\prod_{i,j,k=1}^{n}\left(1-\eta_i^{2s}\eta_j^{2t}\eta_k^{2r}\right)^{\frac{1}{n^3}}}\right)^{1/(s+t+r)} \leqslant \sqrt{\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\gamma_i^2\right)^s\left(1-\gamma_j^2\right)^t\left(1-\gamma_k^2\right)^r\right)^{\frac{1}{n^3}}\right)^{1/(s+t+r)}}. \tag{85}$$

Therefore,

$$\begin{gathered}\left(\sqrt{1-\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\gamma_i^2\right)^s\left(1-\gamma_j^2\right)^t\left(1-\gamma_k^2\right)^r\right)^{\frac{1}{n^3}}\right)^{1/(s+t+r)}}\right)^2 + \\ \left(\left(\sqrt{1-\prod_{i,j,k=1}^{n}\left(1-\eta_i^{2s}\eta_j^{2t}\eta_k^{2r}\right)^{\frac{1}{n^3}}}\right)^{1/(s+t+r)}\right)^2 \leqslant \\ 1-\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\gamma_i^2\right)^s\left(1-\gamma_j^2\right)^t\left(1-\gamma_k^2\right)^r\right)^{\frac{1}{n^3}}\right)^{1/(s+t+r)} \\ +\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\gamma_i^2\right)^s\left(1-\gamma_j^2\right)^t\left(1-\gamma_k^2\right)^r\right)^{\frac{1}{n^3}}\right)^{1/(s+t+r)} = 1.\end{gathered} \tag{86}$$

Thereby completing the proof.

Similar to DHPFBM operator, the DHPFGGBM operator has the same properties. The proofs of these properties are similar to that of the properties of DHPFGGBM, Accordingly, the proofs are omitted to save space.

Property 10 (Idempotency), if d_i $(i = 1, 2, \cdots, n)$ are equal, that is $d_i = d = (h, g)$, then

$$\mathrm{DHPFGGBM}^{s,t,r}\left(d_1, d_2, \cdots, d_n\right) = d. \tag{87}$$

Property 11 (Monotonicity), let $d_j = \left(h_{d_j}, g_{d_j}\right)$ and $b_j = \left(h_{b_j}, g_{b_j}\right)$ $(j = 1, 2, 3, \cdots, n)$ be two sets of DHPFNs, If $\forall \left(\gamma_{d_j}\right)^2 \leqslant \forall \left(\gamma_{b_j}\right)^2$, $\gamma_{d_j} \in h_{d_j}$, $\gamma_{b_j} \in h_{b_j}$ and $\forall \left(\eta_{d_j}\right)^2 \geqslant \forall \left(\eta_{b_j}\right)^2$, $\eta_{d_j} \in g_{d_j}$, $\eta_{b_j} \in g_{b_j}$ then

$$\text{DHPFGGBM}^{s,t,r}\left(d_1, d_2, \cdots, d_n\right) \leqslant \text{DHPFGGBM}^{s,t,r}\left(d_1, d_2, \cdots, d_n\right). \tag{88}$$

Property 12 (Boundedness), let $t, r > 0$ and $d_j = (h_{d_j}, g_{d_j})$ $(j = 1, 2, 3, \cdots, n)$ be a collection of DHPFNs. If

$$d^+ = \bigcup_{\substack{\gamma_j \in h_{d_j},\\ \eta_j \in g_{d_j}}} \left\{\left\{\max_i \left(\gamma_i\right)\right\}, \left\{\min_i \left(\eta_i\right)\right\}\right\} \qquad \text{and}$$

$$d^- = \bigcup_{\substack{\gamma_j \in h_{d_j},\\ \eta_j \in g_{d_j}}} \left\{\left\{\min_i \left(\gamma_i\right)\right\}, \left\{\max_i \left(\eta_i\right)\right\}\right\},$$

then

$$d^- \leqslant \text{DHPFGGBM}^{s,t,r}\left(d_1, d_2, \cdots, d_n\right) \leqslant d^+. \tag{89}$$

5. Dual hesitant Pythagorean fuzzy dual Bonferroni mean operators

In the section, we go on with deriving the dual hesitant Pythagorean fuzzy dual Bonferroni mean (DHPFDBM) operator and dual hesitant Pythagorean fuzzy dual geometric Bonferroni mean (DHPFDGBM) operator.

5.1. DHPFDBM operator

Definition 15 *Let $l_j > 0$ and $d_i = (h_i, g_i)$ $(i = 1, 2, \cdots, n)$ be a set of DHPFNs in which $h_P(x)$ and $g_P(x)$ are two sets of some values in* $[0, 1]$. *If*

$$DHPFDBM^l\left(d_1, d_2, \cdots, d_n\right) = \left(\bigoplus_{i_1, i_2, \cdots, i_n = 1}^{n} \left(\bigotimes_{j=1}^{n} \frac{1}{n} d_{i_j}^{l_j}\right)\right)^{1/\sum_{i=1}^{n} l_j}. \tag{90}$$

Then the DHPFDBM^l is called the DHPFDBM operator.

