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A robust method of quasigeoid modelling in Poland based on GPS/levelling data with support of gravity data

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Abstract: An increased use of global navigation techniques for positioning, and in particular for height determination, led to a growing need for precise models of height reference surface, i.e. geoid or quasigeoid. Geoid or quasigeoid heights at a cm accuracy level, provided on growing number of GPS/levelling sites, can not only be used for quality control of gravimetric geoid but they also can be integrated with gravity data for geoid/quasigeoid modelling. Such a model is of particular use for surveying practice. A method of quasigeoid modelling based on GPS/levelling data with support of geopotential model and gravity data was developed. The components of height anomaly are modelled with the deterministic part that consists of height anomaly based on EGM96 geopotential model and Molodensky's integral, as well as the polynomial representing trend, and from the stochastic part represented by the isotropic covariance function. Model parameters, i.e. polynomial coefficients and covariance function parameters are determined in a single process of robust estimation, resistant to the outlying measurements. The method was verified using almost a thousand height anomalies from the sites of the EUREF-POL, POLREF, EUVN'97 and WSSG (Military Satellite Geodetic Network) networks in Poland as well as geopotential model refined with gravity data in $1' \times 1'$ grid. The estimated average mean square error of quasigeoid height is at the level of 0.01 m. The outlying measurements were efficiently detected.

Keywords: Quasigeoid, GPS/levelling, collocation, robust estimation

1. Introduction

An increased use of global navigation techniques for positioning, and in particular for height determination, led to a growing need for precise models of height reference surface, i.e. geoid or quasigeoid, depending on the height system used in the region. The reference surface for heights, traditionally derived from terrestrial gravity data is not sufficiently accurate for using space techniques for height determination in surveying practice. Accuracy of existing local gravimetric geoid models is estimated as not worse than 20 cm. There is, however, a growing role of geoid models based on geopotential models derived from recent satellite mission dedicated to gravity field recovery, i.e. CHAMP and GRACE, and in the nearest future – GOCE. Spatial resolution of those models is still at the level of a few hundreds km, but their accuracy is quite high, i.e. within the range from single cm to a few decimetres (Featherstone, 2004).

For last ten years geoid or quasigeoid heights at a cm accuracy level, provided at GPS/levelling sites were primarily used for quality control of gravimetric geoid (e.g. Sideris and Li, 1992; Milbert, 1995; Łyszkowicz, 1996; Forsberg, 1998; Haagmans and de Min, 1999; Denker et al., 2000; Ollikainen, 2002; Benahmed Daho et al., 2004; Krynski and Lyszkowicz, 2005). Growing number of GPS/levelling sites of high precision ellipsoidal heights provides an independent observable with respect to gravity data that can be used for modelling quasigeoid on a regional scale (Duquenne et al., 2004).

A model of quasigeoid based on GPS/levelling data is given by a discrete set of height anomalies. Its resolution depends on the distribution of GPS/levelling sites used. When using GPS/levelling quasigeoid then, contrary to a gravimetric geoid model, an interpolation is required for determination of height anomaly at arbitrary site of the area. Pure numerical algorithm seems a rough tool for precise interpolation of such a complicated, locally variable surface as quasigeoid. Moreover, it does not provide any control for height anomalies determined at GPS/levelling sites. Accuracy of interpolated height anomalies is, therefore, in general, overestimated and might mislead the surveyor.

Fitting gravimetric quasigeoid to GPS/levelling height anomalies substantially improves statistical agreement among corresponding quasigeoid heights. There are, however, numerous effects, e.g. systematic errors of spirit levelling, distortions in vertical datum definition based on tide gauge records, the use of GPS-derived ellipsoidal heights from different realizations of the ITRS, long- and medium-wavelength errors in the gravimetric quasigeoid, fundamental incompatibilities between height system and quasigeoid model, that are being neglected by the users of the method (Featherstone, 2004).

Different methods of combination of gravity data with GPS/levelling height anomalies for quasigeoid modelling were discussed in the literature (e.g. Schödelbauer et al., 1991; de Min, 1993; Haagmans and de Min, 1999; Khtreiber, 2001; Blazquez et al., 2001). Two basic approaches of quasigeoid modelling can be specified. One approach is based on least squares collocation applied to gravity and height anomalies data on regional (de Min, 1993) or local (Schödelbauer et al., 1991) scale. The other approach is based on a common adjustment of gravity and height anomalies reduced for signals from geopotential model using point masses and appropriate weight relations (Denker et al., 2000). Recently, the most common approach uses modelling of residual height anomalies represented by differences between height anomalies $\zeta_{GPS/lev}$ from GPS/levelling and gravity-derived ones ζ_{grav}. General form of the model used is as follows

$$\zeta_{GPS/lev} - \zeta_{grav} = t + s + n \tag{1}$$

where t is the trend, s is the signal, and n is the noise. In the simplified models no signal is taken into consideration. The trend of residual height anomalies in the analysis of geoid models over Fennoscandia was modelled using a simple polynomial (Bilker et al., 2002)

$$t = a_{00} + a_{01}(\lambda - \lambda_0) + a_{10}(\varphi - \varphi_0) \tag{2}$$

while in a combined geoid model designed first for modelling geoid in Germany (Ihde, 1994) and then applied in the Netherlands (Haagmans and de Min, 1999) the trend t was modelled with a sum of a bi-linear function

$$a_{00} + b_{00}(\lambda - \lambda_0) + c_{00}(\varphi - \varphi_0) + d_{00}(\varphi - \varphi_0)(\lambda - \lambda_0) \tag{3}$$

and trigonometric functions

$$\begin{aligned}\sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij}\cos[i(\lambda - \lambda_0)]\cos[j(\varphi - \varphi_0)] + b_{ij}\sin[i(\lambda - \lambda_0)]\cos[j(\varphi - \varphi_0)] \\ + c_{ij}\cos[i(\lambda - \lambda_0)]\sin[j(\varphi - \varphi_0)] + d_{ij}\sin[i(\lambda - \lambda_0)]\sin[j(\varphi - \varphi_0)]\end{aligned} \tag{4}$$

