CONTROL AND ROBOTICS

Fractional order model of measured quantity errors

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Abstract. The paper presents an interpretation of fractional calculus for positive and negative orders of functions based on sampled measured quantities and their errors connected with digital signal processing. The derivative as a function limit and the Grünwald-Letnikov differintegral are shown in chapter 1 due to the similarity of the presented definition. Notation of fractional calculus based on the gradient vector of measured quantities and its geometrical and physical interpretation of positive and negative orders are shown in chapters 2 and 3.

Key words: fractional calculus, data processing, measured quantities errors, jitter.

1. Introduction

The Leibniz notation $\frac{d}{dt}f$ of derivative is defined [1]:

$$\frac{d}{dt}f(t) = \lim_{dt \to 0} \frac{f(t+dt) - f(t)}{dt} = f^{(1)}(t) \qquad (1)$$

where: $dt = (t+dt) - t$ is an increment of the independent variable t; $f(t+dt) - f(t)$ is an increment of function dependent on t; $f^{(1)}(t)$ is a first order derivative.

Derivative of n–order is defined:

$$\frac{d}{dt}\left(\frac{d}{dt}...\left(\frac{d}{dt}f(t)\right)\right) = f^{(n)}(t) = \frac{d^n}{dt^n}f(t) =$$
$$\lim_{dt \to 0} \frac{f^{(n-1)}(t+dt) - f^{(n-1)}(t)}{dt}, \qquad (2)$$

where $n \in \mathbb{N}$ is an order of derivative (multiplicity of the first order derivative).

Equation (2) causes to [2–4]:

$$\frac{d^n}{dt^n}f(t) = \lim_{dt \to 0} \frac{\sum_{m=0}^{n}(-1)^m \binom{n}{m} f(t-mdt)}{(dt)^n} \qquad (3)$$

where: $dt = t_m - t_{m-1}$; $t_m = t - mdt$; $\binom{n}{m} = \frac{n!}{m!(n+m)!}$ for $n > m$.

Grünwald-Letnikov differintegrals of any order is obtained by replacing $\binom{n}{m}$ with gamma function and n with $\eta \in \mathbb{R}$:

$$\frac{n!}{m!(n-m)!} = \frac{\Gamma(n+1)}{m!\Gamma(n-m+1)} = \frac{\Gamma(\eta+1)}{m!\Gamma(\eta-m+1)}. \qquad (4)$$

By substituting (4) to (3) the direct Grünwald-Letnikov derivative of non-integer order (G-L) is produced [3, 4]:

$$\frac{d^\eta}{dt^\eta}f(t) = \lim_{dt \to 0} \frac{\sum_{m=0}^{p}(-1)^m \frac{\Gamma(\eta+1)}{m!\Gamma(\eta-m+1)} f(t-mdt)}{(dt)^\eta} \qquad (5)$$

where: $p = \left\lfloor \frac{t_0 - t_n}{dt} \right\rfloor$.

In (5) the notation dt^η is treated as the function power $(dt)^\eta$. This approach leads to problems with clear mathematical and physical interpretation of an order of fractional calculus [2, 5, 6] especially in relation to interpretation of an integer order derivative for example as a tangent to a function or a measurement of a path (derivative of velocities change). The interpretations of G-L differintegral are shown in [7] but they have borders due to the possibility of substitution of any number for η and properties of gamma function. This problem is shown in the work [8].

The problem of determining the derivative should be approached differently than mathematically by changing the integer order to non-integer order using different function (for example Gamma function) because this solution can lead to breaking the laws of physics [9–11]. This radical view is not true because many studies confirm the correctness of fractional models in phenomena of the real world [2, 12–21]. The problem is not breaking the laws of physics by fractional calculus but not clear physical interpretation of how and why changing integer order into fractional order influences the physical meaning of modelled quantities such as mass, length, time, force, electric current, temperature and others.

2. Differintegrals of measured function

2.1. Interval error. Let $f(t): t \in \mathbb{R}$ between t_{n-1} and t_n be a measured function for:

$$t_f = (t_0, t_1, ..., t_n) \qquad (6)$$

where t_n is produced on the basis of known interval:

$$dt = t_1 - t_0 = t_2 - t_1 = \ldots = t_n - t_{n-1}. \qquad (7)$$

Let variable v:

$$v = (v_0, v_1, \ldots, v_n) = \big(f(t_0), f(t_1), \ldots, f(t_n)\big). \qquad (8)$$

Let for every interval dt be added an interval Δt named as the interval error and $0 \le |\Delta t| \le dt$.

If t is time then v is measured quantity in time t by an interval dt. In this case Δt can be interpreted as an absolute error of determining of an interval length.

Let dt_η be sum of an interval and its error [7]:

$$dt_\eta = dt + \Delta t \qquad (9)$$

where:

$$dt_\eta = t_{\eta_1} - t_{\eta_0} = t_{\eta_2} - t_{\eta_1} = \ldots = t_{\eta_n} - t_{\eta_{n-1}} \qquad (10)$$

and

$$t_\eta = \big(t_{\eta_0}, t_{\eta_1}, \ldots, t_{\eta_n}\big) \qquad (11)$$

where:

$$t_{\eta_n} = t_n + n\Delta t. \qquad (12)$$

Let g be a function where its values are equal values of $f(t)$ in t_η:

$$g\big(t_\eta\big) = g\big(t_\eta + n\Delta t\big) = v. \qquad (13)$$

Functions $f(t)$ and $g(t)$ are shown in Fig. 1.

Assuming $f(t_f)$ is a function obtained by measured quantities v every dt, $g(t_\eta)$ function takes into consideration an interval error. The difference between both functions will be bigger if an interval error Δt is bigger.

In dynamic models of real systems with fast changes of an input signal an interval error is taken into consideration as the most important factor in precise modelling based on the empirical datum. It is used to: models of vibration and velocity transducers, models of gas and liquid flows, models taking into consideration the deviation from true periodicity of a presumably periodic signal (jitter).