Theorem 5 *Let $l_j > 0$ and $d_i = (h_i, g_i)$ $(i = 1, 2, \cdots, n)$, $\gamma_i \in h_i$, $\eta_i \in g_i$ be a collection of DHPFNs. The aggregated result of DHPFDBM is a DHPFN.*

$$DHPFDBM^{l}(d_1, d_2, \cdots, d_n) = \left(\bigoplus_{i_1, i_2, \cdots, i_n = 1}^{n} \left(\bigotimes_{j=1}^{n} \frac{1}{n} d_{i_j}^{l_j} \right) \right)^{1/\sum_{i=1}^{n} l_j}$$
$$= \bigcup_{\gamma \in h, \eta \in g} \left\{ \left\{ \left(\sqrt{1 - \prod_{i_1, i_2, \cdots, i_n = 1}^{n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - \gamma_{i_j}^{2l_j} \right)^{\frac{1}{n}} \right) \right)} \right)^{1/\sum_{j=1}^{n} l_j} \right\}, \right. \tag{91}$$
$$\left. \left\{ \sqrt{1 - \left(1 - \prod_{i_1, i_2, \cdots, i_n = 1}^{n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - \left(1 - \eta_{i_j}^2 \right)^{l_j} \right)^{\frac{1}{n}} \right) \right) \right)^{1/\sum_{j=1}^{n} l_j}} \right\} \right\}.$$

Proof.

$$d_{i_j}^{l_j} = \bigcup_{\substack{\gamma_{i_j} \in h_{i_j}, \\ \eta_{i_j} \in g_{i_j}}} \left\{ \left\{ \left(\gamma_{i_j} \right)^{l_j} \right\}, \left\{ \sqrt{1 - \left(1 - \eta_{i_j}^2 \right)^{l_j}} \right\} \right\}, \tag{92}$$

$$\frac{1}{n} d_{i_j}^{l_j} = \bigcup_{\substack{\gamma_{i_j} \in h_{i_j}, \\ \eta_{i_j} \in g_{i_j}}} \left\{ \left\{ \sqrt{1 - \left(1 - \gamma_{i_j}^{2l_j} \right)^{\frac{1}{n}}} \right\}, \left\{ \left(\sqrt{1 - \left(1 - \eta_{i_j}^2 \right)^{l_j}} \right)^{\frac{1}{n}} \right\} \right\}. \tag{93}$$

Thus,

$$\bigotimes_{j=1}^{n} \frac{1}{n} d_{i_j}^{l_j} = \bigcup_{\substack{\gamma_{i_j} \in h_{i_j}, \\ \eta_{i_j} \in g_{i_j}}} \left\{ \left\{ \prod_{j=1}^{n} \sqrt{1 - \left(1 - \gamma_{i_j}^{2l_j} \right)^{\frac{1}{n}}} \right\}, \right. \tag{94}$$
$$\left. \left\{ \sqrt{1 - \prod_{j=1}^{n} \left(1 - \left(1 - \left(1 - \eta_{i_j}^2 \right)^{l_j} \right)^{\frac{1}{n}} \right)} \right\} \right\}.$$

Thereafter,

$$\bigoplus_{i_1,i_2,\cdots,i_n=1}^{n}\left(\bigotimes_{j=1}^{n}\frac{1}{n}d_{i_j}^{l_j}\right)$$

$$= \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}}\left\{\left\{\sqrt{1-\prod_{i_1,i_2,\cdots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\gamma_{i_j}^{2l_j}\right)^{\frac{1}{n}}\right)\right)}\right\},\right. \tag{95}$$

$$\left.\left\{\prod_{i_1,i_2,\cdots,i_n=1}^{n}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(1-\left(1-\eta_{i_j}^{2}\right)^{l_j}\right)^{\frac{1}{n}}\right)}\right\}\right\}.$$

Therefore,

$$\mathrm{DHPFDBM}^{l}\left(d_1,d_2,\cdots,d_n\right)=\left(\bigoplus_{i_1,i_2,\cdots,i_n=1}^{n}\left(\bigotimes_{j=1}^{n}\frac{1}{n}d_{i_j}^{l_j}\right)\right)^{1/\sum\limits_{i=1}^{n}l_j}$$

$$= \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}}\left\{\left\{\left(\sqrt{1-\prod_{i_1,i_2,\cdots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\gamma_{i_j}^{2l_j}\right)^{\frac{1}{n}}\right)\right)}\right)^{1/\sum\limits_{j=1}^{n}l_j}\right\},\right. \tag{96}$$

$$\left.\left\{\sqrt{1-\left(1-\prod_{i_1,i_2,\cdots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(1-\eta_{i_j}^{2}\right)^{l_j}\right)^{\frac{1}{n}}\right)\right)\right)^{1/\sum\limits_{j=1}^{n}l_j}}\right\}\right\}.$$

Hence, (91) is maintained. Thereafter:

$$0\leqslant\left(\sqrt{1-\prod_{i_1,i_2,\cdots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\gamma_{i_j}^{2l_j}\right)^{\frac{1}{n}}\right)\right)}\right)^{1/\sum\limits_{j=1}^{n}l_j}\leqslant 1, \tag{97}$$

$$0\leqslant\sqrt{1-\left(1-\prod_{i_1,i_2,\cdots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(1-\eta_{i_j}^{2}\right)^{l_j}\right)^{\frac{1}{n}}\right)\right)\right)^{1/\sum\limits_{j=1}^{n}l_j}}\leqslant 1. \tag{98}$$