In the combined solution for quasigeoid model in a test area in Germany the complete model (1) was used (Denker et al., 2000) with the trend modelled by a 3-parameter datum shift

$$t = \cos\varphi \cos\lambda\, \Delta X + \cos\varphi \sin\lambda\, \Delta Y + \sin\varphi\, \Delta Z \tag{5}$$

where ΔX, ΔY, ΔZ are datum shift constants. The de-trended residuals where then modelled using second order Markov covariance function

$$C(d) = C_0\left(1 + \frac{d}{q}\right)\exp\left(-\frac{d}{q}\right) \tag{6}$$

where d is the distance, C_0 is the signal variance, and q is the correlation length.

A method of quasigeoid modelling using a combination of geopotential model and gravity data with GPS/levelling height anomalies based on a model similar to (1) with trend represented by higher order polynomial and de-trended residuals modelled using Gauss covariance function (Osada et al., 2003) is discussed in the paper. Model parameters, i.e. polynomial coefficients and covariance function parameters are determined in a single process of robust estimation, resistant to outlying measurements. The method was verified using few hundreds height anomalies from the sites of the EUREF densification network in Poland as well as geopotential model and gravity data in $1' \times 1'$ grid.

2. **A method developed for quasigeoid determination in Poland**

The relation between the gravity anomaly Δg and height anomaly ζ is given in literature (e.g. Heiskanen and Moritz, 1967; Shimbirev, 1975). Height anomaly ζ that represents the distance of the Earth's surface point P from the corresponding point Q on the telluroid can, neglecting third and higher order terms of Brovar's expansion (Moritz, 1980), be given as

$$\zeta = \frac{1}{4\pi\gamma R}\iint_{\sigma}(\Delta g + G_1)S(\psi)d\sigma \tag{7}$$

where

$$G_1 = \frac{1}{2\pi}\iint_{\sigma}\frac{H - H_P}{r^3}\Delta g d\sigma \tag{8}$$

is "the terrain correction", and

$\Delta g = g_{P'} - \gamma_{Q'}$ is the free-air gravity anomaly in the running point P',
R is the mean radius of the spherical Earth,
γ is normal gravity at the telluroid point, averaged over the investigated area,
$S(\psi)$ is Stokes' kernel function,
ψ is a spherical distance between the running point P' and the point P,
$r = 2R\sin(\psi/2)$,
$d\sigma$ is the surface integration element,
H and H_P are the normal heights of the running point P' and the point P, respectively.

The free-air anomaly can be expressed as follows

$$\begin{aligned}\Delta g &= g_P - \gamma_Q + g_P^{GM} - g_P^{GM} \\ &= (g_P^{GM} - \gamma_Q) + (g_P - g_P^{GM}) \\ &= \Delta g^{GM} + \delta g\end{aligned} \tag{9}$$

where

$$\Delta g^{GM} = g_P^{GM} - \gamma_Q \tag{10}$$

is the major component of the free-air anomaly Δg computed with the use of gravity g_P^{GM} determined from global geopotential model GM (EGM96) at point P, and

$$\delta g = g_P - g_P^{GM} \tag{11}$$

is a small term corresponding to the gravity disturbance.

Substituting (9) into (7) one obtains

$$\zeta = \frac{1}{4\pi\gamma R}\iint_{\sigma}(\Delta g^{GM} + \delta g + G_1)S(\psi)d\sigma \tag{12}$$

what can also be expressed as

$$\zeta = \zeta_{GM} + \zeta_{\delta g} + \zeta_{G_1} \tag{13}$$

where:

$$\zeta_{GM} = \frac{1}{4\pi\gamma R}\iint_{\sigma}\Delta g^{GM}S(\psi)d\sigma \tag{14}$$

is the global geopotential model component of the height anomaly,

$$\zeta_{\delta g} = \frac{1}{4\pi\gamma R}\iint_{\sigma} \delta g S(\psi) d\sigma \tag{15}$$

is its gravimetric component, and

$$\zeta_{G_1} = \frac{1}{4\pi\gamma R}\iint_{\sigma} G_1 S(\psi) d\sigma \tag{16}$$

is its terrain component.

The global geopotential model component of the height anomaly (14) can practically be computed on the basis of the definition (Heiskanen and Moritz, 1967)

$$\zeta_{GM} = \frac{W_P^{GM} - U_P}{\gamma_Q} \tag{17}$$

where W_P^{GM} is the gravity potential at P calculated from the geopotential model, and U_P is the normal potential at P obtained from GRS80 (Moritz, 1984).

Quasigeoid model (13) is next fitted to the quasigeoid heights at GPS/levelling sites with use of least-squares robust estimation, resistant to the outlying measurements. The residua are modelled by a trend t of a polynomial form

$$t = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + \ldots + a_{15} y^4 \tag{18}$$

and by a random signal s with the expected value equal to zero and with the isotropic covariance function (6).

Quasigeoid model used has thus the form

$$\zeta = \zeta_{GM} + \zeta_{\delta g} + \zeta_{G_1} + t + s \tag{19}$$

where the global, gravimetric and terrain components, i.e. ζ_{GM}, $\zeta_{\delta g}$ and ζ_{G_1}, are given by (14), (15) and (16), respectively.

