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On a finding the coefficient of one nonlinear wave equation in the mixed problem

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The paper is devoted to the finding of the coefficient of one nonlinear wave equation in the mixed problem. The considered problem is reduced to the optimal control problem with proper functional. Differentiability of functional is proved and the necessary optimality conditions are derived in the form of the variational inequality. Existence of the optimal control is proved.

Key words: hyperbolic equation, nonlinear wave equation, optimal control problem, necessary condition

1. Introduction

Usually wave processes are described by the hyperbolic equations and also different type Schrodinger and Korteweg-de Vries equations. When the process has nonlinear character the mathematical model becomes more complicated and involves different type nonlinear terms. A nonlinear term as, for example, u^p will tend to magnify the size of u, where u is large, and to be negligible where u is small. It can make a solution blow up in finite time, it can produce a solitary wave, or (it involve derivative of u) it can produce a shock wave. Some generalizations of the nonlinear wave equations describe free-electron laser operation in higher harmonics; this significantly extends their tunable range to shorter wavelengths. The dynamics of the laser field's amplitude and phase are explored for a wide range of parameters. Such a parameter can be the coefficients of the equation that indeed may describe various properties of the medium under investigation and such problems arise in different fields of in nature [1]. The close problem

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All the results obtained in this work are valid if in equation (1) instead of the nonlinear term $|u|u$ set u^3 and take $n = 3$.

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was considered in [2], where the singular system of optimality was obtained. In [3] the author studies optimal control problems for the various nonlinear systems in partial derivatives. In [4] the optimal control problem with control at the coefficient is considered for the nonlinear wave equation occurring in the relativistic quantum mechanics. In that work the existence of the optimal control and Gateaux differentiability of the functional is proved and necessary optimality condition is derived. In [5, 6] the problem of determining the coefficient at the lowest term in the equation of oscillations and determination of the initial functions from the observed values of the boundary functions for the second-order hyperbolic equation is considered. In the works [8] optimal boundary control problem is considered for the nonlinear hyperbolic equation. In [7, 9] solution of the nonlinear hyperbolic equations is investigated stimulated by the strong relation of such problems with different applications. In this work the problem of finding the coefficients of the nonlinear wave equation is reduced to the optimal control problem. The existence of the optimal control, continuous Frechet differentiability of the functional is proved and necessary optimality condition in the form of variational inequality is derived.

2. Formulation of the problem

Consider the problem of finding of the pair $\{u(x,t), v(x)\} \in U \times V$ subject to

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + |u|u + vu = f(x,t), \qquad (x,t) \in Q, \tag{1}$$

$$u = 0, \quad (x,t) \in S, \quad u|_{t=0} = u_0(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = u_1(x), \quad x \in \Omega, \tag{2}$$

$$u(x,T) = \varphi(x), \qquad x \in \Omega, \tag{3}$$

where Δ is the Laplace operator with respect to x, $f(x,t)$, $u_0(x)$, $u_1(x)$, $\varphi(x)$ are given functions; $Q = \Omega \times (0,T)$ is a cylinder in R^{n+1}; Ω is a bounded domain in R^n, $n = 3,4$ with smooth enough boundary Γ; $S = \Gamma \times (0,T)$ is a lateral surface of the cylinder Q, $T > 0$ is a given number,

$$U = \left\{u\colon u \in L_\infty(0,T; H_0^1(\Omega)),\ \frac{\partial u}{\partial t} \in L_\infty(0,T;\ L_2(\Omega))\right\},$$

$$V = \{v\colon v \in L_2(\Omega),\ a \leqslant v(x) \leqslant b \ \text{ a.e. in } \Omega\}, \tag{4}$$

where a and b are some constants, moreover $a < b$.

Note that boundary value problem (1)–(3) is an inverse to direct problem (1), (2) by the given function $v(x)$. We reduce this problem to the following optimal control problem: find the minimum of the functional

$$J_0(v) = \frac{1}{2} \int_\Omega [u(x,T;v) - \varphi(x)]^2 \, dx \tag{5}$$

in the class of functions V subject to (1), (2), where $u(x,t;v)$ is a solution of problem (1), (2) for the function $v(x)$. The function $v(x)$ we call a control, and the set V – a class of admissible controls.