2.2. Derivative of function with an interval error. Let s_η be a function of variables t_η (11) and v (8) from Section 2.1.

The gradient of $s_\eta(t_\eta, v)$ function can be formulated as follows [1]:

$$\nabla s_\eta (t_\eta, v) = \frac{\partial}{\partial t} s_\eta (t_\eta, v) i + \frac{\partial}{\partial v} s_\eta (t_\eta, v) j \qquad (14)$$

where ∇s_η is the gradient defined on points t_η situated on the curve g, i and j are the standard unit vectors.
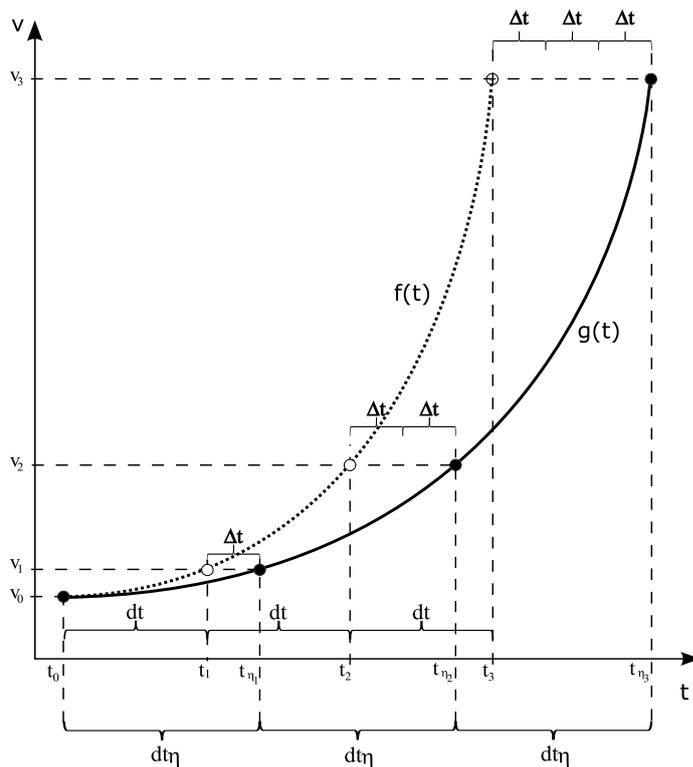

Fig. 1. Functions f and g

Function g is taking into consideration an interval error Δt (12). Gradient $\nabla s_\eta(t_\eta, v)$ for $dt \to 0$ is a linear approximation of g between next values of t_η (Fig. 2).
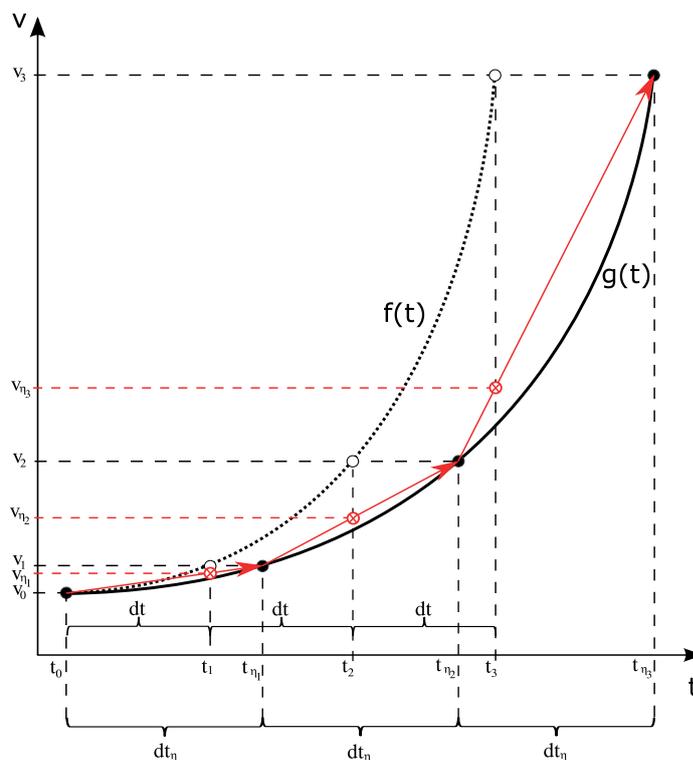

Fig. 2. Gradient vectors of g function

Let v_{η_n} be values situated on the gradient vector ∇s_η in t_{η_n} (Fig. 2):

$$v_\eta = \left(v_{\eta_1}, v_{\eta_2}, \ldots, v_{\eta_n}\right) = \left(v_0(t_0), v_{\eta_1}(t_1), \ldots, v_{\eta_n}(t_n)\right). \quad (15)$$

Derivative of g in t_η is produced, where g between t_{η_n} and $t_{\eta_{n+1}}$ is approximated by the gradient vector:

$$\frac{d}{dt}g(t_{\eta_n}) = \lim_{dt_\eta \to 0} \frac{v_{n+1} - v_n}{t_{\eta_{n+1}} - t_{\eta_n}} = \frac{d}{dt}g(t_{n+1}) =$$
$$= \lim_{(t_{\eta_{n+1}} - t_{n+1}) \to 0} \frac{v_{n+1} - v_{\eta_{n+1}}}{t_{\eta_{n+1}} - t_{n+1}}. \quad (16)$$

By substituting (12) to (16):

$$\frac{d}{dt}g(t_{\eta_n}) = \lim_{dt_\eta \to 0} \frac{v_{n+1} - v_n}{dt + \Delta t} = \frac{d}{dt}g(t_{n+1}). \quad (17)$$