Normal gravity γ in (15), (16) i (17) can practically be computed in Earth's surface point P instead of the telluroid point Q. The error due to that simplification does not exceed 1 mm in Poland.

Numerical calculations were performed using the plane approximations of (15) and (16). The Stokes' kernel was expressed as $S(\psi) = 2/\psi = 2R/r$ what causes a negligible distortion in the area of Poland. The formulae (15) and (16) become

$$\zeta_{\delta g} = \frac{1}{2\pi\gamma}\iint_{\sigma} \frac{\delta g}{r} d\sigma \tag{20}$$

$$\zeta_{G_1} = \frac{1}{2\pi\gamma}\iint_{\sigma} \frac{G_1}{r} d\sigma \tag{21}$$

where r is the planar distance between the running point P' and the point P.

The same assumption with respect to r was used for practical realization of (8). The discrete forms of the operational formulae were obtained by substituting the integral with summation operators.

3. Data used in numerical experiment

The height anomalies $\zeta_{GPS/lev} = h - H^N$ at 924 sites of the EUVN'97 network (58 sites), EUREF-POL network (11 sites)[1], POLREF network (331 sites) and WSSG (Military Satellite Geodetic Network) network (528 sites) computed at each point (φ, λ) as a difference between GPS-derived ellipsoidal heights h and corresponding normal heights H^N obtained by spirit levelling were used for quasigeoid modelling. Coverage of Poland with precisely determined height anomalies (Fig. 1) is rather dense and almost uniform; average distance between neighbouring sites is within the range of 15–20 km.

Fig. 1. Distribution of EUVN'97, EUREF-POL, POLREF and WSSG sites used for modelling quasigeoid

Considering the internal accuracy of network adjustment, accuracy of levelling ties to the vertical control as well as the lengths of GPS observing sessions at sites the following *a priori* standard deviations $\sigma_{\zeta(GPS/lev)}$ of height anomalies were taken as 0.01 m for the EUVN'97 and POLREF sites, and 0.02 m for the WSSG sites.

Gravity data in $1' \times 1'$ grid in the geodetic coordinate system (φ, λ), covering the whole area of Poland (Fig. 2) was used to generate gravity disturbances δg at grid points.

[1] In numerical experiments EUREF-POL sites were joint with POLREF sites

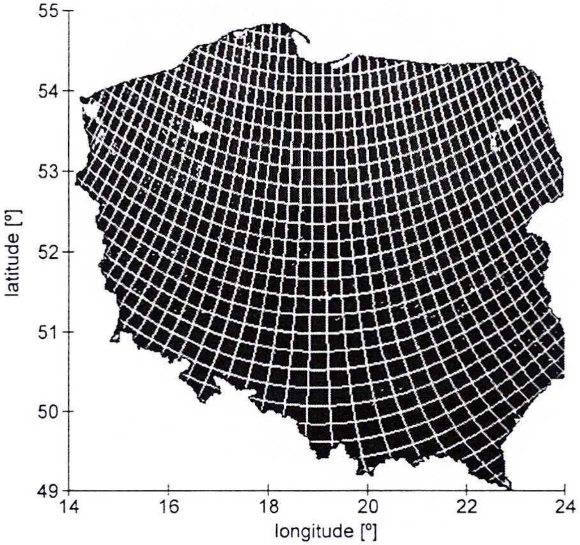

Fig. 2. Grid of gravity data used for modelling quasigeoid

For each grid point its geodetic coordinates (φ, λ), normal height H^N as well as observed gravity g_{obs} are given. The ellipsoidal heights h of the grid points were computed using the quasigeoid model ζ_{2001} "levelling geoid 2001" (Pażus et al., 2002) as $h = H^N + \zeta_{2001}$.

The histogram of the gravity disturbances $\delta g = g_P - g_P^{GM}$ computed at the grid of the terrain points (φ, λ, h) is shown in Fig. 3. The statistics of δg is given in Table 1.

Table 1. The statistics of δg [mGal]

Min	Max	Mean	Std dev.
–49	115	–1	9

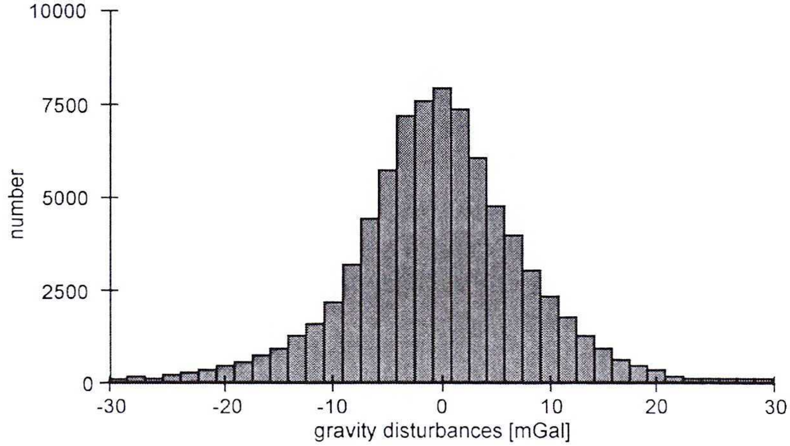

Fig. 3. The histogram of gravity disturbances δg

The components ζ_{GM} (17), $\zeta_{\delta g}$ (15) and ζ_{G_1} (16) of the height anomaly (12), computed at the terrain points (φ, λ, h) are shown in Fig. 4, Fig. 5, and Fig. 6, respectively.

