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Controllabilty and stability analysis on a group associated with Black-Scholes equation

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In this paper we have studied the driftless control system on a Lie group which arises due to the invariance of Black-Scholes equation by conformal transformations. These type of studies are possible as Black-Scholes equation can be mapped to one dimensional free Schrödinger equation. In particular we have studied the controllability, optimal control of the resulting dynamics as well as stability aspects of this system. We have also found out the trajectories of the states of the system through two unconventional integrators along with conventional Runge-Kutta integrator.

Key words: Black-Scholes equation, Schrödinger equation, Lie group, optimal control, stability, numerical integration

1. Introduction

The Black-Scholes equation [6, 17] is a partial differential equation (PDE) governing the price evolution of a European call or European put under the Black-Scholes-Merton model. The equation states that over any infinitesimal time interval the loss from theta(a time decay term) and the gain from the gamma (term involving the second spatial derivative) term offset each other, so that the result is a return at the risk less rate. The key monetary understanding is that one can superbly hedge the option by purchasing and offering the basic resource in simply the correct way and subsequently wipe out the risk. This support suggests that there is just a single right cost for the alternative, as returned by the Black-Scholes equation. This type of hedging is called “continuously revised delta hedging” and is the basis of more complicated hedging strategies such as those engaged in by investment banks and hedge funds.

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The Black-Scholes-Merton model's suppositions have been loose and summed up in numerous ways, prompting a plenty of models that are right now utilized in derivative pricing and risk management [3, 5]. It is the bits of knowledge of the model, as exemplified in the Black-Scholes formula, that are every now and again utilized by market participants, as recognized from the genuine costs. These bits of knowledge incorporate no-arbitrage limits and risk unbiased estimating (on account of persistent modification). Further, the Black-Scholes equation, a PDE that administers the cost of the option, empowers estimating utilizing numerical techniques when an explicit formula isn't conceivable.

The Black-Scholes model can be interpreted as a Schrödinger equation for a free particle, where mass is identified as the inverse of square of the volatility [13]. It is well known that when symmetries are present in a physical system we can get at the properties of a system without completely solving all the equations that describes the system. This technique that arises in quantum mechanics can now be applied to our problem because it has been already shown that Black-Scholes equation is invariant under conformal transformation giving rise to a group from which Schrödinger algebra representation in terms of finance variables is constructed [10].

In the next section, we have briefly reviewed the relationship between Black Scholes Equation and the Schrödinger group. In section 3, we have constructed a drift-free control system, and the controllability and optimal control aspects are studied. We have also derived Casimir functions of our Poisson system and stability analysis of the resulting dynamics is discussed in details. Section 4 shows the numerical integration performed on the dynamics. Section 5 demonstrates the trajectories resulting from numerical integrators. Section 6 concludes the paper.

2. Black Scholes equation and the Schrödinger group

This section deals with the study of Black-Scholes equation and its relation with the Schrödinger equation, which is reviewed from [10]. The Black-Scholes equation is given by

$$\frac{\partial C(s,t)}{\partial t} = \frac{-\sigma^2}{2} s^2 \frac{\partial^2 C(s,t)}{\partial s^2} - rs\frac{\partial C(s,t)}{\partial s} + rC(s,t). \tag{1}$$

The price of a derivative is denoted by C, s is the stock price, volatility is σ and the annualized risk free interest rate is given by r.

The equation of Black-Scholes is equivalent to Schrödinger equation. Under the change of variable $s = e^x$ and if

$$C(x,t) = e^{\left[\frac{1}{\sigma^2}\left(\frac{\sigma^2}{2}-r\right)x+\frac{1}{2\sigma^2}\left(\frac{\sigma^2}{2}+r\right)^2 t\right]}\psi(x,t) \tag{2}$$

the equation (1) reduces to

$$\frac{\partial\psi(x,t)}{\partial t} = \frac{-\sigma^2}{2}\frac{\partial^2\psi(x,t)}{\partial t^2} \tag{3}$$

which is similar to the Schrödinger equation. The quantities like the Momentum P, the Hamiltonian H, the Galileo's transformation G, Anisotropic scaling K_1, and Special conformal transformation K_2 for one dimensional non-relativistic particle can be represented by the following operators

$$\begin{aligned}
\hat{P} &= -i\hbar\frac{\partial}{\partial x},\\
\hat{H} &= \frac{\hat{P}^2}{2m},\\
\hat{G} &= t\hat{P} - mx,\\
\hat{K}_1 &= t\hat{H} - \frac{1}{4}\left(x\hat{P} + \hat{P}x\right),\\
\hat{K}_2 &= t^2\hat{H} - \frac{t}{2}\left(x\hat{P} + \hat{P}x\right) + \frac{m}{2}x^2.
\end{aligned} \tag{4}$$

Under conformal coordinate transformations with the change of variable $s = e^x$, the Black-Scholes equation (3) is invariant as the Schrödinger equation is found to be invariant under the same coordinate transformation.

The Black-Scholes equation (1) is now represented as

$$\frac{\partial C(s,t)}{\partial t} = \hat{\mathbf{H}}C(s,t) \tag{5}$$

with

$$\hat{\mathbf{H}} = \frac{-\sigma^2}{2}s^2\frac{\partial^2}{\partial s^2} - rs\frac{\partial}{\partial s} + r.$$

Using the operator

$$\hat{\Pi} = -is\frac{\partial}{\partial s} + \frac{i}{\sigma^2}\left(\frac{\sigma^2}{2} - r\right), \tag{6}$$

$\hat{\mathbf{H}}$ is written as

$$\hat{\mathbf{H}} = \frac{\sigma^2}{2}\hat{\Pi}^2 + \frac{1}{2\sigma^2}\left(\frac{\sigma^2}{2} + r\right)^2. \tag{7}$$

Here $\hat{\mathbf{H}}$ is equivalent to Hamiltonian $\hat{H}$ and $\frac{1}{\sigma^2}$ is associated with the particle mass m.