Because $\gamma^2 + \eta^2 \leqslant 1$,

$$\left(\left(\sqrt{1-\prod_{i_1,i_2,\cdots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\gamma_{i_j}^{2l_j}\right)^{\frac{1}{n}}\right)\right)}\right)^{1/\sum\limits_{j=1}^{n} l_j}\right)^2 + \left(\sqrt{1-\left(1-\prod_{i_1,i_2,\cdots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(1-\eta_{i_j}^2\right)^{l_j}\right)^{\frac{1}{n}}\right)\right)\right)^{1/\sum\limits_{j=1}^{n} l_j}}\right)^2 \leqslant$$
$$\left(1-\prod_{i_1,i_2,\cdots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\gamma_{i_j}^{2l_j}\right)^{\frac{1}{n}}\right)\right)\right)^{1/\sum\limits_{j=1}^{n} l_j} + \left(1-\left(1-\prod_{i_1,i_2,\cdots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\gamma_{i_j}^{2l_j}\right)^{\frac{1}{n}}\right)\right)\right)^{1/\sum\limits_{j=1}^{n} l_j}\right) = 1. \tag{99}$$

Property 13 (Idempotency), if d_i $(i = 1, 2, \cdots, n)$ are equal, that is $d_i = d = (h, g)$, then

$$\mathrm{DHPFDBM}^l\left(d_1, d_2, \cdots, d_n\right) = d. \tag{100}$$

Proof.

$$\begin{aligned} &\mathrm{DHPFDBM}^l\left(d_1, d_2, \cdots, d_n\right) \\ &= \left(\bigoplus_{i_1,i_2,\cdots,i_n=1}^{n}\left(\bigotimes_{j=1}^{n}\frac{1}{n}d^{l_j}\right)\right)^{1/\sum\limits_{i=1}^{n} l_j} \\ &= \left(\bigoplus_{i_1,i_2,\cdots,i_n=1}^{n}\left(\left(\bigotimes_{j=1}^{n}\frac{1}{n}\right)\cdot d^{\sum\limits_{i=1}^{n} l_j}\right)\right)^{1/\sum\limits_{i=1}^{n} l_j} \\ &= \left(\bigoplus_{i_1,i_2,\cdots,i_n=1}^{n}\left(\bigotimes_{j=1}^{n}\frac{1}{n}\right)\right)^{1/\sum\limits_{i=1}^{n} l_j}\cdot d = d. \end{aligned} \tag{101}$$

Property 14 (Monotonicity), let $d_j = \left(h_{d_j}, g_{d_j}\right)$ and $b_j = \left(h_{b_j}, g_{b_j}\right)$ $(j = 1, 2, 3, \cdots, n)$ be two sets of DHPFNs, if $\forall \left(\gamma_{d_j}\right)^2 \leqslant \forall \left(\gamma_{b_j}\right)^2$, $\gamma_{d_j} \in h_{d_j}$, $\gamma_{b_j} \in h_{b_j}$ and $\forall \left(\eta_{d_j}\right)^2 \geqslant \forall \left(\eta_{b_j}\right)^2$, $\eta_{d_j} \in g_{d_j}$, $\eta_{b_j} \in g_{b_j}$ then

$$\text{DHPFDBM}^l\left(d_1, d_2, \cdots, d_n\right) \leqslant \text{DHPFDBM}^l\left(b_1, b_2, \cdots, q_n\right). \tag{102}$$

Proof.

$$\gamma_{d_{i_j}} \leqslant \gamma_{b_{i_j}}, \tag{103}$$

$$\left(1 - \gamma_{d_{i_j}}^{2l_j}\right)^{\frac{1}{n}} \geqslant \left(1 - \gamma_{b_{i_j}}^{2l_j}\right)^{\frac{1}{n}}, \tag{104}$$

$$\prod_{j=1}^{n}\left(1 - \left(1 - \gamma_{p_{i_j}}^{2l_j}\right)^{\frac{1}{n}}\right) \leqslant \prod_{j=1}^{n}\left(1 - \left(1 - \gamma_{b_{i_j}}^{2l_j}\right)^{\frac{1}{n}}\right). \tag{105}$$

Therefore:

$$\begin{gathered}\prod_{i_1, i_2, \cdots, i_n=1}^{n}\left(1 - \prod_{j=1}^{n}\left(1 - \left(1 - \gamma_{d_{i_j}}^{2l_j}\right)^{\frac{1}{n}}\right)\right) \geqslant \\ \prod_{i_1, i_2, \cdots, i_n=1}^{n}\left(1 - \prod_{j=1}^{n}\left(1 - \left(1 - \gamma_{b_{i_j}}^{2l_j}\right)^{\frac{1}{n}}\right)\right).\end{gathered} \tag{106}$$

Thus:

$$\begin{gathered}\left(\sqrt{1 - \prod_{i_1, i_2, \cdots, i_n=1}^{n}\left(1 - \prod_{j=1}^{n}\left(1 - \left(1 - \gamma_{d_{i_j}}^{2l_j}\right)^{\frac{1}{n}}\right)\right)}\right)^{1/\sum_{j=1}^{n} l_j} \leqslant \\ \left(\sqrt{1 - \prod_{i_1, i_2, \cdots, i_n=1}^{n}\left(1 - \prod_{j=1}^{n}\left(1 - \left(1 - \gamma_{b_{i_j}}^{2l_j}\right)^{\frac{1}{n}}\right)\right)}\right)^{1/\sum_{j=1}^{n} l_j},\end{gathered} \tag{107}$$

which means $\forall \gamma_d^2 \leqslant \forall \gamma_b^2$, similarly, we can obtain $\forall \eta_d^2 \geqslant \forall \eta_b^2$.