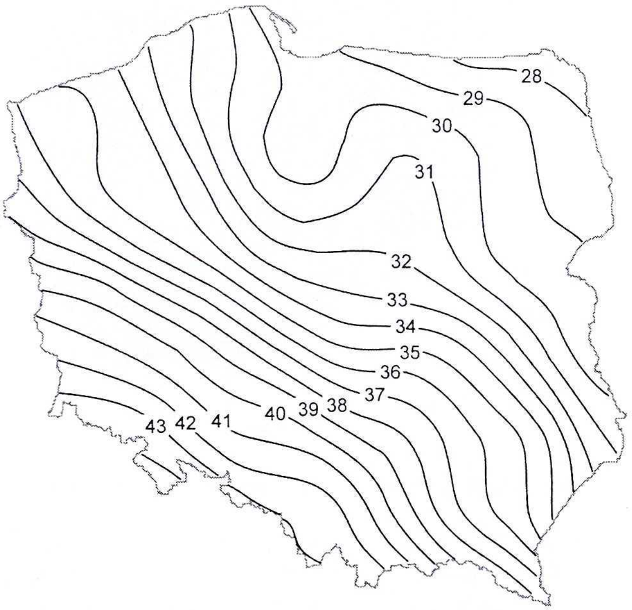

Fig. 4. The global geopotential model component ζ_{GM} [m]

Fig. 5. The gravimetric component $\zeta_{\delta g}$ [cm]

Fig. 6. The terrain component ζ_{G_1} [cm]

4. Robust estimation of parameters of the model, resistant to the outlying measurements

Least squares collocation is used for fitting the quasigeoid model ($\zeta_{GM} + \zeta_{\delta g} + \zeta_{G_1}$) to height anomalies $\zeta_{GPS/lev}$ obtained at GPS/levelling sites with simultaneous determination of model parameters. The mathematical model of the least squares collocation, based on the quasigeoid model (7) is given by

$$\mathbf{v} = \mathbf{AX} + \mathbf{s} - \mathbf{L} \tag{22}$$

where **A** is a known design matrix, **X** is the vector of the trend parameters, **s** is the vector of signals, and **L** is the vector of observed residuals $l = \zeta_{GPS/lev} - (\zeta_{GM} + \zeta_{\delta g} + \zeta_{G_1})$. The **AX** component of (22) represents the vector of the trend t (18) in (19) while vector **v** represents measuring errors that correspond to the noise n in (1) (Moritz, 1972).

The explicit form of (22) is as follows

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 & \dots & y_1^4 \\ 1 & x_2 & y_2 & x_2^2 & x_2 y_2 & y_2^2 & \dots & y_2^4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & y_n & x_n^2 & x_n y_n & y_n^2 & \dots & y_n^4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ \vdots \\ a_{15} \end{bmatrix} + \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{bmatrix} \tag{23}$$

The stochastic model is given by

$$\begin{gathered} \mathrm{E}(\mathbf{v}) = \mathbf{0};\ \mathbf{C}_{\mathrm{vv}} = \mathrm{E}(\mathbf{v}\mathbf{v}^{\mathrm{T}});\ \mathbf{P}_{\mathrm{vv}} = \mathbf{C}_{\mathrm{vv}}^{-1} \\ \mathrm{E}(\mathbf{s}) = \mathbf{0};\ \mathbf{K}_{\mathrm{ss}} = \mathrm{E}(\mathbf{s}\mathbf{s}^{\mathrm{T}});\ \mathbf{P}_{\mathrm{ss}} = \mathbf{K}_{\mathrm{ss}}^{-1} \\ \mathbf{C}_{\mathrm{sv}} = \mathrm{E}(\mathbf{s}\mathbf{v}^{\mathrm{T}}) = \mathbf{0} \end{gathered} \tag{24}$$

where E is an expectation operator, **P** denotes the weight matrix, and **C**, **K** denote the covariance matrices.

The covariance matrix $\mathbf{C}_{\mathrm{vv}}$ is defined with *a priori* standard deviations of height anomalies:

$$\mathbf{C}_{\mathrm{vv}} = \begin{bmatrix} \sigma^2_{\zeta_1} & & & \\ & \sigma^2_{\zeta_2} & & \\ & & \ddots & \\ & & & \sigma^2_{\zeta_n} \end{bmatrix} \tag{25}$$

while the covariance matrix $\mathbf{K}_{\mathrm{ss}}$ of the signal **s** may be written in the form

$$\mathbf{K}_{\mathrm{ss}} = \begin{bmatrix} \sigma^2_{s_1} & \sigma_{s_1 s_2} & \cdots & \sigma_{s_1 s_n} \\ \sigma_{s_1 s_2} & \sigma^2_{s_2} & \cdots & \sigma_{s_2 s_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{s_1 s_n} & \sigma_{s_2 s_n} & \cdots & \sigma^2_{s_n} \end{bmatrix} \tag{26}$$

Two assumptions are used when modelling the covariance matrix $\mathbf{K}_{\mathrm{ss}}$. First one states that standard deviations of all signals are equal. Thus (26) becomes

$$\mathbf{K}_{\mathrm{ss}} = C_0 \begin{bmatrix} 1 & \rho_{s_1 s_2} & \cdots & \rho_{s_1 s_n} \\ \rho_{s_1 s_2} & 1 & \cdots & \rho_{s_2 s_n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{s_1 s_n} & \rho_{s_2 s_n} & \cdots & 1 \end{bmatrix} \tag{27}$$

where $C_0 = \sigma_s^2$ is the variance of the signal s, and $\rho_{s_i s_j}$ denotes the correlation coefficient. The second assumption concerns a constrain on the correlation coefficient. It states that ρ is expressed by the increasing function of a distance d between a pair of surveyed sites. Such function is represented in the method by Gauss covariance function $C(d) = C_0\rho(d) = C_0(1 + d/q)$ that represents a particular case of Markov's function (6). The *a priori* parameters $\sigma_s = 0.04$ m and $q = 60$ km were obtained iteratively, together with parameters of the trend, i.e. the polynomial (18) coefficients, starting with initial values: $\sigma_s = 0.1$ m, $q = 10$ km. The starting value $\sigma_s = 0.1$ m was taken as standard deviation of the observed residuals l. The computed Gauss correlation function $\rho(d)$ and isotropic covariance function $C(d)$, are shown in Fig. 7a and Fig. 7b, respectively.