Using operator (6), the following operators are constructed, which is similar to the conserved quantities (4) of the Schrödinger equation

$$
\begin{aligned}
\hat{\Pi} &= -is\frac{\partial}{\partial s} + \frac{i}{\sigma^2}\left(\frac{\sigma^2}{2} - r\right),\\
\hat{\mathbf{H}}_0 &= \frac{\sigma^2}{2}\hat{\Pi}^2,\\
\hat{\mathbf{G}} &= t\hat{\Pi} - \frac{1}{\sigma^2}ln\ s,\\
\hat{\mathbf{K}}_1 &= t\hat{\mathbf{H}}_0 - \frac{1}{4}\left(\ln\ s\hat{\Pi} + \hat{\Pi}\ln\ s\right),\\
\hat{\mathbf{K}}_2 &= t^2\hat{\mathbf{H}}_0 - \frac{t}{2}\left(\ln\ s\hat{\Pi} + \hat{\Pi}\ln\ s\right) + \frac{1}{2\sigma^2}(\ln\ s)^2 .
\end{aligned}
\tag{8}
$$

Using the commutation relation

$$\left[\ln\ s,\ \hat{\Pi}\right] = i, \tag{9}$$

the following algebra

$$
\begin{aligned}
\left[\hat{\Pi},\ \hat{\mathbf{H}}_0\right] &= 0,\\
\left[\hat{\Pi},\ \hat{\mathbf{K}}_1\right] &= \frac{i}{2}\hat{\Pi},\\
\left[\hat{\Pi},\ \hat{\mathbf{K}}_2\right] &= i\hat{\mathbf{G}},\\
\left[\hat{\Pi},\ \hat{\mathbf{G}}\right] &= \frac{i}{\sigma^2},\\
\left[\hat{\mathbf{H}}_0,\ \hat{\mathbf{K}}_1\right] &= i\hat{\mathbf{H}}_0,\\
\left[\hat{\mathbf{H}}_0,\ \hat{\mathbf{G}}\right] &= i\hat{\Pi},\\
\left[\hat{\mathbf{H}}_0,\ \hat{\mathbf{K}}_2\right] &= 2i\hat{\mathbf{K}}_1,\\
\left[\hat{\mathbf{K}}_1,\ \hat{\mathbf{K}}_2\right] &= i\hat{\mathbf{K}}_2,\\
\left[\hat{\mathbf{K}}_1,\ \hat{\mathbf{G}}\right] &= \frac{i}{2}\hat{\mathbf{G}},\\
\left[\hat{\mathbf{K}}_2,\ \hat{\mathbf{G}}\right] &= 0
\end{aligned}
\tag{10}
$$

is obtained which satisfies the Schrödinger algebra.

Now onwards we denote $B_1 = \hat{\mathbf{H}}_0$, $B_2 = \hat{\mathbf{K}}_1$, $B_3 = \hat{\mathbf{K}}_2$, $B_4 = \hat{\Pi}$, $B_5 = \hat{\mathbf{G}}$ and $B_6 = \dfrac{1}{\sigma^2}$ as the generators of the algebra and the commutation table is given in Table 1.

Table 1: Lie bracket commutation table

[.,.]	B_1	B_2	B_3	B_4	B_5	B_6
B_1	0	B_1	$2B_2$	0	B_4	0
B_2	$-B_1$	0	B_3	$\frac{-1}{2}B_4$	$\frac{1}{2}B_5$	0
B_3	$-2B_2$	$-B_3$	0	$-B_5$	0	0
B_4	0	$\frac{1}{2}B_4$	B_5	0	B_6	0
B_5	$-B_4$	$\frac{-1}{2}B_5$	0	$-B_6$	0	0
B_6	0	0	0	0	0	0

3. Driftless control system

A control system [1] on a n dimensional Lie group is given by [15]

$$\dot{X} = XA + X\left(\sum_{i=1}^{m} u_i B_i\right), qquad m \leqslant n. \tag{11}$$

Here $A, B_1 \cdots, B_m$ are the left invariant vector fields on the Lie group, $u_1, \cdots, u_m$ belongs to the admissible controls U and X represents the state. For $A = 0$ the equation (11) is a driftless control system, defined as

$$\dot{X} = X\left(\sum_{i=1}^{m} u_i B_i\right), \qquad m \leqslant n. \tag{12}$$

We now consider a driftless control system as

$$\dot{X} = X\left(B_1 u_1 + B_3 u_3 + B_4 u_4\right). \tag{13}$$

where B_1, B_3 and B_4 are subset of the Schrödinger algebra as shown in Table 1.

3.1. Controllability

When a system can be steered by a control input $u(t)$ from an initial state $X(t_0) = x_0$ to a desired final state $X(t_f) = x_f$ in a finite time interval, the system is said to be controllable. To analyze the controllability of the driftless system on a Lie group, we use the Chow-Rashevesky Theorem, which states

"If M is a connected manifold and the control distribution $\Delta = \text{span}\{g_1, g_2, \ldots, g_n\}$

is bracket generating, then the drift-free system

$$\dot{X} = \sum_{i=1}^{n} x_i g_i(x), \qquad x \in M \tag{14}$$

is controllable" [7].

The Lie algebra generated by the span of the set $\{B_1, B_3, B_4\}$ coincides with full Schrödinger algebra, as upon Lie bracket operation, $\{B_1, B_3, B_4\}$ generates the whole set of basis elements. Hence by Chow-Rashevesky Theorem, the system in (13) is controllable.