If $\forall \gamma_{d_j}^2 < \forall \gamma_{b_j}^2$ and $\forall \eta_{d_j}^2 > \forall \eta_{b_j}^2$ then

$$\text{DHPFDBM}^l\left(d_1, d_2, \cdots, d_n\right) < \text{DHPFDBM}^l\left(b_1, b_2, \cdots, b_n\right);$$

If $\forall\gamma_{d_j}^2 < \forall\gamma_{b_j}^2$ and $\forall\eta_{d_j}^2 = \forall\eta_{b_j}^2$ then

$$\text{DHPFDBM}^l(d_1, d_2, \cdots, d_n) < \text{DHPFDBM}^l(b_1, b_2, \cdots, b_n);$$

If $\forall\gamma_{d_j}^2 = \forall\gamma_{b_j}^2$ and $\forall\eta_{d_j}^2 > \forall\eta_{b_j}^2$ then

$$\text{DHPFDBM}^l(d_1, d_2, \cdots, d_n) < \text{DHPFDBM}^l(b_1, b_2, \cdots, b_n);$$

If $\forall\gamma_{d_j}^2 = \forall\gamma_{b_j}^2$ and $\forall\eta_{d_j}^2 = \forall\eta_{b_j}^2$ then

$$\text{DHPFDBM}^l(d_1, d_2, \cdots, d_n) = \text{DHPFDBM}^l(b_1, b_2, \cdots, b_n).$$

Therefore, the proof of property 14 is completed.

Property 15 (Boundedness), let $t, r > 0$ and $d_j = \left(h_{d_j}, g_{d_j}\right)$ $(j = 1, 2, 3, \cdots, n)$ be a collection of DHPFNs. If

$$d^+ = \bigcup_{\substack{\gamma_j \in h_{d_j},\\ \eta_j \in g_{d_j}}} \left\{\left\{\max_i(\gamma_i)\right\}, \left\{\min_i(\eta_i)\right\}\right\} \qquad \text{and}$$

$$d^- = \bigcup_{\substack{\gamma_j \in h_{d_j},\\ \eta_j \in g_{d_j}}} \left\{\left\{\min_i(\gamma_i)\right\}, \left\{\max_i(\eta_i)\right\}\right\},$$

according to the property, there is

$$d^- \leqslant \text{DHPFDBM}^l(d_1, d_2, \cdots, d_n) \leqslant d^+. \tag{108}$$

Proof. From property 13 we can obtain

$$\begin{aligned} \text{DHPFDBM}^l(d^+, d^+, \cdots, d^+) &= d^+,\\ \text{DHPFDBM}^l(d^-, d^-, \cdots, d^-) &= d^-. \end{aligned} \tag{109}$$

From property 14, we can obtain

$$\begin{aligned} d^- = \text{DHPFDBM}^l(d^-, d^-, \cdots, d^-) &\leqslant\\ \text{DHPFDBM}^l(d_1, d_2, \cdots, d_n) &\leqslant\\ \text{DHPFDBM}^l(d^+, d^+, \cdots, d^+) &= d^+. \end{aligned} \tag{110}$$

Therefore,

$$d^- \leqslant \text{DHPFDBM}^l(d_1, d_2, \cdots, d_n) \leqslant d^+. \tag{111}$$

5.2. DHPFDGBM operator

Thereafter, we extend DGBM to DHPFN and introduced the dual hesitant Pythagorean fuzzy DGBM (DHPFDGBM) operator.

Definition 16 *Let $l_j > 0$ and $d_i = (h_i, g_i)$ $(i = 1, 2, \cdots, n)$ be a set of DHPFN in which $h_P(x)$ and $g_P(x)$ are two sets of some values in* $[0, 1]$. *Then*

$$DHPFDGBM^{l}\left(d_1, d_2, \cdots, d_n\right) = \frac{1}{\sum\limits_{i=1}^{n} l_j}\left(\bigotimes_{i_1,i_2,\cdots,i_n=1}^{n}\left(\bigoplus_{j=1}^{n}\left(l_j d_{i_j}\right)\right)^{\prod\limits_{j=1}^{n}\frac{1}{n}}\right). \tag{112}$$

Then $DHPFDGBM^{l}_{\omega}$ is called DHPFDGBM operator.

Theorem 6 *Let $l_j > 0$ and $d_i = (h_i, g_i)$ $(i = 1, 2, \cdots, n)$, $\gamma_i \in h_i$, $\eta_i \in g_i$ be a collection of DHPFNs. The aggregated result of DHPFDGBM is also a DHPFN*

$$DHPFDGBM^{l}\left(d_1, d_2, \cdots, d_n\right) = \frac{1}{\sum\limits_{i=1}^{n} l_j}\left(\bigotimes_{i_1,i_2,\cdots,i_n=1}^{n}\left(\bigoplus_{j=1}^{n}\left(l_j d_{i_j}\right)\right)^{\prod\limits_{j=1}^{n}\frac{1}{n}}\right)$$

$$= \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}}\left\{\left\{\sqrt{1-\left(1-\prod_{i_1,i_2,\cdots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\gamma_{i_j}^{2}\right)^{l_j}\right)^{\prod\limits_{j=1}^{n}\frac{1}{n}}\right)^{1/\sum\limits_{i=1}^{n} l_j}}\right\}, \tag{113}$$

$$\left\{\left(\sqrt{1-\prod_{i_1,i_2,\cdots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\eta_{i_j}^{2l_j}\right)^{\prod\limits_{j=1}^{n}\frac{1}{n}}}\right)^{1/\sum\limits_{i=1}^{n} l_j}\right\}\right\}.$$