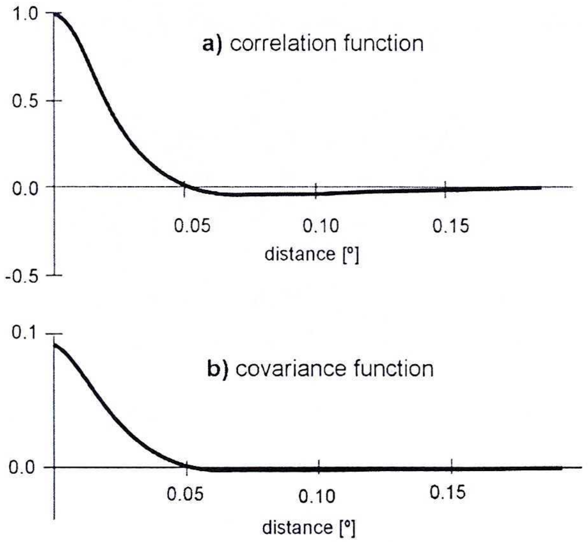

Fig. 7. Gauss correlation function a), and isotropic covariance function b)

The classical algorithm of least squares solution of the system of equations (22) or (23) is derived by using the minimum condition $\mathbf{v}^T\mathbf{P}_{vv}\mathbf{v} + \mathbf{s}^T\mathbf{P}_{ss}\mathbf{s}$ = min, together with (24) (Moritz, 1980). In the equivalent approach the signal s is treated as a measurement ε, i.e. $\varepsilon_i = s_i \pm \sigma_{si}$ (Osada, 2002; Osada et al., 2003). The system of observation equations becomes then

$$\mathbf{v} = \mathbf{AX} + \mathbf{s} - \mathbf{L} \tag{28}$$

$$\boldsymbol{\varepsilon} = \mathbf{s}$$

with

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \tag{29}$$

what can also be written as

$$\begin{bmatrix} \mathbf{v} \\ \boldsymbol{\varepsilon} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{s} \end{bmatrix} - \begin{bmatrix} \mathbf{L} \\ \mathbf{0} \end{bmatrix} \tag{30}$$

with

$$\begin{bmatrix} \mathbf{P}_{vv} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{ss} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{vv} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{ss} \end{bmatrix}^{-1} \tag{31}$$

The solution of (30) using the minimum condition $\mathbf{v}^T\mathbf{P}_{vv}\mathbf{v} + \boldsymbol{\varepsilon}^T\mathbf{P}_{ss}\boldsymbol{\varepsilon} = \min$, together with (31) is given as

$$\begin{bmatrix} \mathbf{X} \\ \mathbf{s} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{XX} & \mathbf{C}_{Xs} \\ \mathbf{C}_{sX} & \mathbf{C}_{ss} \end{bmatrix} \begin{bmatrix} \mathbf{A}^T\mathbf{P}_{vv}\mathbf{L} \\ \mathbf{P}_{vv}\mathbf{L} \end{bmatrix} \tag{32}$$

where

$$\begin{bmatrix} \mathbf{C}_{XX} & \mathbf{C}_{Xs} \\ \mathbf{C}_{sX} & \mathbf{C}_{ss} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^T\mathbf{P}_{vv}\mathbf{A} & \mathbf{A}^T\mathbf{P}_{vv} \\ \mathbf{P}_{vv}\mathbf{A} & \mathbf{P}_{vv} + \mathbf{P}_{ss} \end{bmatrix}^{-1} \tag{33}$$

Denoting $\mathbf{C} = \mathbf{C}_{vv} + \mathbf{K}_{ss}$, the covariance matrices of parameters $\mathbf{X}$ and signal $\mathbf{s}$ in (33) are defined as follows

$$\mathbf{C}_{XX} = (\mathbf{A}^T\mathbf{C}^{-1}\mathbf{A})^{-1} \tag{34}$$

$$\mathbf{C}_{ss} = \mathbf{C}_{vv}\mathbf{C}^{-1}\mathbf{K}_{ss} + \mathbf{K}_{ss}\mathbf{C}^{-1}\mathbf{A}\mathbf{C}_{XX}\mathbf{A}^T\mathbf{C}^{-1}\mathbf{K}_{ss} \tag{35}$$

$$\mathbf{C}_{Xs} = \mathbf{C}_{sX}^T = -\mathbf{C}_{XX}\mathbf{A}^T\mathbf{C}^{-1}\mathbf{K}_{ss} \tag{36}$$

The vector of parameters, i.e. polynomial coefficients, is

$$\mathbf{X} = \mathbf{C}_{XX}\mathbf{A}^T\mathbf{C}^{-1}\mathbf{L} \tag{37}$$

while the vector $\mathbf{s}$ of signals, i.e. the discrepancy between GPS/levelling height anomaly and height anomaly from the model is

$$\mathbf{s} = \mathbf{K}_{ss}\mathbf{C}^{-1}(\mathbf{L} - \mathbf{A}\mathbf{X}) \tag{38}$$

and the *a posteriori* standard deviation of s_i is

$$\sigma_{s_i} = \sqrt{\mathbf{C}_{ss_{ii}}} \tag{39}$$

Each row of covariance matrix $\mathbf{K}_{ss}$ corresponds to the vector of covariance between the signal at the arbitrary point (x, y) and the signal at the surveyed point

$$\mathbf{k}(x, y) = \left[C\left(\sqrt{(x_1 - x)^2 + (y_1 - y)^2}\right), \ldots, C\left(\sqrt{(x_n - x)^2 + (y_n - y)^2}\right)\right] \tag{40}$$