Second derivative of g function in $t_{\eta_{n+1}}$ is produced:

$$\frac{d^2}{dt^2}g(t_n) = \lim_{dt_\eta \to 0} \frac{\frac{d}{dt}g(t_{n+1}) - \frac{d}{dt}g(t_n)}{t_{\eta_{n+1}} - t_{\eta_n}} =$$
$$= \lim_{dt_\eta \to 0} \frac{v_{n+2} - 2v_{n+1} + v_n}{(dt + \Delta t)^2}. \quad (18)$$

Generally, the n–order derivative of g in t_0 is formulated as:

$$\frac{d^n}{dt^n}g(t_0) = \lim_{dt_\eta \to 0} \frac{\frac{d^{n-1}}{dt^{n-1}}g(t_1) - \frac{d^{n-1}}{dt^{n-1}}g(t_0)}{dt + \Delta t} =$$
$$= \lim_{dt_\eta \to 0} \frac{\sum_{m=0}^n (-1)^m \binom{n}{m} v_{n-m}}{(dt + \Delta t)^n}. \quad (19)$$

2.3. Derivative of non-integer positive order and its interpretation. Let dt^γ stands for change of dt:

$$dt + \Delta t = dt_\eta = dt^\gamma. \quad (20)$$

By substituting (20) to (19):

$$\frac{d^n}{dt^n}g(t_0) = \lim_{dt_\eta \to 0} \frac{\frac{d^{n-1}}{dt^{n-1}}g(t_1) - \frac{d^{n-1}}{dt^{n-1}}g(t_0)}{dt_\eta} =$$
$$= \lim_{dt^\gamma \to 0} \frac{\sum_{m=0}^n (-1)^m \binom{n}{m} v_{n-m}}{(dt^\gamma)^n}. \quad (21)$$

Since dt^n on the left of (21) is the notation of multiplicity of dt other than $dt_\eta = dt^\gamma$ on the right, instead of dt^n the notation dt^η is inserted where:

$$\eta = \gamma n \quad (22)$$

and: γ is a factor of changing derivative by Δt (20); n is a multiplicity of dt^γ; $\eta : \mathbb{R}^+$ is a fractional order of derivative.

The fractional order derivative is defined by substituting equation (22) to (21):

$$\frac{d^n}{dt^\eta}g(t_0) = \lim_{dt^\gamma \to 0} \frac{\sum_{m=0}^n (-1)^m \binom{n}{m} v_{n-m}}{(dt^\gamma)^n} \quad (23)$$

where: $\frac{d^n}{dt^\eta}$ is a notation of fractional derivative; n is a natural order of derivative (multiplicity of an interval dt); η is a fractional order of derivative (for $\Delta t \neq 0$) connected with a natural order by a factor γ (22).

In the case where $\Delta t = 0$ then $\eta = n$ and:

$$\left.\frac{d^n}{dt^\eta}g(t_0)\right|_{\Delta t = 0} = \frac{d^n}{dt^n}f(t_0). \quad (24)$$

Factor γ depends on an interval dt and an interval error Δt (20). Because $dt \in [0, dt]$ then on the branch borders:

$$\gamma = \begin{cases} 1 & \text{for} \quad \Delta t = 0 \\ \log_{dt} 2dt & \text{for} \quad \Delta t = dt \end{cases}. \quad (25)$$

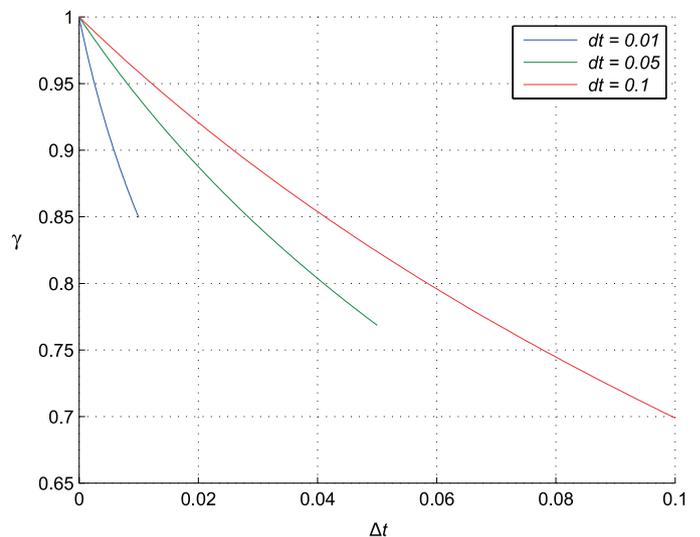Characteristics of $\gamma(\Delta t, dt)$ are shown in Fig. 3.

Fig. 3. Characteristics of factor $\gamma(\Delta t, dt)$

Because for every $n \in \mathbb{N}$:

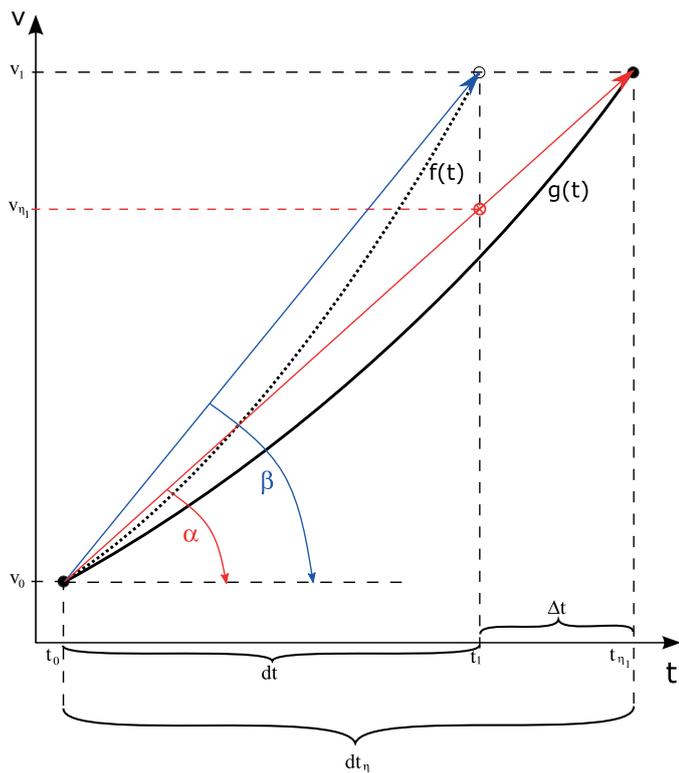$$(dt + \Delta t)^n = \sum_{m=0}^n \binom{n}{m} dt^{n-m} \Delta t^m \quad (26)$$