3.2. Optimal control

Optimal control considers the problem of finding a control input with the end goal that a specific optimal standard is accomplished. An optimal control problem includes a cost functional, i.e., a function of state and control variables. There are different methods of optimization of the control input, like, minimization of time, minimization of effort, infinite horizon optimal control, etc. Here we have considered minimization of effort and designed the input choices in such a way that minimization of input cost function is accomplished. The cost function for our system is

$$\mathcal{F}(u_1, u_3, u_4) = \frac{1}{2} \int_{0}^{t_f} \left(a_1 {u_1}^2 + a_3 {u_3}^2 + a_4 {u_4}^2 \right) \mathrm{d}t, \qquad a_1, a_3, a_4 > 0.$$

In order to minimize $\mathcal{F}$, the constructed controlled Hamiltonian is

$$\overline{H_c} = x_1 u_1 + x_3 u_3 + x_4 u_4 - \frac{1}{2}(a_1 u_1^2 + a_3 u_3^2 + a_4 u_4^2). \tag{15}$$

As per Krishnaprasad's theorem [16], if $\overline{H_c}$ is differential, then the necessary condition for the optimal input for minimizing the cost is $\dfrac{\partial \overline{H_c}}{\partial u_i} = 0$, i.e.,

$$\frac{\partial \overline{H_c}}{\partial u_1} = \frac{\partial \overline{H_c}}{\partial u_3} = \frac{\partial \overline{H_c}}{\partial u_4} = 0. \tag{16}$$

The optimal control inputs obtained from (16) are

$$u_1 = \frac{x_1}{a_1}, \qquad u_3 = \frac{x_3}{a_3}, \qquad u_4 = \frac{x_4}{a_4}\,.$$

Substituting the values of optimal control inputs in equation (15), we obtain the optimal Hamiltonian as follows

$$H_c(x_1, x_2, x_3, x_4, x_5, x_6) = \frac{1}{2}\left(\frac{x_1^2}{a_1} + \frac{x_3^2}{a_3} + \frac{x_4^2}{a_4}\right).$$

During control input optimization, the system is steered such that it complies with the dynamics found applying the Krishnaprasad's theorem. According to the theorem, the restricted dynamics is given by

$$[\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4, \dot{x}_5, \dot{x}_6]^t = \Omega_- \cdot \nabla H_c. \tag{17}$$

Ω_- denotes the minus Lie Poisson matrix and it is expressed as

$$\Omega_- = \begin{bmatrix} 0 & -x_1 & -2x_2 & 0 & -x_4 & 0 \\ x_1 & 0 & -x_3 & \frac{x_4}{2} & \frac{-x_5}{2} & 0 \\ 2x_2 & x_3 & 0 & x_5 & 0 & 0 \\ 0 & \frac{-x_4}{2} & -x_5 & 0 & -x_6 & 0 \\ x_4 & \frac{x_5}{2} & 0 & x_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The system dynamics of (17) is expressed explicitly as follows

$$\begin{cases} \dot{x}_1 &= -\dfrac{2}{a_3}x_2x_3\,, \\ \dot{x}_2 &= \dfrac{1}{a_1}x_1^2 - \dfrac{1}{a_3}x_3^2 + \dfrac{1}{2a_4}x_4^2, \\ \dot{x}_3 &= \dfrac{2}{a_1}x_1x_2 + \dfrac{1}{a_4}x_4x_5\,, \\ \dot{x}_4 &= -\dfrac{1}{a_3}x_3x_5\,, \\ \dot{x}_5 &= \dfrac{1}{a_1}x_1x_4 + \dfrac{1}{a_4}x_4x_6\,, \\ \dot{x}_6 &= 0. \end{cases} \tag{18}$$

3.3. Casimir

Casimir is a function which is in involution with all smooth functions on the Poisson manifold, i.e., $\{C, f\} = 0$, for all smooth functions f. In other words, the

Casimir function is constant along the orbits of all hamiltonian vector field. In order to find the Casimir function of a Poisson system, the following equation is solved [9]

$$\Omega_- \cdot (\nabla C(x_1, x_2, x_3, x_4, x_5, x_6))^t = 0. \tag{19}$$

The system of PDEs obtained from Eq. (19) is

$$\begin{aligned}
x_1 \frac{\partial C}{\partial x_2} + 2x_2 \frac{\partial C}{\partial x_3} + x_4 \frac{\partial C}{\partial x_5} &= 0, \\
-x_1 \frac{\partial C}{\partial x_1} + x_3 \frac{\partial C}{\partial x_3} - \frac{x_4}{2} \frac{\partial C}{\partial x_4} + \frac{x_5}{2} \frac{\partial C}{\partial x_5} &= 0, \\
-2x_2 \frac{\partial C}{\partial x_1} - x_3 \frac{\partial C}{\partial x_2} - x_5 \frac{\partial C}{\partial x_4} &= 0, \\
\frac{x_4}{2} \frac{\partial C}{\partial x_2} + x_5 \frac{\partial C}{\partial x_3} + x_6 \frac{\partial C}{\partial x_5} &= 0, \\
-x_4 \frac{\partial C}{\partial x_1} - \frac{x_5}{2} \frac{\partial C}{\partial x_2} - x_6 \frac{\partial C}{\partial x_4} &= 0.
\end{aligned} \tag{20}$$

The Casimirs obtained by solving (20) are

$$C_1 = x_6, \tag{21}$$

$$C_2 = \left(-2x_3x_6 + x_5^2\right) x_1 + 2x_2^2 x_6 - 2x_2x_4x_5 + x_3x_4^2. \tag{22}$$