Proof. Through Definition 4, we can obtain

$$l_j d_{i_j} = \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}}\left\{\left\{\sqrt{1-\left(1-\gamma_{i_j}^{2}\right)^{l_j}}\right\}, \left\{\eta_{i_j}^{l_j}\right\}\right\}. \tag{114}$$

Thereafter,

$$\bigoplus_{j=1}^{n} l_j d_{i_j} = \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{\left\{\sqrt{1-\prod_{j=1}^{n}\left(1-\gamma_{i_j}^{2}\right)^{l_j}}\right\},\left\{\prod_{j=1}^{n}\eta_{i_j}^{l_j}\right\}\right\}. \tag{115}$$

Thereafter,

$$\begin{aligned}\left(\bigoplus_{j=1}^{n} l_j d_{i_j}\right)^{\prod_{j=1}^{n}\frac{1}{n}} &= \\ &= \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{\left\{\left(\sqrt{1-\prod_{j=1}^{n}\left(1-\gamma_{i_j}^{2}\right)^{l_j}}\right)^{\prod_{j=1}^{n}\frac{1}{n}}\right\},\right. \\ &\qquad \left.\left\{\sqrt{1-\left(1-\prod_{j=1}^{n}\eta_{i_j}^{2l_j}\right)^{\prod_{j=1}^{n}\frac{1}{n}}}\right\}\right\}.\end{aligned} \tag{116}$$

Therefore,

$$\begin{aligned}\bigotimes_{i_1,i_2,\cdots,i_n=1}^{n}\left(\bigoplus_{j=1}^{n} l_j d_{i_j}\right)^{\prod_{j=1}^{n}\frac{1}{n}} &= \\ &= \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{\left\{\prod_{i_1,i_2,\cdots,i_n=1}^{n}\left(\sqrt{1-\prod_{j=1}^{n}\left(1-\gamma_{i_j}^{2}\right)^{l_j}}\right)^{\prod_{j=1}^{n}\frac{1}{n}}\right\},\right. \\ &\qquad \left.\left\{\sqrt{1-\prod_{i_1,i_2,\cdots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\eta_{i_j}^{2l_j}\right)^{\prod_{j=1}^{n}\frac{1}{n}}}\right\}\right\}.\end{aligned} \tag{117}$$

Thus

$$\mathrm{DHPFDGBM}^{l}\left(d_1, d_2, \cdots, d_n\right) = \frac{1}{\sum_{i=1}^{n} l_j}\left(\bigotimes_{i_1,i_2,\cdots,i_n=1}^{n}\left(\bigoplus_{j=1}^{n}\left(l_j d_{i_j}\right)\right)^{\prod_{j=1}^{n}\frac{1}{n}}\right)$$

$$= \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{\left\{\sqrt{1-\left(1-\prod_{i_1,i_2,\cdots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\gamma_{i_j}^2\right)^{l_j}\right)^{\prod_{j=1}^{n}\frac{1}{n}}\right)^{1/\sum_{i=1}^{n} l_j}}\right\},\right.$$
$$\left.\left\{\left(\sqrt{1-\prod_{i_1,i_2,\cdots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\eta_{i_j}^{2l_j}\right)^{\prod_{j=1}^{n}\frac{1}{n}}}\right)^{1/\sum_{i=1}^{n} l_j}\right\}\right\}. \tag{118}$$

Hence, (113) is maintained. Thereafter:

$$1 \leqslant \sqrt{1-\left(1-\prod_{i_1,i_2,\cdots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\gamma_{i_j}^2\right)^{l_j}\right)^{\prod_{j=1}^{n}\frac{1}{n}}\right)^{1/\sum_{i=1}^{n} l_j}} \leqslant 0, \tag{119}$$

$$1 \leqslant \left(\sqrt{1-\prod_{i_1,i_2,\cdots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\eta_{i_j}^{2l_j}\right)^{\prod_{j=1}^{n}\frac{1}{n}}}\right)^{1/\sum_{i=1}^{n} l_j} \leqslant 0. \tag{120}$$

Because of $\gamma^2 + \eta^2 \leqslant 1$,

$$\left(\sqrt{1-\prod_{i_1,i_2,\cdots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\eta_{i_j}^{2l_j}\right)^{\prod_{j=1}^{n}\frac{1}{n}}}\right)^{1/\sum_{i=1}^{n} l_j} \leqslant \sqrt{1-\prod_{i_1,i_2,\cdots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\gamma_{i_j}^2\right)^{l_j}\right)^{\prod_{j=1}^{n}\frac{1}{n}}}^{\,1/\sum_{i=1}^{n} l_j}. \tag{121}$$

Therefore,

$$\left(\sqrt{1-\left(1-\prod_{i_1,i_2,\cdots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\gamma_{i_j}^2\right)^{l_j}\right)^{\prod_{j=1}^{n}\frac{1}{n}}\right)^{1/\sum_{i=1}^{n}l_j}}\right)^2+$$

$$\left(\left(\sqrt{1-\prod_{i_1,i_2,\cdots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\eta_{i_j}^{2l_j}\right)^{\prod_{j=1}^{n}\frac{1}{n}}}\right)^{1/\sum_{i=1}^{n}l_j}\right)^2\leqslant \tag{122}$$

$$1-\left(1-\prod_{i_1,i_2,\cdots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\gamma_{i_j}^2\right)^{l_j}\right)^{\prod_{j=1}^{n}\frac{1}{n}}\right)^{1/\sum_{i=1}^{n}l_j}+$$

$$\left(1-\prod_{i_1,i_2,\cdots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\gamma_{i_j}^2\right)^{l_j}\right)^{\prod_{j=1}^{n}\frac{1}{n}}\right)^{1/\sum_{i=1}^{n}l_j}=1.$$

Thereby completing the proof.