It should be noted that there exists a bold relation between problems (1)–(3) and (1), (2), (4), (5) – if the minimum of functional (5) in problem (1), (2), (4), (5) is equal to zero, then the additional condition (3) holds true. In future to avoid the possible degeration in the obtained necessary condition of optimality we consider the following problem: find the control $v \in V$ that gives minimum to the functional

$$J_\alpha(v) = J_0(v) + \frac{\alpha}{2} \int_\Omega |v(x)|^2 \, dx \tag{6}$$

subject to (1), (2), where $\alpha > 0$ is a given number [4, p. 45].

This problem we call problem (1), (2), (4), (6).

Let the following conditions on the data of problem (1), (2), (4), (6) hold true

$$f \in L_2(Q), \qquad u_0, \ \varphi \in H_0^1(\Omega), \qquad u_1 \in L_2(\Omega). \tag{7}$$

Similarly to [10, pp. 20–29] may be proved that under conditions (7) problem (1), (2) at the given function $v(x)$ from V has the unique generalized solution $u = u(v) = u(x,t;v)$ in the space U, moreover for the solution of problem (1), (2) is valid the estimation

$$\|u\|^2_{H_0^1(\Omega)} + \left\| \frac{\partial u}{\partial t} \right\|^2_{L_2(\Omega)} \leqslant c \left[\|u_0\|^2_{H_0^1(\Omega)} + \|u_1\|^2_{L_2(\Omega)} + \|f\|^2_{L_2(Q)} \right], \qquad t \in [0,T]. \tag{8}$$

Here and hereinafter, by c we will denote various constants independent of the estimated quantities and of the admissible controls.

In [10], is considered the following equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + |u|^\rho u = f,$$

where $\rho > 0$ is a given number, and the space U is taken as

$$U = \left\{ u | u \in L_\infty(0,T; \ H_0^1(\Omega) \cap L_p(\Omega)), \quad \frac{\partial u}{\partial t} \in L_\infty(0,T; \ L_2(\Omega)) \right\},$$

with $p = \rho + 2$. In our case $n = 3$ or 4, $\rho = 1$, therefore by the embedding theorem

$$H_0^1(\Omega) \subset L_q(\Omega), \qquad q = \frac{2n}{n-2}$$

we have $H_0^1(\Omega) \cap L_p(\Omega) = H_0^1(\Omega)$.

As a generalized solution of problem (1), (2) for the given $v(x) \in V$ we take the function $u = u(v)$ from U which for $t = 0$ satisfies the condition $u(x, 0; v) = u_0(x))$ and integral identity

$$\int_Q \left[-\frac{\partial u}{\partial t}\frac{\partial n}{\partial t} + \sum_{i=1}^{n} \frac{\partial u}{\partial k_i}\frac{\partial \eta}{\partial x_i} + |u|u\eta + vu\eta \right] \mathrm{d}x\mathrm{d}t$$
$$- \int_\Omega u_1(x)\eta(x,0)\mathrm{d}x = \int_Q f\eta\mathrm{d}x\mathrm{d}t$$

$\forall \eta \in U$, moreover is equal to zero at $t = T$.

3. Some auxiliary facts and the solvability of problem (1), (2), (4), (6)

Lemma 1 *Under condition (7), the mapping $u(v) : L_2(\Omega) \to U$ for problem (1), (2) is weakly continuous.*

Proof. Let $v_k \to v$ weakly in $L_2(\Omega)$ at $k \to \infty$. Then due to estimate (8), the corresponding sequence $\{u_k \equiv u(v_k)\}$ of the solutions of boundary value problem (1), (2) is bounded in the space U. Therefore after choosing the subsequence (we keep the same denotations) we get weak in Uconvergence $u(v_k) \to u$. This convergence is weak on $H^1(Q)$. Then by the Rellich theorem [11, p. 64] $u_k \to u$ strongly in $L_2(Q)$ and a.e. in Q. By the embedding theorem $H_0^1(\Omega) \subset L_4(\Omega)$ the sequence $\{|u_k|\, u_k\}$ is bounded in $L_2(Q)$. Therefore by lemma 1.3 from [10, p. 25] we get that $|u_k|\, u_k \to u|u$ weakly on $L_2(Q)$.

Using the compactness theorem [11, p. 71] we conclude that $u_k(x, T) \equiv u(x, T; v_k) \to u(x, T)$ strongly in $L_2(\Omega)$ at $k \to \infty$. Now we show that $u = u(x, t; v)$. Since $u_k,\ \eta \in U$ and $n = 3$ or 4, $u_k,\ \eta \in L_\infty(0, T; L_4(\Omega))$.