3.4. Stability

The equilibrium points of a dynamical system represents a stationary condition for the dynamics. It is one of the most significant features of a system, as the equilibrium points defines the state, corresponding to the constant operating conditions. These are the points where motion of the body freezes. For a dynamical system $\dot{x} = F(x)$, a point e is an equilibrium point if $F(e) = 0$. For the ease of computation of the equilibrium states, the a_i's have been replaced with a. The equilibrium states $\mathcal{E} = \{e_i, i = 1, \cdots, 5\}$ computed here are not unique, since $\mathcal{E} \subset E$, where E denotes the set of all possible equilibrium states. The elements of set $\mathcal{E}$ are:

$$\begin{aligned}
e_1^{m_1,m_2} &= (m_1, 0, m_1, 0, 0, m_2), && m_1, m_2 \in \mathbb{R}, \\
e_2^{m_1} &= \left(0, 0, m_1, \sqrt{2}m_1, 0, 0\right), && m_1 \in \mathbb{R}, \\
e_3^{m_1,m_2,m_3} &= (0, m_1, 0, 0, m_2, m_3), && m_1, m_2, m_3 \in \mathbb{R}, \\
e_4^{m_1,m_2} &= \left(m_1, 0, \sqrt{m_1^2 + \frac{m_2^2}{2}}, m_2, 0, -m_1\right), && m_1, m_2 \in \mathbb{R},
\end{aligned}$$

$$e_5^{m_1} = \left(0, 0, \frac{m_1}{\sqrt{2}}, m_1, 0, 0\right), \qquad m_1 \in \mathbb{R}.$$

The linearized Jacobian matrix for equation (18) is

$$J = \begin{bmatrix} 0 & \frac{-2x_3}{a} & \frac{-2x_2}{a} & 0 & 0 & 0 \\ \frac{2x_1}{a} & 0 & \frac{-2x_3}{a} & \frac{x_4}{a} & 0 & 0 \\ \frac{2x_2}{a} & \frac{2x_1}{a} & 0 & \frac{x_5}{a} & \frac{x_4}{a} & 0 \\ 0 & 0 & \frac{-x_5}{a} & 0 & \frac{-x_3}{a} & 0 \\ \frac{x_4}{a} & 0 & 0 & \frac{x_1}{a} + \frac{x_6}{a} & 0 & \frac{x_4}{a} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Next we have computed the eigenvalues corresponding to each equilibrium state. The type(real or complex) and sign of roots of the characteristic polynomial gives a clear insight on the stability of the system. For all the propositions given below we have considered the case $a \in \mathbb{R} - \{0\}$, because for $a = 0$, all the eigen values are not defined, which makes the system stability inconclusive. Also $m_1, m_2, m_3 \in \mathbb{R}$.

Proposition 1 *The equilibrium states* $e_1^{m_1,m_2}$

1. *Are unstable when* m_1, m_2 *are simultaneously not equal to zero and* $m_1 < |m_2|,\ m_2 < 0$ *or* $|m_1| < m_2,\ m_1 < 0$,

2. *Are spectrally stable when* m_1, m_2 *are both simultaneously positive or simultaneously negative.*

Proof. The eigen values of the system which is linearized about $e_1^{m_1,m_2}$ are:

$$\nu_1 = \nu_2 = 0, \quad \nu_3 = i\frac{\sqrt{m_1(m_1+m_2)}}{a}, \quad \nu_4 = -i\frac{\sqrt{m_1(m_1+m_2)}}{a},$$

$$\nu_5 = i\frac{2\sqrt{2}m_1}{a}, \quad \nu_6 = -i\frac{2\sqrt{2}m_1}{a}.$$

Case I. For $m_1 < |m_2|,\ m_2 < 0$ or $|m_1| < m_2,\ m_1 < 0$ the characteristic polynomial has at least one root with a positive real part. So the system is unstable.

Case II. For m_1, m_2 both positive or both negative, all the roots of the characteristic polynomial are imaginary. So the system is spectrally stable.

Also if $m_1 = 0$ then the stability of the system is uncertain. □

Proposition 2 *The equilibrium states $e_2^{m_1}$ are unstable for $m_1 \neq 0$.*

Proof. The eigen values of the system which is linearized about $e_2^{m_1}$ are:

$$\nu_1 = \nu_2 = 0, \quad \nu_3 = \frac{2m_1}{a}, \quad \nu_4 = -\frac{2m_1}{a}, \quad \nu_5 = i\frac{2m_1}{a}, \quad \nu_6 = -i\frac{2m_1}{a}.$$

The system is unstable as there is at least one root of the characteristic polynomial whose real part is positive for $m_1 \neq 0$. The system stability is uncertain when $m_1 = 0$. □

Proposition 3 *The equilibrium states $e_3^{m_1,m_2,m_3}$ are spectrally stable for m_1, m_2 simultaneously not equal to zero.*

Proof. The eigen values of the system which is linearized about $e_3^{m_1,m_2,m_3}$ are:

$$\nu_1 = \nu_2 = \nu_3 = \nu_4 = 0, \quad \nu_5 = i\frac{\sqrt{4m_1^2 + m_2^2}}{a}, \quad \nu_6 = -i\frac{\sqrt{4m_1^2 + m_2^2}}{a}.$$

For $4m_1^2 + m_2^2 \neq 0$, the system is spectrally stable because all the roots of the characteristic polynomial are purely imaginary. The system stability is uncertain when $m_1 = m_2 = 0$. □