Fig. 4. Geometric interpretation of a non-integer positive order derivative

then the derivative (23) can have the formula:

$$\frac{d^n}{dt^\eta} g(t_0) = \lim_{dt^\gamma \to 0} \frac{\sum_{m=0}^{n} (-1)^m v_{n-m}}{\sum_{m=0}^{n} \binom{n}{m} dt^{n-m} \Delta t^m}. \tag{27}$$

For $dt^\gamma \to 0$ and $n = 1$ the geometric interpretation of the fractional orders derivative (23) is an angle of inclination of tangent to g in t_n, the same as the first order derivative. In the case where the approximation of a tangent is a secant (Fig. 4) then the geometric interpretations of the derivative are: an angle of inclination of a secant joining points (t_0, v_0) and (t_{η_1}, v_1) for the fractional order derivative and an angle of inclination of a secant joining points (t_0, v_0) and (t_1, v_1) for the first order derivative. Difference between the angle β of inclination a secant on f and the angle α of inclination a secant on g will be the bigger as the bigger is a value of interval error Δt. For $\Delta t = 0$ the angle of inclination secants on f and g will be the same (characteristics will overlap). Other interpretation can found in [5–7, 22].

2.4. Derivative of non-integer minus order and its interpretation. Let $n = 1$. On the basis of (22, 23) and $\eta = \gamma$ the derivative is:

$$\frac{d}{dt^\gamma} g(t_0) = \lim_{dt^\gamma \to 0} \frac{v(t_{\eta_1}) - v(t_0)}{dt^\gamma}. \tag{28}$$

Equation (28) for minus order γ has the formula:

$$\frac{d}{dt^{-\gamma}} g(t_0) = \lim_{dt^\gamma \to 0} \frac{v(t_{\eta_1}) - v(t_0)}{dt^{-\gamma}} =$$
$$= \lim_{dt^\gamma \to 0} (v_1 - v_0) dt^\gamma \equiv \int_{t_0}^{t_{\eta_1}} v(t) dt^\gamma. \tag{29}$$

The minus order derivative (29) is equivalent to the integral definition where step and branch of integration equal $dt^\gamma = t_{\eta_0} - t_0$. Geometrical interpretation of (29) as a surface area is shown in Fig. 5.
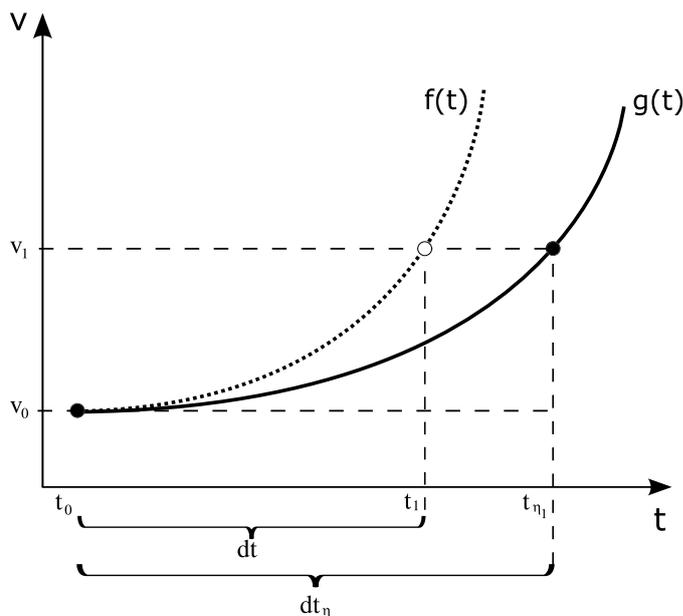

Fig. 5. Geometric interpretation of a non-integer minus order derivative

Let $n = 2$. On the basis of (18) and (23) the minus order derivative of function g in t_0 with multiplicity $n = 2$ is produced:

$$\frac{d^2}{dt^{-2\gamma}} g(t_0) = \lim_{dt_\eta \to 0} \frac{\frac{d}{dt^{-\gamma}} g(t_{\eta_1}) - \frac{d}{dt^{-\gamma}} g(t_0)}{dt^{-\gamma}} =$$
$$= \lim_{dt_\eta \to 0} \frac{\int_{t_{\eta_1}}^{t_{\eta_2}} v(t) dt^\gamma - \int_{t_0}^{t_{\eta_1}} v(t) dt^\gamma}{dt^{-2\gamma}}. \tag{30}$$

Generally, the minus order derivative with multiplicity n is written as:

$$\frac{d^n}{dt^{-n\gamma}} g(t_0) = \lim_{dt_\eta \to 0} \frac{\frac{d^{n-1}}{dt^{-(n-1)\gamma}} g(t_{\eta_1}) - \frac{d^{n-1}}{dt^{-(n-1)\gamma}} g(t_0)}{dt^{-\gamma}} =$$
$$= \lim_{dt_\eta \to 0} \left(\int_{t_{\eta_1}}^{t_{\eta_2}} v(t) dt^\gamma - \int_{t_0}^{t_{\eta_1}} v(t) dt^\gamma \right) dt^{n\gamma}. \tag{31}$$

Because integral is operation opposite to derivative (Newton-Leibniz formula) [1]:

$$\int_a^b f(t)dt = F(b) - F(a) \qquad (32)$$

and

$$F'(a) = \lim_{(b-a)\to 0} \frac{F(b) - F(a)}{b - a} = \lim_{(b-a)\to 0} \frac{\int_a^b f(t)dt}{b - a} \qquad (33)$$

where F is a primitive function of f.