Therefore

$$\lim_{k\to\infty} \int_Q |u_k|\, u_k\eta\mathrm{d}x\mathrm{d}t = \int_Q |u|u\eta\mathrm{d}x\mathrm{d}t, \quad \forall \eta \in U. \tag{9}$$

For any $\eta \in U$ is valid

$$\left|\int\limits_Q v_k u_k \eta \,\mathrm{d}x\mathrm{d}t - \int\limits_Q vu\eta \,\mathrm{d}x\mathrm{d}t\right| \leqslant \left|\int\limits_Q (v_k - v)u\eta \,\mathrm{d}x\mathrm{d}t\right| + \left|\int\limits_Q v_k(u_k - u)\eta \,\mathrm{d}x\mathrm{d}t\right|.$$

Then considering that $u\eta \in L_2(Q)$ and $\{v_k\eta\}$ are bounded in $L_2(Q)$, and also above proved convergence of the sequences $\{v_k\}$ and $\{u_k\}$ weakly and strongly on $L_2(\Omega)$ and $L_2(Q)$, correspondingly, from the last inequality we obtain

$$\lim_{k\to\infty} \int\limits_Q v_k u_k \eta \,\mathrm{d}x\mathrm{d}t = \int\limits_Q vu\eta \,\mathrm{d}x\mathrm{d}t. \tag{10}$$

Then if in the definition of the generalized solution of problem (1), (2)

$$\int\limits_Q \left[-\frac{\partial u(v_k)}{\partial t}\frac{\partial \eta}{\partial t} + \sum_{i=1}^{n} \frac{\partial u(v_k)}{\partial x_i}\frac{\partial \eta}{\partial x_i} + |u(v_k)|\, u(v_k)\eta + v_k u(v_k)\eta\right] \mathrm{d}x\mathrm{d}t$$
$$- \int\limits_\Omega u_1(x)\eta(x,0)\,\mathrm{d}x = \int\limits_Q f\eta \,\mathrm{d}x\mathrm{d}t$$

to pass to limit at $v = v_k$ and consider weak convergence of the sequence $u_k = u(v_k)$ to u in $H^1(Q)$ and relations (9), (10) we can conclude that $u = u(v)$ and $u(x, T) = u(x, T; v)$. Lemma 1 is proved. □

Lemma 2 *Under conditions (7) functional (6) is weak semicontinuous on* $L_2(\Omega)$.

Proof. By Lemma 1, the first term in the expression of functional (6) is weakly continuous in $L_2(\Omega)$. Since the second term in the expression of functional (6) is weak semicontinuous from below on $L_2(\Omega)$, then functional (6) is weak semicontinuous from below in $L_2(\Omega)$. Lemma 2 is proved. □

Theorem 1 *Let the conditions set in the formulation of problem (1), (2), (4), (6) be satisfied. Then the set of optimal controls for this problem* $V_* = \{v_* \in V : J_\alpha(v_*) = \inf\{J_*(v) : v \in V\}\}$ *is not empty, is weak compact on* $L_2(\Omega)$ *and arbitrary minimizing sequence* $\{v_m(x)\}$ *converges to the set* V_* *weakly in* $L_2(\Omega)$.

Proof of the theorem follows from the proof of Theorem 2 from [12, p. 49]. Actually the set V is weak compact in $L_2(\Omega)$ and by the Lemma 2 the functional $J_\alpha(v)$ is weak semicontinuous from below on the set V. Thus all conditions of Theorem 2 from [12, p. 49] are satisfied and so all statements of Theorem 1 are valid. □

4. Differentiability of functional (6) and necessary optimality conditions

Now we investigate Frechet differentiability of functional (6) and get necessary optimality conditions in problem (1), (2), (4), (6). Let $\delta v \in L_\infty(\Omega)$ be an increment of the control on the element $v \in V$ such that $v + \delta v \in V$. Denote $\delta u = \delta u(x,t) = u(x,t;\ v+\delta v) - u(x,t;v)$. It is clear that the function $\delta u(x,t)$ is a generalized solution from U for the boundary value problem

$$\frac{\partial^2 \delta u}{\partial t^2} - \Delta \delta u + 2\,|u + \theta\delta u|\,\delta u + (v+\delta v)\delta u = -u\delta v, \qquad (x,t) \in Q, \tag{11}$$

$$\delta u = 0, \qquad (x,t) \in S, \qquad \delta u|_{t=0} = 0, \qquad \left.\frac{\partial \delta u}{\partial t}\right|_{t=0} = 0, \qquad x \in \Omega, \tag{12}$$

where $0 \leqslant \theta \leqslant 1$.