The signal at the arbitrary point can thus be computed as

$$s(x, y) = \mathbf{k}(x, y)\, \mathbf{K}_{ss}^{-1}\mathbf{s} \tag{41}$$

or using the notation $\mathbf{p} = \mathbf{K}_{ss}^{-1}\mathbf{s}$

$$s(x, y) = \sum_{i=1}^{n} p_i C\left(\sqrt{(x_i - x)^2 + (y_i - y)^2}\right) \tag{42}$$

The model can be considered correct and sufficiently well fitted to the data when the calculated residuals $\mathbf{v} = \mathbf{AX} + \mathbf{s} - \mathbf{L}$ and signals $\boldsymbol{\varepsilon} = \mathbf{s}$ fulfil the quality test

$$m_0 = \sqrt{\frac{\mathbf{v}^T \mathbf{P}_{vv} \mathbf{v} + \boldsymbol{\varepsilon}^T \mathbf{P}_{ss} \boldsymbol{\varepsilon}}{n - k}} \equiv 1 \pm 0.1, \tag{43}$$

and they are within the range of double standard deviation, i.e.

$$|v_i| \le 2\sigma_{v_i}, \quad |\varepsilon_i| \le 2\sigma_{\varepsilon_i} \tag{44}$$

where

$$\sigma_{v_i} = \sqrt{\sigma^2_{\zeta_i} - (\mathbf{A}\mathbf{C}_{XX}\mathbf{A}^T + \mathbf{A}\mathbf{C}_{Xs} + \mathbf{C}^T_{Xs}\mathbf{A}^T + \mathbf{C}_{ss})_{i,i}} \tag{45}$$

$$\sigma_{\varepsilon_i} = \sqrt{\sigma^2_s - \sigma^2_{s_i}} \tag{46}$$

are the standard deviations of observed height anomalies and signals, respectively.

The negative result of the test might indicate the outlier among height anomalies at GPS/levelling sites. It may also indicate improper modelling of covariance function, in particular when the signal field is not isotropic.

Testing for outliers is performed iteratively by fitting the model to height anomalies at GPS/levelling sites. In the current iteration, small weights are assigned to the outliers (height anomalies) detected in the previous iteration step. Those weights are functions of the corresponding residuals v obtained in the previous iteration. The uncertainty σ_{ζ_i} of height anomalies used to generate weights might be expressed in different ways. That uncertainty was defined in the numerical experiments performed as follows ($r = 1, 2, 3$):

$$\sigma_{\zeta_i} = \begin{cases} \sigma_{\zeta_i} & |v_i| \le r\sigma_{\zeta_i} \\ \sigma_{\zeta_i} + |v_i| - r\sigma_{\zeta_i} & |v_i| > r\sigma_{\zeta_i} \end{cases} \tag{47}$$

Iteration process is terminated when computed parameters exhibit no significant variability any more.

The uncertainty of the derived quasigeoid model $\zeta(x, y)$ might be represented by mean square error $m_\zeta(x, y)$ of height anomaly

$$m_\zeta(x, y) = \sqrt{\begin{matrix} \mathbf{g}(x, y)\mathbf{C}_{XX}\mathbf{g}(x, y)^T + \mathbf{k}(x, y)\mathbf{P}_{ss}\mathbf{C}_{Xs}{}^T\mathbf{g}(x, y)^T \\ + \mathbf{g}(x, y)\mathbf{C}_{Xs}\mathbf{P}_{ss}\mathbf{k}(x, y)^T + \mathbf{k}(x, y)\mathbf{P}_{ss}\mathbf{C}_{ss}\mathbf{P}_{ss}\mathbf{k}(x, y)^T \end{matrix}} \tag{48}$$

where

$$\mathbf{g}(x, y) = [1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^4, x^3y, x^2y^2, xy^3, y^4] \tag{49}$$

is the gradient of $\zeta(x, y)$ with respect to the polynomial coefficients.

5. Results

Fitting of the quasigeoid model to GPS/levelling quasigeoid heights was done in two steps (Osada et al., 2003).

First, starting with initial values of parameters σ_s = 0.1 m, and q = 10 km of the covariance function, and using mean square errors of height anomalies σ_ζ = 1 cm for POLREF and EUVN, the parameters σ_s = 0.04 m and q = 60 km were determined after a few iterations; the criteria (43), (44) were fulfilled.

In the second step all data from POLREF, EUVN as well as WSSG sites were used in calculations with fixed values of the parameters σ_s and q of the covariance function. The quasigeoid model was computed after a few iterations with varying the uncertainties (and also weights) of height anomalies at the outlying WSSG sites, according to (47). The final value m_0 (43) equals to 1, and criteria (44) are fulfilled for each EUVN and POLREF site. A number of outliers amongst height anomalies at WSSG sites that could be considered blunders were detected (Fig. 8).

Figures 9, 10 and 11 show the computed trend $t(x, y)$, the signal $s(x, y)$ and quasigeoid model $\zeta(x, y)$, respectively.