Exchanging generally signs $F'(a)$ and $f(t)$ with fractional derivative (23) and $v(t)$:

$$\frac{d^n}{dt^\eta} g(t_0) = \lim_{dt^\gamma\to 0} \frac{\sum_{m=0}^n (-1)^m \binom{n}{m} v_{n-m}}{(dt^\gamma)^n} =$$
$$= \lim_{dt^\gamma\to 0} \frac{\int_{t_0}^{t_{\eta_n}} v(t)dt^\gamma}{(dt^\gamma)^n} \qquad (34)$$

where v is a primitive function of g.

Converting (34):

$$\lim_{dt^\gamma\to 0} \int_{t_0}^{t_{\eta_n}} v(t)dt^\gamma = (dt^\gamma)^n \frac{d^n}{dt^\eta} g(t_0) =$$
$$= \lim_{dt^\gamma\to 0} \sum_{m=0}^n (-1)^m \binom{n}{m} v_{n-m}. \qquad (35)$$

Equation (35) is a relationship between minus order derivative and an integral with an integration step dt^γ and an integration branch $[t_0, t_{\eta_n}]$. Simultaneously (35) is the numerator of positive order derivative (23).

Because minus order derivative has an influence on denominator of (23) only, then minus order derivative is tantamount to subsequent integrations of primitive function v of g function:

$$\frac{d^n}{dt^{-\eta}} g(t_0) = \lim_{dt^\gamma\to 0} (dt^\gamma)^n \sum_{m=0}^n (-1)^m \binom{n}{m} v_{n-m} =$$
$$= \int_{t_{\eta_{n-1}}}^{t_{\eta_n}} \cdots \left(\int_{t_{\eta_1}}^{t_{\eta_2}} \left(\int_{t_{\eta_0}}^{t_{\eta_1}} v(t)dt^\gamma \right) dt^\gamma \cdots \right) dt^\gamma. \qquad (36)$$

2.5. Interpretation of a multiple non-integer minus order derivative. In mathematics there are multiple integrations of more than one variable $x_1, \ldots, x_m$:

$$\iiint \cdots \int_D f(x_1, \ldots, x_m)dx_1 \ldots dx_m. \qquad (37)$$

So that the multiple integration of one variable is produced:

$$\iiint \cdots \int_D f(x)dx^1 \ldots dx^n \qquad (38)$$

where n is a number of iteration of x.

By using double integral of (37) we can calculate a volume. A triple integral can calculate a mass of a solid with known density. Currently no physical interpretations are known regarding the infinity multiple integral of one variable (38). From this an interpretation of presented minus order derivative is limited to the one iteration of minus order derivative (36).

Let dt^γ be an integration step and $[t_0, t_{\eta_n}]$ be an integration branch. Because the definite integral can be written as a sum of definite integrals, thus on the basis of (29):

$$\frac{d^n}{dt^{-\eta}} g(t_0) = \lim_{dt^\gamma\to 0} \int_{t_0}^{t_{\eta_n}} v(t)dt^\gamma =$$
$$= \lim_{dt^\gamma\to 0} \left(\int_{t_0}^{t_{\eta_1}} v(t)dt^\gamma + \int_{t_{\eta_1}}^{t_{\eta_2}} v(t)dt^\gamma + \right.$$
$$\left. + \ldots + \int_{t_{\eta_{n-1}}}^{t_{\eta_n}} v(t)dt^\gamma \right) = \frac{d}{dt^{-\gamma}} g(t_0) +$$
$$+ \frac{d}{dt^{-\gamma}} g(t_{\eta_1}) + \ldots + \frac{d}{dt^{-\gamma}} g(t_{\eta_{n-1}}) =$$
$$= \lim_{dt^\gamma\to 0} (v_1 - v_0)dt^\gamma + \lim_{dt^\gamma\to 0} (v_2 - v_1)dt^\gamma +$$
$$+ \ldots + \lim_{dt^\gamma\to 0} (v_n - v_{n-1})dt^\gamma. \qquad (39)$$

The graphical interpretation of (39) as a surface area is shown in Fig. 6 and 7. The marked area in the Fig. 6 represents $\Delta t = 0$ and natural number of η order. The marked area in the
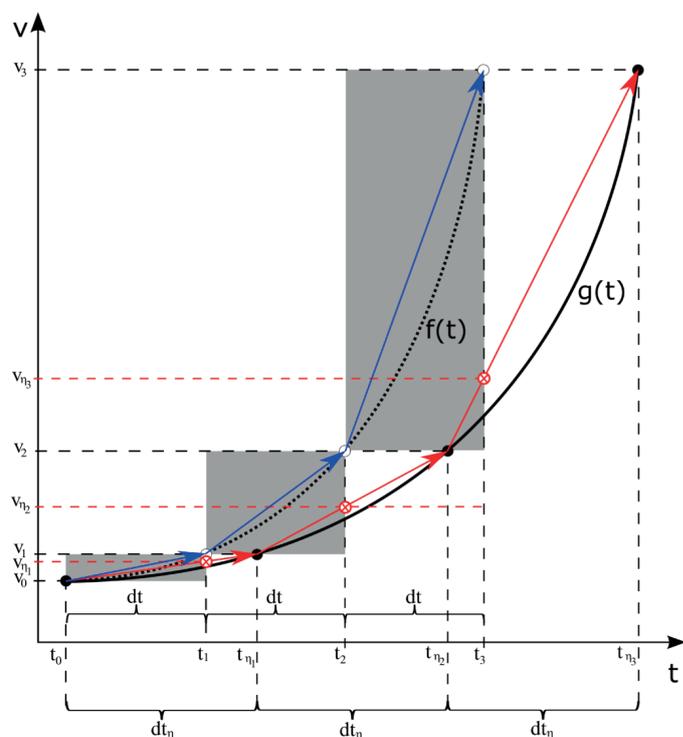