The generalized solution from U of problem (11), (12) is equal to zero at $t = 0$ and satisfy the identity

$$\int\limits_Q \left[\frac{\partial \delta u}{\partial t}\frac{\partial \eta}{\partial t} - \sum_{i=1}^n \frac{\partial \delta u}{\partial x_i}\frac{\partial \eta}{\partial x_i} - 2\,|u+\theta\delta u|\,\delta u\eta - (v+\delta v)\,\delta u\eta\right] \mathrm{d}x\mathrm{d}t = \int\limits_Q u\delta v\eta\,\mathrm{d}x\mathrm{d}t \tag{13}$$

for all $\eta = \eta(x,t) \in U$ equaling to zero at $t = T$.

Lemma 3 *For the solution of problem (11), (12) is valid*

$$\left\|\frac{\partial \delta u}{\partial t}\right\|^2_{L_2(\Omega)} + \|\delta u\|^2_{H_0^1(\Omega)} \leqslant c\|\delta v\|^2_{L_\infty(\Omega)}, \qquad t \in [0,T]. \tag{14}$$

Proof. Let $\{w_k(x)\}_{k=1}^\infty$ be a complete family of the linearly independent elements of the space $H_0^1(\Omega)$. According to the Faedo-Galerkin method the approximate solution of problem (11), (12) of order N we define as follows

$$\delta u^N(x,t) = \sum_{k=1}^N \xi_{kN}(t) w_k(x), \qquad N = 1,2,\ldots,$$

where the functions $\xi_{kN}(t)$ are such that the following relations hold

$$\begin{aligned}\int\limits_\Omega \frac{\partial^2 \delta u^N}{\partial t^2} w_j\,\mathrm{d}x + \int\limits_\Omega \sum_{i=1}^n \frac{\partial \delta u^N}{\partial x_i}\frac{\partial w_j}{\partial x_i}\mathrm{d}x + \int\limits_\Omega 2\left|u+\theta\delta u^N\right|\delta u^N w_j\,\mathrm{d}x + \\ + \int\limits_\Omega (v+\delta v)\delta u^N w_j\,\mathrm{d}x = -\int\limits_\Omega u\delta v w_j\,\mathrm{d}x, \qquad j = 1,\ldots,N,\end{aligned} \tag{15}$$

$$\xi_{kN}(0) = 0, \qquad \dot{\xi}_{kN}(0) = 0. \tag{16}$$

Relation (15) presents a system of nonlinear second order ordinary differential equations with respect to $(\xi_{1N}(t), \ldots, \dot{\xi}_{NN}(t))$. The general results on the nonlinear systems guarantee the existence of the solution for problem (15), (16) on the interval $[0, t_N]$; the a-prior estimate show that $t_N = T$ [10].

Multiplying the j-th equality of (15) by $\dot{\xi}_{jN}(t)$ and summing over $j = 1, \ldots, N$ we get

$$\int\limits_{\Omega} \left[\frac{\partial^2 \delta u^N}{\partial t^2} \frac{\partial \delta u^N}{\partial t} + \sum_{i=1}^{n} \frac{\partial \delta u^N}{\partial x_i} \frac{\partial^2 \delta u^N}{\partial t \partial x_i} \right] \mathrm{d}x = -2 \int\limits_{\Omega} \left| u + \theta \delta u^N \right| \delta u^N \frac{\partial \delta u^N}{\partial t} \mathrm{d}x$$
$$- \int\limits_{\Omega} (v + \delta v) \delta u^N \frac{\partial \delta u^N}{\partial t} \mathrm{d}x - \int\limits_{\Omega} u \delta v \frac{\partial \delta u^N}{\partial t} \mathrm{d}x\,.$$

Adding the term $\int\limits_{\Omega} 2 \left| \theta \delta u^N \right| \delta u^N \frac{\partial \delta u^N}{\partial t} \mathrm{d}x$ to the both sides of this equality after some transformation we obtain

$$\int\limits_{\Omega} \left[\frac{1}{2} \frac{\partial}{\partial t} \left| \frac{\partial \delta u^N}{\partial t} \right|^2 + \frac{1}{2} \frac{\partial}{\partial t} \sum_{i=1}^{n} \left| \frac{\partial \delta u^N}{\partial x_i} \right|^2 \right] \mathrm{d}x + \frac{2\theta}{3} \int\limits_{\Omega} \frac{\partial}{\partial t} \left| \delta u^N \right|^3 \mathrm{d}x \leqslant$$
$$\leqslant 2 \int\limits_{\Omega} \left| \left| \theta \delta u^N \right| - \left| u + \theta \delta u^N \right| \right| \left| \delta u^N \right| \left| \frac{\partial \delta u^N}{\partial t} \right| \mathrm{d}x +$$
$$+ \int\limits_{\Omega} |v + \delta v| \left| \delta u^N \right| \left| \frac{\partial \delta u^N}{\partial t} \right| \mathrm{d}x + \int\limits_{\Omega} |u| \, |\delta v| \left| \frac{\partial \delta u^N}{\partial t} \right| \mathrm{d}x.$$