Similar to DHPFDBM, the DHPFDGBM has the same properties. The proofs of these properties are similar to that of the properties of DHPFDGBM, Accordingly, the proofs are omitted to save space.

Property 16 (Idempotency), if d_i $(i=1,2,\cdots,n)$ are equal, that is $d_i=d=(h,g)$, then

$$\text{DHPFDGBM}^l\left(d_1,d_2,\cdots,d_n\right)=d. \tag{123}$$

Property 17 (Monotonicity), let $d_j=\left(h_{d_j},g_{d_j}\right)$ and $b_j=\left(h_{b_j},g_{b_j}\right)$ $(j=1,2,3,\cdots,n)$ be two sets of DHPFNs, If $\forall\left(\gamma_{d_j}\right)^2\leqslant\forall\left(\gamma_{b_j}\right)^2$, $\gamma_{d_j}\in h_{d_j},\gamma_{b_j}\in h_{b_j}$ and $\forall\left(\eta_{d_j}\right)^2\geqslant\forall\left(\eta_{b_j}\right)^2$, $\eta_{d_j}\in g_{d_j},\eta_{b_j}\in g_{b_j}$ then

$$\text{DHPFDGBM}^l\left(d_1,d_2,\cdots,d_n\right)\leqslant\text{DHPFDGBM}^l\left(b_1,b_2,\cdots,b_n\right). \tag{124}$$

Property 18 (Boundedness), if

$$d^+ = \bigcup_{\substack{\gamma_j \in h_{d_j}, \\ \eta_j \in g_{d_j}}} \left\{ \left\{ \max_i (\gamma_i) \right\}, \left\{ \min_i (\eta_i) \right\} \right\} \qquad \text{and}$$

$$d^- = \bigcup_{\substack{\gamma_j \in h_{d_j}, \\ \eta_j \in g_{d_j}}} \left\{ \left\{ \min_i (\gamma_i) \right\}, \left\{ \max_i (\eta_i) \right\} \right\},$$

according to the property, there is

$$d^- \leqslant \text{DHPFDGBM}^l (d_1, d_2, \cdots, d_n) \leqslant d^+. \tag{125}$$

6. Models for MADM with DHPFNs

Based the DHPFBM and DHPFGBM operators, in this section, we shall propose the model for MADM with DHPFNs. Let $A = \{A_1, A_2, \cdots, A_m\}$ be a discrete set of alternatives, and $G = \{G_1, G_2, \cdots, G_n\}$ be the set of attributes. Suppose that $d = \left(d_{cj}\right)_{m \times n} = \left(h_{cj}, g_{cj}\right)_{m \times n}$ is the Pythagorean fuzzy decision matrix, where h_{cj} set indicates the degree that the alternative A_c satisfies the attribute G_j given by the decision maker, g_{cj} set indicates the degree that the alternative A_c doesn't satisfy the attribute G_j given by the decision maker, $\gamma_{cj} \in h_{cj} \subset [0, 1]$, $\eta_{cj} \in g_{cj} \subset [0, 1]$, $\left(\gamma_{cj}\right)^2 + \left(\eta_{cj}\right)^2 \leqslant 1$, $c = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$.

In the following, we apply the DHPFBM (DHPFGBM) operator to the MADM problems with DHPFNs.

Step 1. We utilize the DHPFNs given in matrix $\widetilde{R}$, and the DHPFBM operator

$$d_c = \text{DHPFBM}^{t,r} (d_{c1}, d_{c2}, \cdots, d_{cn}) = \left(\bigoplus_{i,j=1}^{n} \frac{1}{n^2} \left(d_{ci}^t \otimes d_{cj}^r \right) \right)^{1/(t+r)} =$$

$$\bigcup_{\gamma \in h, \eta \in g} \left\{ \left\{ \left(\sqrt{1 - \prod_{i,j=1}^{n} \left(1 - \gamma_{ci}^{2t} \gamma_{cj}^{2r} \right)^{\frac{1}{n^2}}} \right)^{1/(t+r)} \right\}, \quad c = 1, 2, \cdots, m \right. \tag{126}$$

$$\left. \left\{ \sqrt{1 - \left(1 - \prod_{i,j=1}^{n} \left(1 - \left(1 - \eta_{ci}^2 \right)^t \left(1 - \eta_{cj}^2 \right)^r \right)^{\frac{1}{n^2}} \right)^{1/(t+r)}} \right\} \right\}$$

or

$$d_c = \text{DHPFGBM}_{\omega}^{t,r}\left(d_{c1}, d_{c2}, \cdots, d_{cn}\right) = \bigcup_{\gamma\in h,\eta\in g}\left\{\left\{\left(\sqrt{1-\left(1-\prod_{i,j=1}^{n}\left(1-\left(1-\gamma_{ci}^2\right)^t\left(1-\gamma_{cj}^2\right)^r\right)^{\frac{1}{n^2}}\right)}\right)^{1/(t+r)}\right\}, \left\{\left(\sqrt{1-\prod_{i,j=1}^{n}\left(1-\eta_{ci}^{2t}\eta_{cj}^{2r}\right)^{\frac{1}{n^2}}}\right)^{1/(t+r)}\right\}\right\}, \quad c = 1, 2, \cdots, m \tag{127}$$

to derive the d_c $(c = 1, 2, \cdots, m)$ of the alternative A_c.