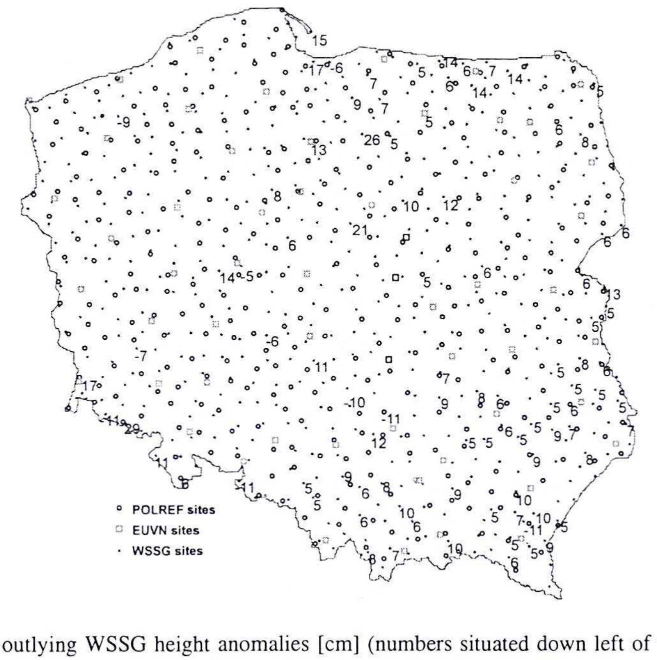

Fig. 8. Detected outlying WSSG height anomalies [cm] (numbers situated down left of the WSSG sites correspond to the outlying values of height anomalies)

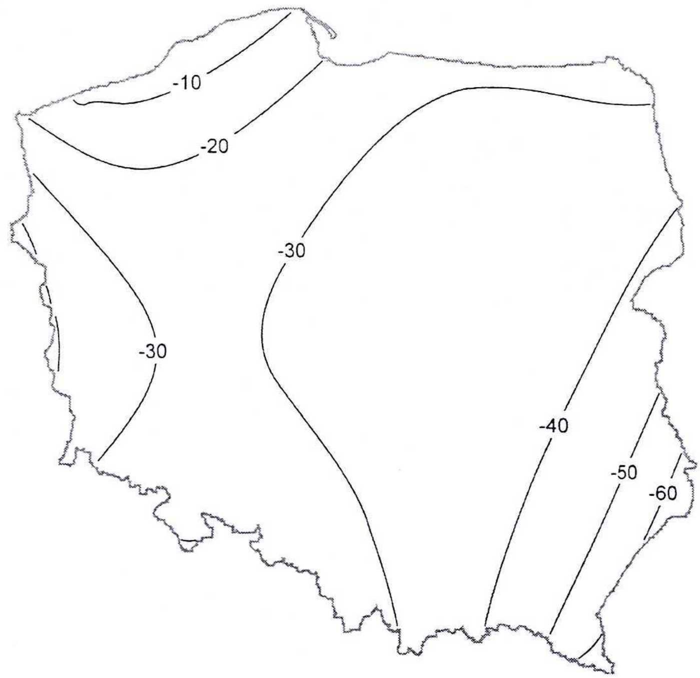

Fig. 9. The trend $t(x, y)$ [cm]

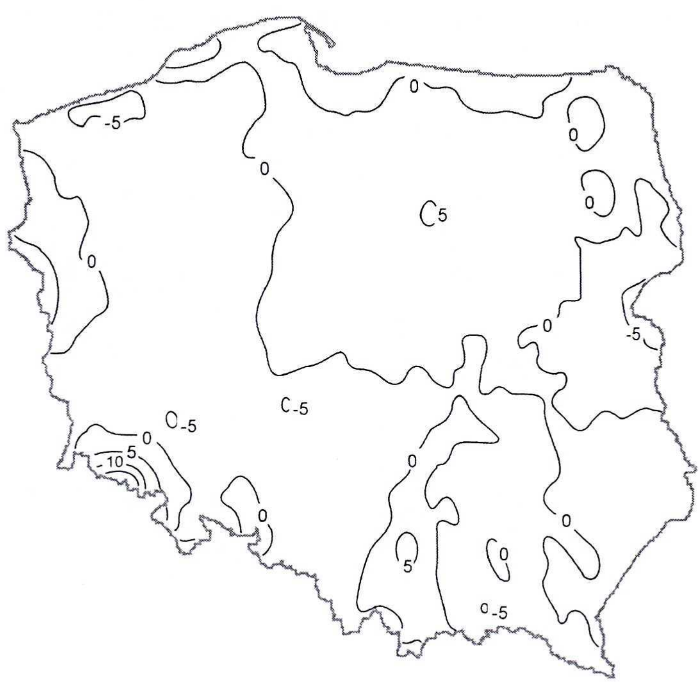

Fig. 10. The signal $s(x, y)$ [cm]

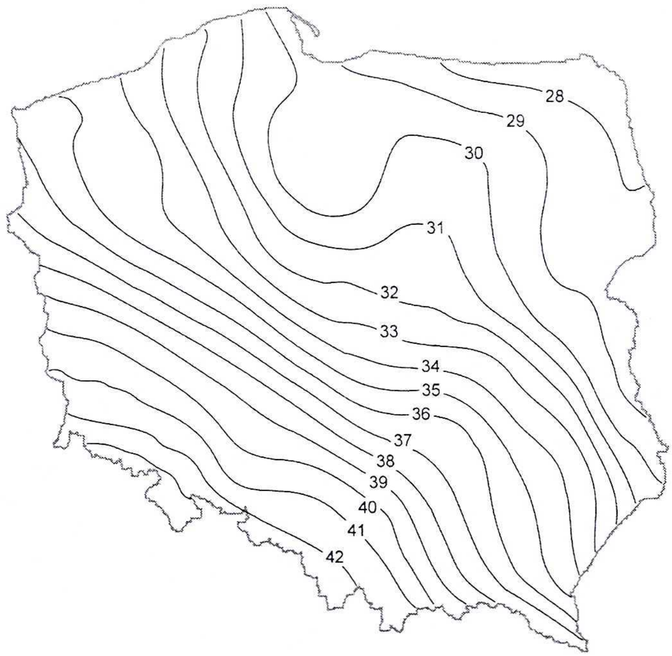

Fig. 11. Quasigeoid model $\zeta = \zeta_{GM} + \zeta_{\delta g} + \zeta_{G_1} + t + s$ [m]

The estimated mean square error of the height of quasigeoid (48) fitted to GPS/levelling sites with support of gravity data is at the level of 1 cm (Fig. 12).