From this integration over t from 0 to t due to (16) gives

$$\int\limits_{\Omega} \left[\left| \frac{\partial \delta u^N}{\partial t} \right|^2 + \sum_{i=1}^{n} \left| \frac{\partial \delta u^N}{\partial x_i} \right|^2 \right] \mathrm{d}x + \frac{4\theta}{3} \int\limits_{\Omega} \left| \delta u^N \right|^3 \mathrm{d}x \leqslant$$
$$\leqslant 4 \int\limits_0^t \int\limits_{\Omega} |u| \left| \delta u^N \right| \left| \frac{\partial \delta u^N}{\partial t} \right| \mathrm{d}x \mathrm{d}s + 2 \int\limits_0^t \int\limits_{\Omega} |v + \delta v| \left| \delta u^N \right| \left| \frac{\partial \delta u^N}{\partial t} \right| \mathrm{d}x \mathrm{d}s +$$
$$+ 2 \int\limits_0^t \int\limits_{\Omega} |u| \, |\delta v| \left| \frac{\partial \delta u^N}{\partial t} \right| \mathrm{d}x \mathrm{d}s.$$

According to elementary inequalities and definition of the class of the admissible controls from this we get

$$\int\limits_{\Omega}\left[\left|\frac{\partial\delta u^N}{\partial t}\right|^2+\sum_{i=1}^{n}\left|\frac{\partial\delta u^N}{\partial x_i}\right|^2\right]\mathrm{d}x\leqslant c\int\limits_0^t\int\limits_{\Omega}|u|\left|\delta u^N\right|\left|\frac{\partial\delta u^N}{\partial t}\right|\mathrm{d}x\mathrm{d}s+$$
$$+c\int\limits_0^t\int\limits_{\Omega}\left[\left|\delta u^N\right|+\sum_{i=1}^{n}\left|\frac{\partial\delta u^N}{\partial x_i}\right|^2+\left|\frac{\partial\delta u^N}{\partial t}\right|^2\right]\mathrm{d}x\mathrm{d}s+c\left\|\delta v\right\|^2_{L_\infty(\Omega)}. \tag{17}$$

It follows from Holder's inequality that

$$\left|\int\limits_{\Omega}\xi\eta\zeta\,\mathrm{d}x\right|\leqslant c\left\|\xi\right\|_{L_p(\Omega)}\left\|\eta\right\|_{L_r(\Omega)}\left\|\zeta\right\|_{L_s(\Omega)}, \tag{18}$$

where $c>0$, $1/p+1/r+1/s=1$.

For $n=3$ or 4 in inequality (18) we set $p=n$, $r=\dfrac{2n}{n-2}$, $s=2$ and $\xi=|u|$, $\eta=\left|\delta u^N\right|$, $\zeta=\left|\dfrac{\partial\delta u^N}{\partial t}\right|$.

Note that, under the conditions of the theorem, it follows from Sobolev's theorem on the continuity of the embedding $H_0^1(\Omega)\subset L_r(\Omega)$ that implies $U\subset L_\infty(0,T;L_r(\Omega))$. Then the following embedding is valid $u\in L_\infty(0,T;L_r(\Omega))$. Considering $n\leqslant r$ we set that $u\in L_\infty(0,T;L_n(\Omega))$.