Step 2. Calculate the scores $S(d_c)$ $(c = 1, 2, \cdots, m)$ of the overall DHPFNs d_c $(c = 1, 2, \cdots, m)$ to rank all the alternatives A_c $(c = 1, 2, \cdots, m)$ and then to select the best one(s). If there is no difference between two scores $S(d_c)$ and $S(d_{c1})$, then we need to calculate the accuracy degrees $H(d_c)$ and $H(d_{c1})$ of the overall DHPFNs d_c and d_{c1}, respectively, and then rank the alternatives A_c and A_{c1} in accordance with the accuracy degrees $H(d_c)$ and $H(d_{c1})$.

Step 3. Rank all the alternatives A_c $(c = 1, 2, \cdots, m)$ and select the best one(s) in accordance with $S(d_c)$ $(c = 1, 2, \cdots, m)$.

Step 4. End.

7. Numerical example and comparative analysis

7.1. Numerical example

In this section, we shall give an application to select green suppliers in green supply chain management with DHPFNs. There are five possible green suppliers in green supply chain management O_i $(i = 1, 2, 3, 4, 5)$ to select. The experts select four attribute to assess the five possible green suppliers:

1) G_1 is the product quality factor;
2) G_2 is environmental factors;
3) G_3 is delivery factor;
4) G_4 is price factor.

Five green suppliers O_i $(i = 1, 2, 3, 4, 5)$ are to be assessed with DHPFNs according to four attributes (whose $t = r = 3$, $s = t = r = 3, l_i = 3, i = 1, \cdots, 4$ as shown in Table 1.

Table 1: DHPFN decision matrix.

	G_1	G_2
O_1	{(0.5,0.4), (0.5,0.3)}	{(0.6,0.5), (0.3,0.2), (0.4,0.2)}
O_2	{(0.3,0.2), (0.4,0.2)}	{(0.6,0.1), (0.4,0.3)}
O_3	{(0.5,0.3), (0.8,0.3)}	{(0.7,0.3), (0.5,0.4)}
O_4	{(0.4,0.6), (0.5,0.4)}	{(0.6,0.5), (0.6,0.7)}
O_5	{(0.5,0.3), (0.6,0.5)}	{(0.5,0.4), (0.6,0.4)}
	G_3	G_4
O_1	{(0.3,0.5), (0.4,0.3)}	{(0.4,0.3), (0.5,0.3)}
O_2	{(0.4,0.3), (0.6,0.4)}	{(0.4,0.6), (0.3,0.4), (0.5,0.6)}
O_3	{(0.6,0.2), (0.5,0.4), (0.6,0.1)}	{(0.6,0.3), (0.5,0.3)}
O_4	{(0.6,0.3), (0.7,0.4)}	{(0.5,0.3), (0.5,0.4)}
O_5	{(0.6,0.4), (0.6,0.5)}	{(0.2,0.3), (0.3,0.4)}

Step 1. We utilize the DHPFNs given in matrix $\widetilde{R}$, and the DHPFBM operator to get aggregation results, we illustrate one of alternative for save space.

$$
\begin{aligned}
O_1 &= \mathrm{DHPFBM}^{t,r}(G_1, G_2, G_3, G_4) \\
&= \{\{(0.5, 0.4), (0.5, 0.3)\}, \{(0.6, 0.5), (0.3, 0.2), (0.4, 0.2)\}, \\
&\quad \{(0.3, 0.5), (0.4, 0.3)\}, \{(0.4, 0.3), (0.5, 0.3)\}\} \\
&= \left\{\begin{matrix}
(0.506, 0.4215), (0.5195, 0.4215), (0.5101, 0.3713), (0.5232, 0.3713), \\
(0.4175, 0.3442), (0.4489, 0.3442), (0.4279, 0.2983), (0.4562, 0.2983), \\
(0.4279, 0.3442), (0.4562, 0.3442), (0.4371, 0.2983), (0.4631, 0.2983), \\
(0.506, 0.3945), (0.5195, 0.3945,), (0.5101, 0.3456), (0.5232, 0.3456), \\
(0.4175, 0.3187), (0.4489, 0.3187), (0.4279, 0.2747), (0.4562, 0.2747), \\
(0.4279, 0.3187), (0.4562, 0.3187), (0.4371, 0.2747), (0.4631, 0.2747)
\end{matrix}\right\}.
\end{aligned}
$$

Step 2. According to the aggregating results and the score functions of the green suppliers are shown in Table 2.

Table 2: The rank and score of green suppliers by using DHPF operators.