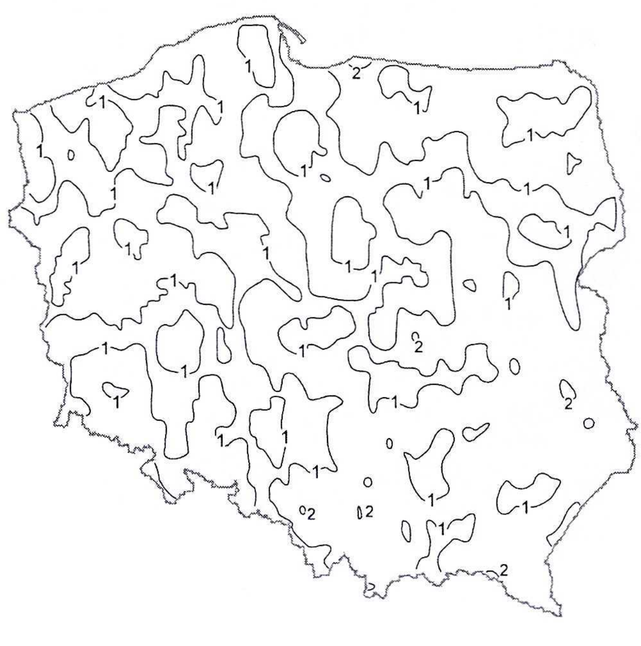

Fig. 12. Mean square error of the quasigeoid height $m_\zeta(x, y)$ [cm]

6. Conclusions

Quality of quasigeoid model based on GPS/levelling data depends on the distribution of GPS/levelling sites, and on the quality of heights from both GPS and levelling. Distribution of GPS/levelling sites determines spatial resolution of the model. Outlying heights can eventually be detected using purely numerical procedures. Additional use of geopotential model as well as gravity data when developing GPS/levelling quasigeoid model is beneficial.

The analytical method developed provides a robust estimation of parameters of quasigeoid model, resistant to the outlying measurements. It also allows for efficient detection of outliers. Finally it provides more reliable prediction of quasigeoid heights than pure numerical interpolation methods.

The uncertainty of the hights of quasigeoid in Poland fitted to GPS/levelling sites with support of gravity data is at the level of 1 cm_{ζ}.

The analytical method developed for GPS/levelling quasigeoid $\zeta(x, y)$ modelling with additional use of gravity data should further be examined using new GPS/levelling data. The obtained new quasigeoid model can efficiently be applied for developing a regular discrete quasigeoid model, e.g. a grid of for example 100×100 m. Such a model could be incorporated to GPS receivers to convert in real time ellipsoidal heights into the normal heights.

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Odporna metoda wyznaczenia quasigeoidy na podstawie danych wysokości quasigeoidy w punktach POLREF, EUVN i WSSG oraz siatki anomalii grawimetrycznych

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Streszczenie

Wraz ze wzrostem zastosowań precyzyjnych satelitarnych technik wyznaczania pozycji, w szczególności wyznaczania wysokości wzrosło zapotrzebowanie na precyzyjne modele powierzchni odniesienia dla wysokości, tj. geoidy lub quasigeoidy. Wysokości geoidy lub quasigeoidy nad elipsoidą z centymetrową dokładnością znane na coraz większej liczbie stacji GPS o dokładnie wyznaczonej wysokości ortometrycznej lub normalnej mogą być wykorzystane nie tylko do kontroli jakości geoidy grawimetrycznej, ale również w połączeniu z danymi grawimetrycznymi mogą być wykorzystane do modelowania geoidy/quasigeoidy. Model taki ma szczególne znaczenie dla praktyki geodezyjnej. W opracowanej metodzie modelowania quasigeoidy opartej na danych satelitarno-niwelacyjnych wykorzystywane są również dane grawimetryczne. Przyjęty model wysokości quasigeoidy składa się z części deterministycznej, która zawiera długofalową składową pochodzącą od modelu geopotencjału EGM96 i składową grawimetryczną wyrażoną całką Molodenskiego oraz części stochastycznej opisanej izotropową funkcją kowariancji, a także wielomianowego trendu. Parametry modelu – współczynniki wielomianu oraz parametry funkcji kowariancji są wyznaczane w jednym procesie estymacji, odpornej na odstające punkty pomiarowe GPS. Metoda została zweryfikowana przy użyciu niemal tysiąca anomalii wysokości na punktach krajowych sieci satelitarno-niwelacyjnych GPS: POLREF, EUVN i WSSG, modelu geopotencjału EGM96 oraz danych grawimetrycznych w siatce $1' \times 1'$ z obszaru Polski. Błąd średni wysokości obliczonej quasigeoidy szacowany jest na poziomie 0.01 m. Opracowana metoda stwarza możliwość efektywnej detekcji odstających obserwacji wysokości na punktach satelitarno-niwelacyjnych.