Using inequality (18) and the equivalence of the norms in the space $H_0^1(\Omega)$, from (17) we obtain

$$\left\|\frac{\partial\delta u^N(t)}{\partial t}\right\|^2_{L_2(\Omega)}+\left\|\delta u^N(t)\right\|^2_{H_0^1(\Omega)}\leqslant$$
$$\leqslant c\int\limits_0^t\left\|u(s)\right\|_{L_n(\Omega)}\left\|\delta u^N(s)\right\|_{L_r(\Omega)}\left\|\frac{\partial\delta u^N(s)}{\partial t}\right\|_{L_2(\Omega)}\mathrm{d}s+$$
$$+\;c\int\limits_0^t\left[\left\|\frac{\partial\delta u^N(s)}{\partial t}\right\|^2_{L_2(\Omega)}+\left\|\delta u^N(s)\right\|^2_{H_0^1(\Omega)}\right]\mathrm{d}s+c\left\|\delta v\right\|^2_{L_\infty(\Omega)}\leqslant$$
$$\leqslant c\int\limits_0^t\left[\left\|\frac{\partial\delta u^N(s)}{\partial t}\right\|^2_{L_2(\Omega)}+\left\|\delta u^N(s)\right\|^2_{H_0^1(\Omega)}+\left\|\delta u^N(s)\right\|^2_{L_r(\Omega)}\right]\mathrm{d}s+c\left\|\delta v\right\|^2_{L_\infty(\Omega)}.$$

According to the embedding $H_0^1(\Omega) \subset L_r(\Omega)$ this implies

$$\left\|\frac{\partial \delta u^N(t)}{\partial t}\right\|^2_{L_2(\Omega)} + \left\|\delta u^N(t)\right\|^2_{H_0^1(\Omega)} \leqslant c\int\limits_0^t \left[\left\|\frac{\partial \delta u^N(s)}{\partial t}\right\|^2_{L_2(\Omega)} + \left\|\delta u^N(s)\right\|^2_{H_0^1(\Omega)}\right] \mathrm{d}s + \\ + c\,\|\delta v\|^2_{L_\infty(\Omega)}.$$

Using the Gronwall's lemma we get

$$\left\|\frac{\partial \delta u^N(t)}{\partial t}\right\|^2_{L_2(\Omega)} + \left\|\delta u^N(t)\right\|^2_{H_0^1(\Omega)} \leqslant c\,\|\delta v\|^2_{L_\infty(\Omega)}, \qquad t \in [0,T]. \tag{19}$$

Then as follows from (19) the sequence $\{\delta u^N(x,t)\}$ is bounded in U. Therefore we can take

$$\delta u^N \to \delta u \qquad * - \text{weekly in } L_\infty(0,T;\ H_0^1(\Omega)),$$
$$\frac{\partial \delta u^N}{\partial t} \to \frac{\partial \delta u}{\partial t} \qquad * - \text{weekly in } L_\infty(0,T;\ L_2(\Omega)).$$

Since the norm is weak lower semicontinuous in Banach spaces, estimate (14) holds for the limit function $\delta u(x,t)$and it can easily be shown that it is a generalized solution to problem (11), (12). Lemma 3 is proved. □

Let $\psi = \psi(x,t;v)$ be a generalized solution from U for the adjoint problem

$$\frac{\partial^2 \psi}{\partial t^2} - \Delta\psi + 2|u|\psi + v\psi = 0, \qquad (x,t) \in Q, \tag{20}$$

$$\psi = 0, \quad (x,t) \in S, \quad \psi|_{t=T} = 0, \quad \left.\frac{\partial \psi}{\partial t}\right|_{t=T} = -[u(x,T;v) - \varphi(x)], \quad x \in \Omega. \tag{21}$$

Equation (20) is linear relative to the function $\psi = \psi(x,t;v)$. One may prove that problem (20), (21) under conditions set on the data of problem (1), (2), (4), (6) in U has the only generalized solution. As a generalized solution of problem (20), (21) at given $v \in V$ we take the function U that is equal to zero $t = T$ and satisfies the integral identity

$$\int\limits_Q \left[-\frac{\partial \psi}{\partial t}\frac{\partial g}{\partial t} + \sum_{i=1}^n \frac{\partial \psi}{\partial x_i}\frac{\partial g}{\partial x_i} + 2|u|\psi g + v\psi g\right] \mathrm{d}x\mathrm{d}t - \\ - \int\limits_\Omega [u(x,T;v) - \varphi(x)]\, g(x,T)\mathrm{d}x = 0, \quad \forall g = g(x,t) \in U. \tag{22}$$

Applying the same technique as in the proof of estimation (14) one can obtain the following estimate for the solution of problem (20), (21)

$$\|\psi\|^2_{H^1_0(\Omega)} + \left\|\frac{\partial\psi}{\partial t}\right\|^2_{L_2(\Omega)} \leqslant c\,\|u(x,T;v)-\varphi(x)\|^2_{H^1_0(\Omega)}, \qquad t\in[0,T]. \tag{23}$$