	O_1	O_2	O_3	O_4	O_5	Order
DHPFBM	0.5527	0.5709	0.6540	0.5717	0.5703	$O_3 > O_4 > O_2 > O_5 > O_1$
DHPFGBM	0.5174	0.4966	0.6160	0.5095	0.5228	$O_3 > O_5 > O_1 > O_4 > O_2$
DHPFGBM	0.5514	0.5672	0.6527	0.5694	0.5696	$O_3 > O_5 > O_4 > O_2 > O_1$
DHPFGGBM	0.5185	0.4981	0.6166	0.5104	0.5270	$O_3 > O_5 > O_1 > O_4 > O_2$
DHPFDBM	0.5934	0.6117	0.6949	0.6100	0.6095	$O_3 > O_2 > O_4 > O_5 > O_1$
DHPFDGBM	0.5190	0.4987	0.6168	0.5107	0.5285	$O_3 > O_5 > O_1 > O_4 > O_2$

According the result of green suppliers order, we can know that the best choice is supplier 3, we get same result by different aggregation, that proved the effectiveness of result.

7.2. Influence of the Parameter on the Final Result

The aggregation method of extend DHPFS with BM has two advantages, one is that it can reduce the bad effects of the unduly high and low assessments on the final result, the other is that it can capture the interrelationship between dual hesitate Pythagorean fuzzy numbers. These aggregation operators have a parameter vector, which make extended operator more flexible, so the different vector lead to different aggregation results, different scores and ranking results. In order to illustrate the influence of the parameter vector l_i on the ranking result, we discuss the influence with several parameter vectors, the result you can find in Table 3 and Table 4.

Table 3: Ranking results by utilizing different parameter vector l_i in the DHPFDBM operator.

$l_i, i = 1, \cdots, 6$	Scores					Order
	O_1	O_2	O_3	O_4	O_5	
(1,1,1,1)	0.6074	0.6131	0.7188	0.6731	0.6468	$O_3 > O_4 > O_5 > O_2 > O_1$
(2,2,2,2)	0.5991	0.6105	0.6971	0.6276	0.6244	$O_3 > O_4 > O_5 > O_2 > O_1$
(3,3,3,3)	0.5934	0.6117	0.6949	0.6100	0.6095	$O_3 > O_2 > O_4 > O_5 > O_1$
(4,4,4,4)	0.5890	0.6122	0.6947	0.6044	0.6014	$O_3 > O_2 > O_4 > O_5 > O_1$
(5,5,5,5)	0.5857	0.600	0.6951	0.6037	0.5973	$O_3 > O_4 > O_2 > O_5 > O_1$
(6,6,6,6)	0.5135	0.5921	0.6908	0.6050	0.5955	$O_3 > O_4 > O_5 > O_2 > O_1$

Table 4: Ranking results by utilizing different parameter vector l_i in the DHPFDGBM operator.

$l_i, i = 1, \cdots, 6$	Scores					Order
	O_1	O_2	O_3	O_4	O_5	
(1,1,1,1)	0.5348	0.5361	0.6325	0.5439	0.5443	$O_3 > O_5 > O_4 > O_2 > O_1$
(2,2,2,2)	0.5264	0.5146	0.6240	0.5261	0.5365	$O_3 > O_5 > O_1 > O_4 > O_2$
(3,3,3,3)	0.5190	0.4987	0.6168	0.5107	0.5285	$O_3 > O_5 > O_1 > O_4 > O_2$
(4,4,4,4)	0.5199	0.4872	0.6248	0.4985	0.5207	$O_3 > O_5 > O_1 > O_4 > O_2$
(5,5,5,5)	0.5256	0.5005	0.6629	0.4890	0.5322	$O_3 > O_5 > O_1 > O_2 > O_4$
(6,6,6,6)	0.5902	0.4944	0.6602	0.5061	0.6035	$O_3 > O_5 > O_1 > O_4 > O_2$

We can see that the different parameters lead to different result and different ranking order. More attributes we consider more bigger the scores, more bigger the attribute value more lower the scores. Therefore, the parameter vector can be considered as decision maker's risk preference.

8. Conclusion

In this paper, we investigate the multiple attribute decision making (MADM) problem based on the Bonferroni mean operators with dual Pythagorean hesitant fuzzy information. Firstly, we introduce the concept and basic operations of the dual hesitant Pythagorean fuzzy sets, which is a new extension of Pythagorean fuzzy sets. Then, motivated by the idea of Bonferroni mean operators, we have developed some Bonferroni mean aggregation operators for aggregating dual hesitant Pythagorean fuzzy information: dual hesitant Pythagorean fuzzy Bonferroni mean (DHPFBM) operator, dual hesitant Pythagorean fuzzy geometric Bonferroni mean (DHPFGBM) operator, dual hesitant Pythagorean fuzzy generalized Bonferroni mean (DHPFGBM) operator, dual hesitant Pythagorean fuzzy generalized geometric Bonferroni mean (DHPFGGBM) operator, dual hesitant Pythagorean fuzzy dual Bonferroni mean (DHPFDBM) operator and dual hesitant Pythagorean fuzzy dual geometric Bonferroni mean (DHPFDGBM) operator. The prominent characteristic of these proposed operators are studied. Then, we have utilized these operators to develop some approaches to solve the dual hesitant Pythagorean fuzzy multiple attribute decision making problems. Finally, a practical example for supplier selection in supply chain management is given to verify the developed approach and to demonstrate its practicality and effectiveness and we gave a comparative analysis with existing models. In the future, we shall continue working in the extension and application of the developed operators to other domains [62–66] and other uncertain environments [67–73].

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