Proposition 4 *The equilibrium states $e_4^{m_1,m_2}$*

1) *are unstable for any m_1, m_2 simultaneously not equal to zero,*

2) *are spectrally stable when $m_2 = 0$.*

Proof. The eigen values of the system which is linearized about $e_4^{m_1,m_2}$ are:
$\nu_1 = \nu_2 = 0$,

$$\nu_3 = -\sqrt{\frac{2\left(2m_1^2 + m_2^2 - 2m_1\sqrt{m_1^2 + \frac{m_2^2}{2}}\right)}{a^2}},$$

$$\nu_4 = -i\sqrt{\frac{2\left(2m_1^2 + m_2^2 + 2m_1\sqrt{m_1^2 + \frac{m_2^2}{2}}\right)}{a^2}},$$

$$\nu_5 = \sqrt{\frac{2\left(2m_1^2 + m_2^2 - 2m_1\sqrt{m_1^2 + \frac{m_2^2}{2}}\right)}{a^2}},$$

$$\nu_6 = i\sqrt{\frac{2\left(2m_1^2 + m_2^2 + 2m_1\sqrt{m_1^2 + \frac{m_2^2}{2}}\right)}{a^2}}.$$

Case I. For m_1, m_2 not simultaneously equal to zero, the characteristic polynomial has at least one root whose real part is positive. So the system is unstable.

Case II. For $m_2 = 0$, all the roots of the characteristic polynomial are imaginary. So the system is spectrally stable.

Also, the system stability is found to be uncertain when $m_1 = m_2 = 0$. □

Proposition 5 *The equilibrium states $e_5^{m_1}$ are unstable for any $m_1 \neq 0$.*

Proof. The eigen values of the system which is linearized about $e_5^{m_1}$ are:

$$\nu_1 = \nu_2 = 0, \quad \nu_3 = \frac{\sqrt{2}m_1}{a}, \quad \nu_4 = -\frac{\sqrt{2}m_1}{a}, \quad \nu_5 = i\frac{\sqrt{2}m_1}{a}, \quad \nu_6 = -i\frac{\sqrt{2}m_1}{a}.$$

The system is found to be unstable due to the existence of at least one root of the characteristic polynomial whose real part is positive for $m_1 \neq 0$. The system stability is found to be uncertain when $m_1 = 0$. □

4. Numerical integration of dynamics

The system (18) has simultaneous nonlinear ordinary differential equations(ODEs) which are very difficult to solve analytically. Hence, the ODEs are solved applying numerical approaches. In this work, the ODEs have been solved numerically using unconventional integrators which gives insight on the Poisson structure and its geometry. The properties used to select the integrators for numerical computations of system (18) are:

Poisson preservation

An integrator $\varphi : \mathbb{R}^q \to \mathbb{R}^q$ is found to be Poisson preserving if for smooth function $F_1, F_2 : \mathbb{R}^q \to \mathbb{R}$, it preserves the poisson bracket, i.e., $\{F_1, F_2\} \circ \varphi = \{F_1 \circ \varphi, F_2 \circ \varphi\}$, which is equivalent to the following condition [12]:

$$\varphi_\eta(\eta) Z(\eta) \varphi_\eta^T(\eta) = Z(\varphi(\eta)) \tag{23}$$

where Z is the Poisson tensor (skew symmetric) matrix, which satisfies the Jacobi identity. Let $\eta(t) = X^n = [x_1^n \;\; x_2^n \;\; \cdots\cdots \;\; x_q^n]$ be the solution of dynamics, then

$$\eta(t+h) = X^{n+1}$$

where h is the step length. So,

$$\varphi_\eta(\eta) = \frac{\partial(\eta(t+h))}{\partial(\eta(t))} = \frac{\partial X^{n+1}}{\partial X^n} = Z' \tag{24}$$

where Z' is the Fréchet derivative. Subsequently, Eq. (23) reduces to

$$Z' \cdot Z(X^n) \cdot (Z')^T = Z(X^{n+1}). \tag{25}$$

Casimir preservation

The Casimir C_i, $i = 1, 2 \ldots n$ of a system is preserved when C_i remains constant along $\eta(t)$, that is

$$\frac{\mathrm{d}}{\mathrm{d}t} C_i(\eta(t)) = 0. \tag{26}$$

The discretized form of (26) is

$$\begin{aligned} & \frac{C_i(\eta(t+h)) - C_i(\eta(t))}{h} = 0 \\ \Rightarrow \quad & C_i(\eta(t+h)) - C_i(\eta(t)) = 0 \\ \Rightarrow \quad & C_i(X^{n+1}) = C_i(X^n). \end{aligned} \tag{27}$$

Hamiltonian or Energy preservation

Hamiltonian H_c of a system is preserved if the Hamiltonian H_c remains constant along $\eta(t)$, that is

$$H_c(\eta(t)) = \text{constant} \qquad \text{or} \qquad \frac{\mathrm{d}}{\mathrm{d}t} H_c(\eta(t)) = 0. \tag{28}$$

The discretized form of (28) is

$$\begin{aligned} & \frac{H_c(\eta(t+h)) - H_c(\eta(t))}{h} = 0 \\ \Rightarrow \quad & H_c(\eta(t+h)) - H_c(\eta(t)) = 0 \\ \Rightarrow \quad & H_c(X^{n+1}) = H_c(X^n). \end{aligned} \tag{29}$$

4.1. Kahan's integrator

Kahan integrator is an unconventional discretization method which preserves the affine preserving symmetries, foliations and all affine symmetric integrals [11, 18]. In this work, different properties of the sympletic poisson system has been analyzed using Kahan integrator.