Then it follows from (8) and (23) that

$$\begin{aligned}\|\psi\|^2_{H^1_0(\Omega)} + \left\|\frac{\partial\psi}{\partial t}\right\|^2_{L_2(\Omega)} \leqslant c\Big[\|u_0\|^2_{H^1_0(\Omega)} + \|u_1\|^2_{L_2(\Omega)} + {}&\\ + \|f\|^2_{L_2(Q)} + \|\varphi\|^2_{H^1_0(\Omega)}\Big], &\qquad t\in[0,T].\end{aligned} \tag{24}$$

Theorem 2 *Let the conditions set on the problem (1), (2), (4), (6) are valid. Then functional (6) is continuously Frechet differentiable on V and its differential at $v\in V$ with the increment $\delta v\in L_\infty(\Omega)$ is defined by the expression*

$$\langle J'_\alpha(v),\delta v\rangle = \int\limits_\Omega\left[\alpha v - \int\limits_0^T u\psi\,\mathrm{d}t\right]\delta v\,\mathrm{d}x. \tag{25}$$

Proof. Consider the increment of functional (6)

$$\begin{aligned}J_\alpha(v) &= J_\alpha(v+\delta v) - J_\alpha(v) = \\ &= \alpha\int\limits_\Omega v\delta v\,\mathrm{d}x + \int\limits_\Omega[u(x,T;v)-\varphi(x)]\,\delta u(x,T)\,\mathrm{d}x + \\ &\quad + \frac{\alpha}{2}\int\limits_\Omega|\delta v|^2\,\mathrm{d}x + \frac{1}{2}\int\limits_\Omega|\delta u(x,T)|^2\,\mathrm{d}x.\end{aligned} \tag{26}$$

If in (13) take $\eta=\psi(x,t;v)$ and in (22) take $g=\delta u(x,t)$ and sum the obtained expressions we get

$$\begin{aligned}&\int\limits_Q 2\psi\delta u\,[|u|-|u+\theta\delta u|]\,\mathrm{d}x\mathrm{d}t - \int\limits_Q \psi\delta u\delta v\,\mathrm{d}x\mathrm{d}t \\ &\qquad - \int\limits_\Omega[u(x,T;v)-\varphi(x)]\,\delta u(x,T)\,\mathrm{d}x = \int\limits_Q u\psi\delta v\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

It follows from this

$$\int\limits_{\Omega} [u(x,T;v) - \varphi(x)]\, \delta u(x,T)\mathrm{d}x = -\int\limits_{Q} u\psi\delta v \,\mathrm{d}x\mathrm{d}t + \\ + 2\int\limits_{Q} \psi\delta u\,[|u| - |u+\theta\delta u|]\,\mathrm{d}x\mathrm{d}t - \int\limits_{Q} \psi\delta u\delta v\,\mathrm{d}x\mathrm{d}t. \tag{27}$$

If consider formula (27) in (26) we obtain the following formula for the increment of functional (6)

$$\Delta J_\alpha(v) = \int\limits_{\Omega} \left[\alpha v(x) - \int\limits_0^T u(x,t;v)\psi(x,t;v)\mathrm{d}t\right] \delta v(x)\mathrm{d}x + R, \tag{28}$$

where

$$R = \frac{\alpha}{2}\int\limits_{\Omega} |\delta v(x)|^2\,\mathrm{d}x + \frac{1}{2}\int\limits_{\Omega} |\delta u(x,T)|^2\,\mathrm{d}x - \int\limits_{Q} \psi\delta u\delta v\,\mathrm{d}x\mathrm{d}t + \\ + 2\int\limits_{Q} \psi\delta u\,[|u| - |u+\theta\delta u|]\,\mathrm{d}x\mathrm{d}t$$

is a remainder term.

From this expression we obtain

$$|R| \leqslant c\|\delta v\|^2_{L_\infty(\Omega)} + \frac{1}{2}\|\delta u(x,T)\|^2_{L_\infty(\Omega)} + \int\limits_{Q} |\psi|\,|\delta u|\,\mathrm{d}x\mathrm{d}t\|\delta v\|_{L_\infty(\Omega)} + \\ + c\int\limits_{Q} |\psi|\,|\delta u|^2\,\mathrm{d}x\mathrm{d}t \leqslant c\,\|\delta v\|^2_{L_\infty(\Omega)} + \frac{1}{2}\|\delta u(x,T)\|^2_{L_2(\Omega)} + \\ + \int\limits_0^T \left(\int\limits_{\Omega} |\psi|^2\mathrm{d}x\right)^{1/2} \left(\int\limits_{\Omega} |\delta u|^2\mathrm{d}x\right)^{1/2} \mathrm{d}t\|\delta v\|_{L_\infty(\Omega)} + \\ + c\int\limits_0^T \left(\int\limits_{\Omega} |\psi|^2\mathrm{d}x\right)^{1/2} \left(\int\limits_{\Omega} |\delta u|^4\mathrm{d}x\right)^{1/2} \mathrm{d}t.$$