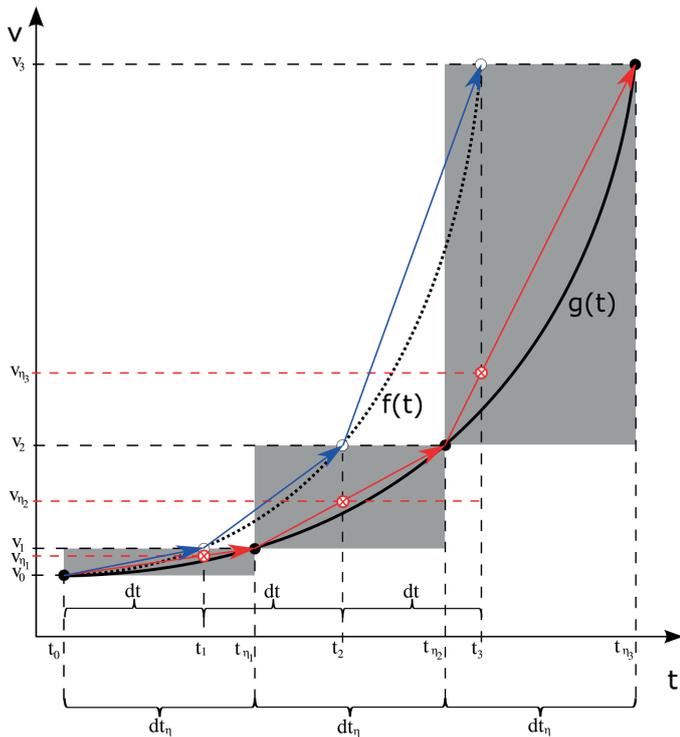Fig. 6. Geometric interpretation of multiple non-integer minus order derivative for $\Delta t = 0$

Fig. 7. Geometric interpretation of multiple non-integer minus order derivative for $\Delta t \neq 0$

Fig. 7 represents $\Delta t \neq 0$ where η is not a natural number. In both cases the negative order derivative is the one step upper Darboux integral, where an integral step and an integral branch are equal.

2.6. Jitter effect. Jitter definition and its effect can be found in works [23–25]. Generally, jitter effect changes real measured input signal by inaccuracy of work of sampled systems and the read signal (behind the sampled system) is different than the measured signal (before the sampled system). In this case next operation has result with jitter error, for example an integration operation of signal. Block diagram of this example is shown in the Fig. 8 where input signal is sampled with and without jitter effect and obtained signals are integrated by classical integrator and fractional integrator built on base of the equation (39). Jitter modelling (Fig. 9) was created on the base of work [26].

Signals (output signals from Signal Sampling block) with and without jitter effect are shown in the Fig. 10. Signals after integration (classical and fractional) are shown in the Fig. 11 and 12. Absolute error of signals (classical and fractional) are shown in the Fig. 13 (in relation to integration signal without jitter – the ideal signal). From Fig. 13 it arises that fractional integration where jitter error is coded into its order has lower absolute error values (0.01–0.14%) than classical integration (0.03–0.3%).
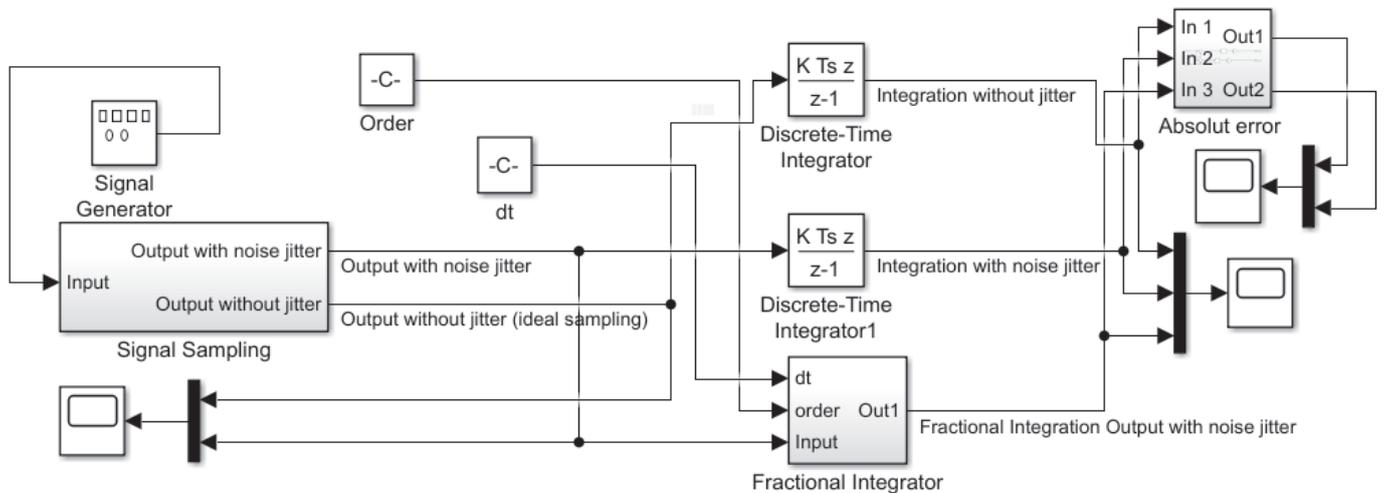