Kahan's integrator for (18) is

$$\begin{cases} x_1^{n+1} - x_1^n &= -\dfrac{h}{a_3}\left(x_2^{n+1}x_3^n + x_3^{n+1}x_2^n\right), \\ x_2^{n+1} - x_2^n &= \dfrac{h}{a_1}\left(x_1^{n+1}x_1^n\right) - \dfrac{h}{a_3}\left(x_3^{n+1}x_3^n\right) + \dfrac{h}{2a_4}\left(x_4^{n+1}x_4^n\right), \\ x_3^{n+1} - x_3^n &= \dfrac{h}{a_1}\left(x_1^{n+1}x_2^n + x_2^{n+1}x_1^n\right) + \dfrac{h}{2a_4}\left(x_4^{n+1}x_5^n + x_5^{n+1}x_4^n\right), \\ x_4^{n+1} - x_4^n &= -\dfrac{h}{2a_3}\left(x_3^{n+1}x_5^n + x_5^{n+1}x_3^n\right), \\ x_5^{n+1} - x_5^n &= \dfrac{h}{2a_1}\left(x_1^{n+1}x_4^n + x_4^{n+1}x_1^n\right) + \dfrac{h}{2a_4}\left(x_4^{n+1}x_6^n + x_6^{n+1}x_4^n\right), \\ x_6^{n+1} - x_6^n &= 0. \end{cases} \tag{30}$$

Proposition 6 *We have the following observations for the Kahan's integrator of the system under study*

1. *The Poisson structure is not preserved.*
2. *The Casimir C_1 is preserved but Casimir C_2 is not preserved.*
3. *Hamiltonian H_c is not preserved.*

Proof. From the computations of X^{n+1} in Eqs. (30) we have found

$$Z' \cdot Z(X^n) \cdot (Z')^T \neq Z(X^{n+1}).$$

It is evident that Poisson structure is not preserved for our system.

Also

$$C_1(X^{n+1}) = C_1(X^n),$$

$$C_2(X^{n+1}) \neq C_2(X^n),$$

$$H_c(X^{n+1}) \neq H_c(X^n).$$

Hence the Casimir C_2 and Hamiltonian are not preserved for our system.

4.2. Lie-Trotter integrator

The Lie Trotter integrator [8] is another unconventional numerical integration scheme which involves splitting of the Hamiltonian H_c into H_{c1}, H_{c2}, $\cdots$, H_{cn} for explicit computations of the dynamics generated by $H_{c1}, H_{c2}, \cdots, H_{cn}$. The Poisson structure of phase space as well as symplectic leaves of the Poisson manifold are preserved by this integrator. Therefore, this integrator is employed to analyze the preservation of certain properties of the sympletic Poisson system.

The Hamiltonian vector field X_{H_c} has been split as

$$X_{H_c} = X_{H_{c1}} + X_{H_{c3}} + X_{H_{c4}}$$

where

$$H_{c1} = \frac{x_1^2}{2a_1}, \qquad H_{c3} = \frac{x_3^2}{2a_3}, \qquad H_{c4} = \frac{x_4^2}{2a_4}.$$

The integral curves are as under

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix} = \Lambda_i \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \\ x_5(0) \\ x_6(0) \end{bmatrix}, \qquad i = 1, 3, 4,$$

where

$$\Lambda_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \alpha t & 1 & 0 & 0 & 0 & 0 \\ \alpha^2 t^2 & 2\alpha t & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \alpha t & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \alpha = \frac{x_1(0)}{a_1};$$

$$\Lambda_3 = \begin{bmatrix} 1 & -2\beta t & \beta^2 t^2 & 0 & 0 & 0 \\ 0 & 1 & -\beta t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\beta t & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \beta = \frac{x_3(0)}{a_3};$$

$$\Lambda_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2}\gamma t & 0 & 0 \\ 0 & 0 & 1 & 0 & \gamma t & \frac{1}{2}\gamma^2 t^2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \gamma t \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \gamma = \frac{x_4(0)}{a_4}.$$

The Lie-Trotter integrator is given by

$$\begin{bmatrix} x_1^{n+1} \\ x_2^{n+1} \\ x_3^{n+1} \\ x_4^{n+1} \\ x_5^{n+1} \\ x_6^{n+1} \end{bmatrix} = \Lambda_1 \Lambda_3 \Lambda_4 \begin{bmatrix} x_1^n \\ x_2^n \\ x_3^n \\ x_4^n \\ x_5^n \\ x_6^n \end{bmatrix};$$

i.e

$$\begin{cases} x_1^{n+1} &= x_1^n - 2\beta t x_2^n + \beta^2 t^2 x_3^n - \beta\gamma t^2 x_4^n + \beta^2\gamma t^3 x_5^n + \dfrac{1}{2}\beta^2\gamma^2 t^4 x_6^n, \\ x_2^{n+1} &= \alpha t x_1^n + \left(1 - 2\alpha\beta t^2\right) x_2^n + \left(\alpha\beta^2 t^3 - \beta t\right) x_3^n \\ & \quad + \dfrac{1}{2}\gamma t \left(1 - 2\alpha\beta t^2\right) x_4^n + \gamma t \left(\alpha\beta^2 t^3 - \beta t\right) x_5^n \\ & \quad + \dfrac{1}{2}\gamma^2 t^2 \left(\alpha\beta^2 t^3 - \beta t\right) x_6^n, \\ x_3^{n+1} &= \alpha^2 t^2 x_1^n + \left(2\alpha t - 2\alpha^2\beta t^3\right) x_2^n + \left(\alpha^2\beta^2 t^4 - 2\alpha\beta t^2 + 1\right) x_3^n \\ & \quad + \gamma t \left(\alpha t - \alpha^2\beta t^3\right) x_4^n + \gamma t \left(\alpha^2\beta^2 t^4 - 2\alpha\beta t^2 + 1\right) x_5^n \\ & \quad + \dfrac{1}{2}\gamma^2 t^2 \left(\alpha^2\beta^2 t^4 - 2\alpha\beta t^2 + 1\right) x_6^n, \\ x_4^{n+1} &= x_4^n - \beta t x_5^n - \beta\gamma t^2 x_6^n, \\ x_5^{n+1} &= \alpha t x_4^n + \left(1 - \alpha\beta t^2\right) x_5^n + \gamma t \left(1 - \alpha\beta t^2\right) x_6^n, \\ x_6^{n+1} &= x_6^n. \end{cases} \tag{31}$$