Considering estimates (14) and (24) and the embedding theorem $H_0^1(\Omega) \subset L_4(\Omega)$it gives

$$|R| \leqslant c\,\|\delta v\|^2_{L_\infty(\Omega)}\,. \tag{29}$$

Then as follows from (28) and (29) functional (6) is Frechet differentiable on V and formula (25) is valid. Show that the mapping $v \to J'_\alpha(v)$ defined by expression (25) acts continuously from V to the adjoint $(L_\infty(\Omega))^*$ of the space $L_\infty(\Omega)$. Let $\delta\psi(x,t) = \psi(x,t;v+\delta v) - \psi(x,t;v)$. From (20), (21) we get that $\delta\psi(x,t)$ is a generalized solution from U for the boundary value problem

$$\frac{\partial^2 \delta\psi}{\partial t^2} - \Delta\delta\psi + 2\left[|u+\delta u|\psi(x,t;v+\delta v) - |u|\psi(x,t;v)\right] + \\ + (v+\delta v)\delta\psi = -\psi\delta v, \qquad (x,t) \in Q, \tag{30}$$

$$\delta\psi = 0, \quad (x,t) \in S, \quad \delta\psi|_{t=T} = 0, \quad \left.\frac{\partial\psi}{\partial t}\right|_{t=T} = -\delta u(x,T), \quad x \in \Omega. \tag{31}$$

From (30), (31) one may obtain the estimate

$$\|\delta\psi\|^2_{H^1_0(\Omega)} + \left\|\frac{\partial\delta\psi}{\partial t}\right\|^2_{L_2(\Omega)} \leqslant c\,\|\delta v\|^2_{L_\infty(\Omega)}, \quad t \in [0,T]. \tag{32}$$

In addition, using (25) we can verify the inequality

$$\|J'_\alpha(v+\delta v) - J'_\alpha(v)\|_{(L_\infty(\Omega))^*} \leqslant \\ \leqslant \int_\Omega \left\{\alpha|\delta v| + \int_0^T \left[|u||\delta\psi| + |\psi||\delta u| + |\delta u||\delta\psi|\right] \mathrm{d}t\right\} \mathrm{d}x \leqslant \\ \leqslant c\|\delta v\|_{L_\infty(\Omega)} + c\left[\|\delta u\|_{L_2(Q)} + \|\delta\psi\|_{L_2(Q)} + \|\delta u\|_{L_2(Q)}\|\delta\psi\|_{L_2(Q)}\right].$$

By virtue of estimates (14) and (32), the right-hand side of this inequality tends to zero for $\|\delta v\|_{L_\infty(\Omega)} \to 0$. From this it follows that $v \to J'_\alpha(v)$ is a continuous mapping from V to $(L_\infty(\Omega))^*$. Theorem 2 is proved. □

Theorem 3 *Let the conditions of Theorem 2 are valid. Then for the optimality of the control $v_* = v_*(x) \in V$ in problem (1), (2), (4), (6) it is necessary fulfilment of the inequality*

$$\int_\Omega \left[\alpha u_*(x) - \int_0^T u_*(x.t)\psi_*(x,t)\mathrm{d}x\right](v(x) - v_*(x))\mathrm{d}x \geqslant 0 \tag{33}$$

for arbitrary control $v = v(x) \in V$, where $u_*(x,t) = u(x,t;v_*)$, $\psi_*(x,t) = \psi(x,t;v)$ is a solution of problems (1), (2) and (20), (21) correspondingly at $v = v_*(x)$.

Proof. The set V defined by relation (4) is convex in $L_\infty(\Omega)$. In addition by virtue of Theorem 2 the functional $J_\alpha(u)$ is continuously Frechet differentiable on V and its differential in the point $v \in V$ is defined by equality (25). By virtue of Theorem 5 [11, p. 28] on the element $v_* \in V$ is necessary fulfilment of the inequality $\langle J'_\alpha(v_*), v - v_* \rangle \geqslant 0$, $\forall v \in V$. From this and (25) follows the validity of inequality (33). Theorem 3 is proved. □

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