Fig. 8. Block diagram of system integration with jitter: $dt = 0.0001$ (sample time), $\eta = -1.001$ (fractional order)
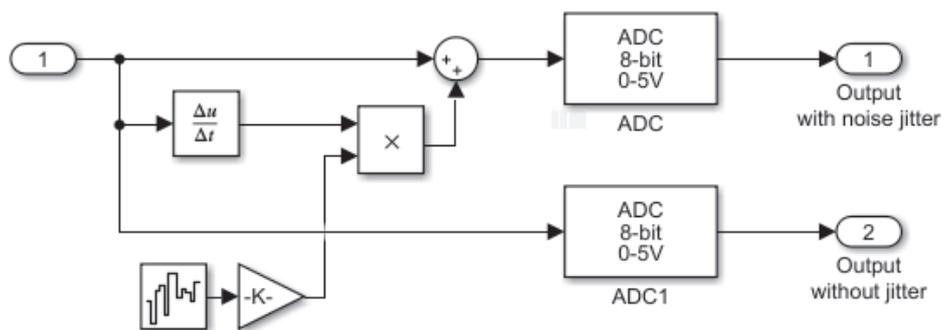

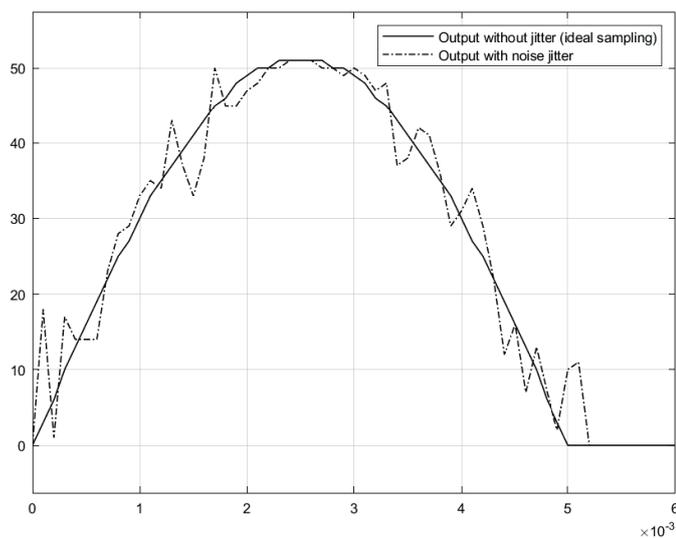Fig. 9. Jitter modelling (signal sampling block)
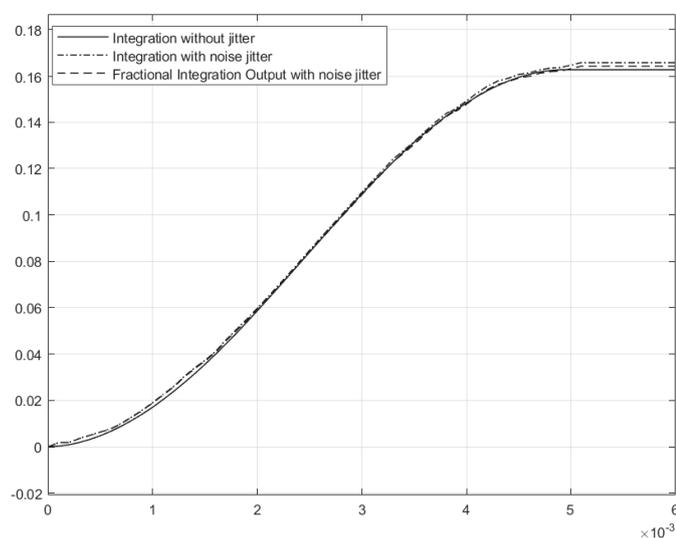
Fig. 10. Signals with and without jitter effect

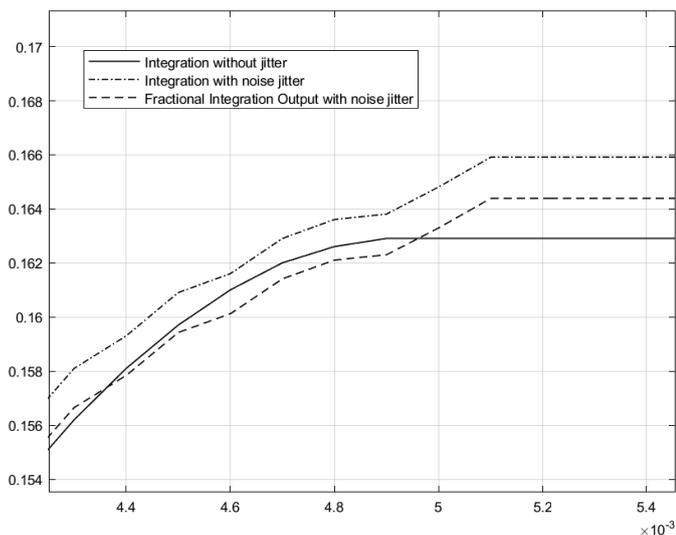Fig. 11. Integration of signals with and without jitter effect

Fig. 12. Integration of signals with and without jitter effect – zoom

3. Conclusions

A definition of fractional order derivative and integral dedicated to describe a measured quantity error and its notation $\frac{d^n}{dt^\eta}$ have been proposed.

In the definition an order of derivative is a product of natural order and γ factor which determines a relation between an interval (an increment of independent variable t) and an interval error (20). In the special case where an interval error is equal 0 ($\Delta t = 0$), $\gamma = 1$ and the real positive order derivative (23) becomes natural order derivative. From this the natural order derivative is special case of the real order derivative.

The minus order derivative (29–31) has been defined on the basis of the definition of positive order derivative of function (23). If an integral step and an integral branch are equal the negative order derivative of function is the upper Darboux integral.