Proposition 7 *We have the following observations for the Lie-Trotter integrator of the system under study*

1. *The Poisson structure is preserved.*

2. *The Casimir C_1 and C_2 is preserved.*

3. *Hamiltonian H_c is not preserved.*

Proof. From Eq. (31), we have found that

$$Z' \cdot Z(X^n) \cdot (Z')^T = Z(X^{n+1}).$$

It is evident that Poisson structure is preserved for our system. Also

$$C_1(X^{n+1}) = C_1(X^n),$$
$$C_2(X^{n+1}) = C_2(X^n).$$

It is deduced from the above findings that the Casimirs C_1 and C_2 are preserved. Since $\{H_{c1}, H_{c2}\} \neq 0$, the integrator does not preserve the Hamiltonian. □

5. Results and discussion

Trajectory of the dynamical system

Trajectory of dynamical system is the path which is followed by a system when driven from an initial state to a desired state. Trajectory tracking and subsequent analysis of its geometrical shape is essential for different applications such as, object detection for autonomous navigation [2], detection of crowd activities [4]. In our case the price of the option is the state function.

The unconventional numerical integration methods are applied to determine the trajectories of the system (18). The constant values used for each integration scheme are given in Table 2.

Table 2: Constants values for Integrators

Integrators	a_1	a_2	a_3
Kahan	1	1	1
Lie-Trotter	1	1	1
4th step R-K	1	1	1

The initial values considered for each method are $x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 1$. Trajectories obtained using Kahan and Lie-Trotter integrator are shown

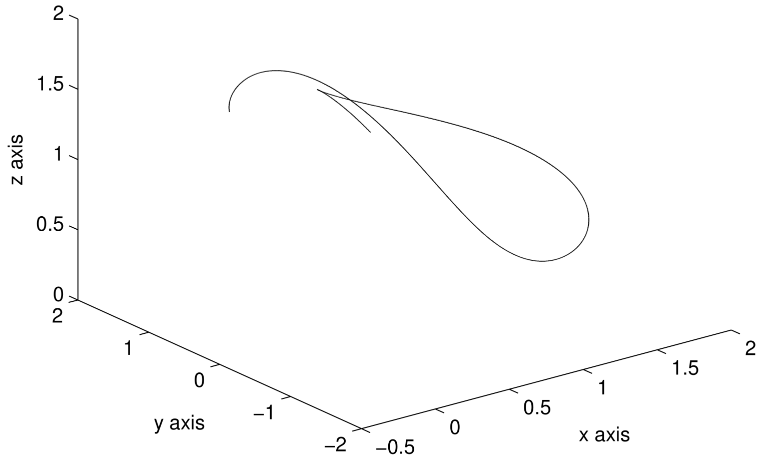

Figure 1: Trajectory of Kahan's integrator

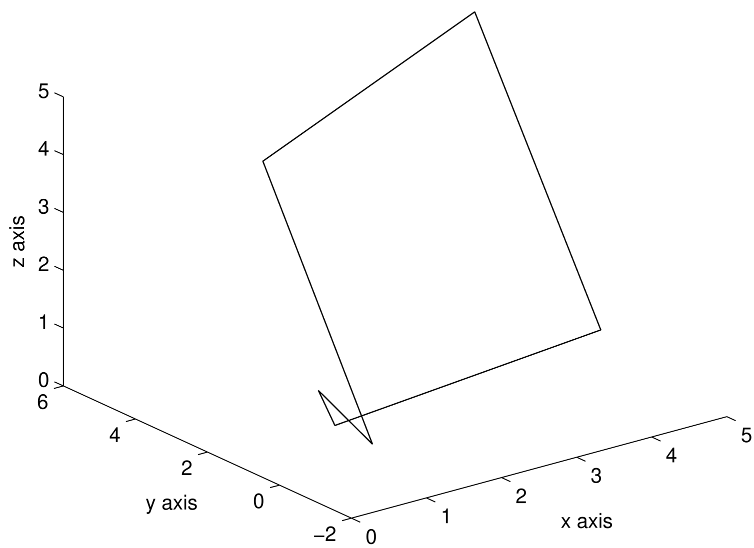

Figure 2: Trajectory of Lie Trotter integrator

in Fig. 1 and Fig. 2. In addition, Runge Kutta 4th step method has been applied to the system and the trajectory is shown in Fig. 3.

The trajectories derived from the unconventional integration techniques are compared to that of the Fourth step Runge-Kutta method. It is found from Fig. 4 that the trajectory obtained from Kahan integrator is close to that obtained from Runge Kutta integrator, therefore Kahan integrator contributes a good approximation of the system dynamics.

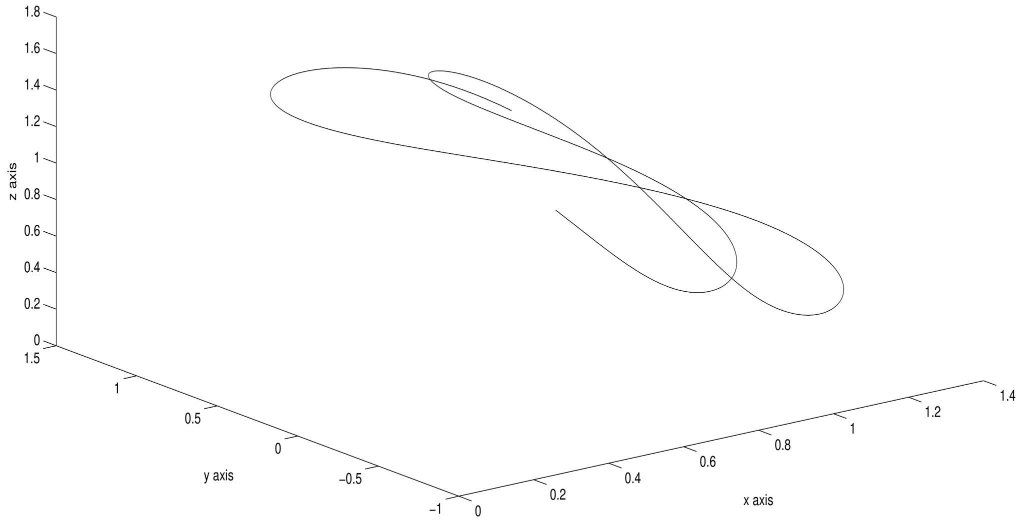

Figure 3: Trajectory of Runge Kutta integrator

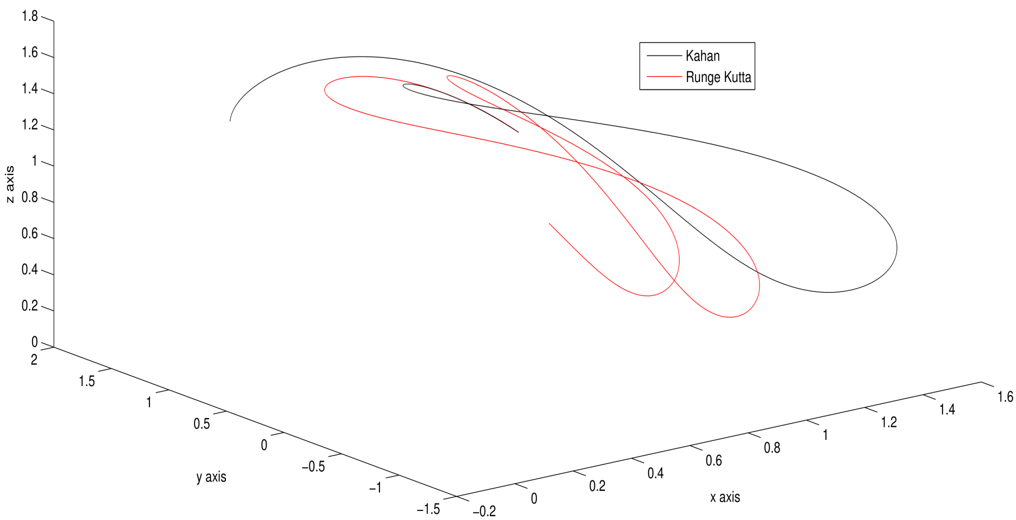

Figure 4: Trajectory of Kahan and Runge Kutta integrator

Fig. 5 shows that the trajectory of Lie-trotter integrator does not follow that of the Fourth step Runge-Kutta method. This implies that the approximation of the system dynamics by Lie-trotter integrator is weak. It is inferred from the analysis of the trajectories that Kahan integrator is a better numerical integration method compared to Lie trotter integrator for this system.

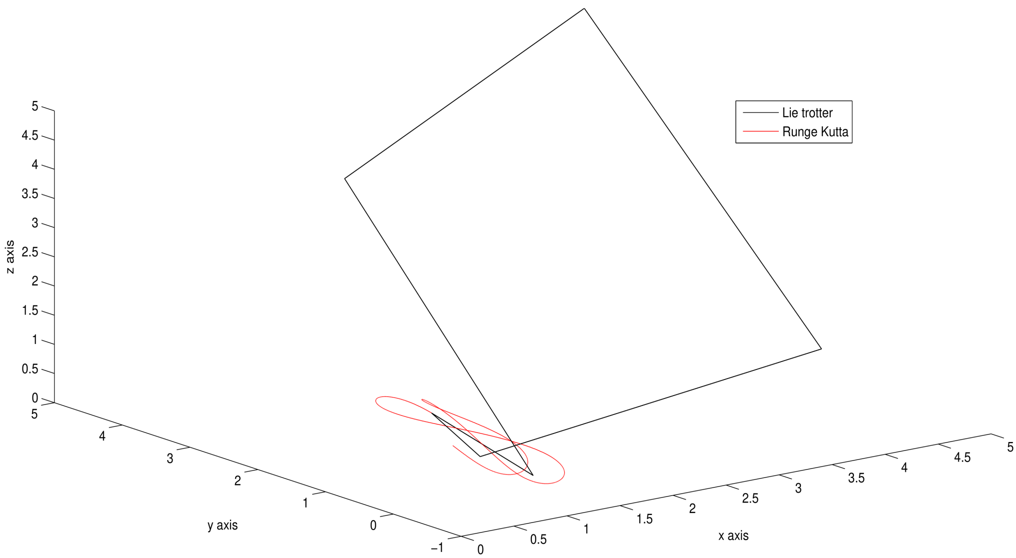

Figure 5: Trajectory of Lie-Trotter and Runge Kutta integrator

6. Conclusion

In this article we have analysed some controllability and stability aspects of a driftless control system of a group which arises due to invariance of Black-Scholes equation through the knowledge of differential geometry and Lie algebra. We have also determined the trajectories of the control system using two unconventional integrators and compared it with conventional integrators. We hope it will help in studying a new direction in quantum finance. Now a days many modified model of Black-Scholes-Merton model taking many options of financial markets have been evolved. So it is quite interesting to study these type of equations to get a more realistic answers.

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