In the case of positive and negative order of derivative of measured function, an interval error can interpreted as an error of determining of an increment dt of independent measured variable t.

The definition presented can be used in various ways. Generally the definition takes into consideration an interval error of independent measured value. In real measurements an interval error can be interpreted as an error of determining of sample time branch of measuring signal arising for example from an action of electronic parts of sampling systems (from not ideal sampling of measuring cards and jitter effect). In mathematics it can be an error arising from rounded values of a differential to defined place after comma. In physics an interval error can be influenced by factors connected with drag air and friction of moving objects.

References

[1] T.M. Apostol, *Calculus*, John Wiley & Sons (1968).
[2] S. Das, Functional Fractional Calculus for System Identification and Controls, Springer-Verlag Berlin Heidelberg (2008).
[3] K. Miller and R. Bertram, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley & Sons, Inc. (1993).
[4] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego (1999).
[5] I. Podlubny, "Geometric and physical interpretation of fractional integration and fractional differentiation", *Fract. Calc. Appl. Anal*. 5(4) (2002).
[6] R.S. Rutman, "On physical interpretation of fractional integration and differentiation", *Theor. Math. Phys*. 105(3) (1995).
[7] R. Cioć, "Physical and geometrical interpretation of Grünwald-Letnikov differintegrals, measurement of path and acceleration", *Fract. Calc. Appl. Anal*. 19(1) (2016), DOI: /10.1515/fca-2016–0009.
[8] R. Cioć, "Grünwald-Letnikov derivative, analysis in range of first order", *Frontiers in Fractional Calculus. Book Series, Current Developments in Mathematical Sciences* 1, Bentham Science Publishers (2017).
[9] R. Sikora, "Fractional derivatives in electrical circuit theory – critical remarks", *Archives of Electrical Engineering* 66(1) (2017).

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[10] P. Ostalczyk, "Jednak, Rachunek różniczkowo-całkowy niecał-kowitych rzędów. Riposta na artykuł Ryszarda Sikory pt., Pochodne ułamkowe w teorii obwodów elektrycznych Uwagi krytyczne. Przegląd Elektrotechniczny, R.92, Nr 10/2016", *Przegląd Elektrote*chniczny 3 (2017).

[11] R. Sikora, "Riposta na ripostę prof. Piotra Ostalczyka", *Przegląd Elektrote*chniczny 4 (2017).

[12] R. Cioć, "Digital fractional integrator", *Lecture Notes in Electrical Engineering, Theory and Applications of Non-integer Order Systems, 8th Conference on Non-integer Calculus and Its Applications*, 407 (2017), DOI: 0.1007/978–3–319–45474–0_16.

[13] A. Dzieliński, G. Sarwas, and D. Sierociuk, "Comparison and validation of integer and fractional order ultracapacitor models" *Adv. Differ. Equ.*, 2011:11 (2011), DOI: 10.1186/1687-1847-2011-11.

[14] H. El Brouji, J.-M. Vinassa, et al., "Ultracapacitors self discharge modelling using a physical description of porous electrode impedance", *IEEE Vehicle Power and Propulsion Conference*, 3–5 Sept. (2008), DOI: 10.1109/VPPC.2008.4677493.

[15] Z. Lukasik, J. Kozyra et al., "Guaranteed power supply for the purpose of automated technological line of powder coating, *Electr. Eng.* 100 (2018), DOI: 10.1007/s00202–016–0489–8.

[16] C. Monje, B. Vinagre, V. Feliu, and C. YangQuan, "Tuning and auto-tuning of fractional order controllers for industry applications", *Control Engineering Practice* 16(7) (2008), DOI: 10.1016/j.conengprac.2007.08.006.

[17] P. Ostalczyk and T. Rybicki, "Variable-fractional-order deadbeat control of an electromagnetic servo", *Journal of Vibration and Control* 14(9–10) (2008), DOI: /10.1177/1077546307087437.

[18] M. Rodolfo, J.J. Quintana, et al., "Modeling electrochemical double layer capacitor, from classical to fractional impedance", *MELECON 2008 – The 14th IEEE Mediterranean Electrotechnical Conference*, (2008), DOI: 10.1109/MELCON. 2008.4618411.

[19] H. Sheng, H. Sun, et al., "Physical experimental study of variable-order fractional integrator and differentiator", *Eur. Phys. J. Spec. Top*. 193(1) (2010), 93–104; DOI: 10.1140/epjst/e2011–01384–4.

[20] M.D. Ortigueira and J.T.M. Machado, "On fractional vectorial calculus", *Bull. Pol. Ac.: Tech*. 66(4), 389–402 (2018), DOI: 10.24425/124254.

[21] M. Lewandowski and M. Orzyłkowski, "Fractional-order models: The case study of the supercapacitor capacitance measurement", *Bull. Pol. Ac.: Tech*. 65(4), 439–447 (2017), DOI: 10.1515/bpasts-2017–0050.

[22] P. Ostalczyk, *Discrete Fractional Calculus*, World Scientific (2016).

[23] F. Maloberti, *Data Converters*, Springer-Verlag (2007).

[24] K.S. Oh and X. Yuan, *High-Speed Signaling Jitter Modeling, Analysis, and Budgeting*, Prentice Hall (2011).

[25] M. Mauerer, A. Tüysüz, and J.W. Kolar, "Low-jitter GaN E-HEMT gate driver with high common-mode voltage transient immunity" *IEEE Transactions on Industrial Electronics* 64(11), 9043–9051 (2017), DOI: 10.1109/tie.2017.2677354.

[26] P. Malcovati, S. Brigati, F. Francesconi, F. Maloberti, P. Cusinato, and A. Baschirotto, "Behavioral modeling of switchedcapacitor sigma-delta modulators" *IEEE Transactions on Circuits and Systems I, Fundamental Theory and Applications* 50(3), 352–364 (2003).