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The correct length of a sight line for determination of a refraction factor from synchronic and bi-directional observations

The paper presents discusses a formula for the mean error m_k of the refraction factor, determined basing on synchronic and bi-directional observations of vertical angles, performed for a section of trigonometric levelling. It turns out from analysis of the formula that the mean error of this factor is mostly influenced by mean errors of vertical angles and heights of distance meters and prisms over survey points. The error of a distance may be neglected; this leads to the simple working formula:

$$m_k^2 = 2\left(\frac{R}{S}m_\alpha\right)^2 + 4\left(\frac{R}{S^2}m_i\right)^2$$

The m_k value was calculated for assumed values of mean errors of a vertical angle m_α = 5" and the height of an instrument m_i = 0.02 m, and for changing values of the S distance. It turns out from the results that the refraction factor k should be determined basing on sight lines, which are longer than 3000 m. In the case of shorter sight lines, mean errors of the refraction factor are of the same order as the value of this factor, which is commonly used, i.e. $k = 0.13$ or they are bigger than this value.

Utilisation of precise methods of measurements of heights of instruments will not change the above conclusion and the condition S>3000 is still valid.

A method of synchronic and bi-directional observations, which allows to calculate the height difference of survey points, without using of a refraction factor k, is well known in trigonometric levelling. It requires that vertical angles and, possibly, inclined distances, are simultaneously measured from both ends of a section, and auxiliary values [2] are calculated basing on measured values:

$$\overline{\Delta H}_{PK} = S_P \sin\alpha_P + i_P - w_K \tag{1}$$

$$\overline{\Delta H}_{KP} = S_K \sin\alpha_K + i_K - w_P \tag{2}$$

The height difference of survey points

$$\Delta H_{PK} = H_K - H_P \tag{3}$$

is calculated as the half difference of corresponding auxiliary values

$$\Delta H_{PK} = 0.5(\overline{\Delta H_{PK}} - \overline{\Delta H_{KP}}) \tag{4}$$

i.e.

$$\Delta H_{PK} = 0.5(S_P \sin\alpha_P + i_P - w_K - S_K \sin\alpha_K - i_K + w_P) \tag{5}$$

The following notation has been assumed:

S_P – the inclined distance observed by means of an instrument located over the point P to the prism located over the point K,

S_K – the inclined distance observed by means of an instrument located over the point K to the prism located over the point P,

α_P – the vertical angle observed by means of an instrument located over the point P to the prism located over the point K,

α_K – the vertical angle observed by means of an instrument located over the point K to the prism located over the point P,

i_P, i_K – heights of location of an instrument over points P and K, respectively,

w_P, w_K – heights of location of prisms over points P and K, respectively,

H_P, H_K – levels of survey points P,K, i.e. their altitudes over the reference surface,

R – the radius of a reference sphere, equal to 6382 km.

If co-ordinates of the points P, K are known, measurements of inclined distances is not required and it is sufficient to simultaneously measure vertical points α_P, α_K from both ends of the section. Values of D_P, D_K, required for distance calculations, which are equal to lengths of arcs at the instrument level over the points P and K, respectively, are calculated by adding corrections due to projection and the instrument level, to the distance, which has been calculated basing on co-ordinates of points. In this case formulas used for calculation of auxiliary values [2] have the following form:

$$\overline{\Delta H_{PK}} = D_P \operatorname{tg}\alpha_P + i_P - w_K \tag{6}$$

$$\overline{\Delta H_{KP}} = D_K \operatorname{tg}\alpha_K + i_K - w_P \tag{7}$$

D_P, D_K are not observations.

The difference of levels of survey points will be calculated from (4); after substitution the following relation is obtained:

$$\Delta H_{PK} = 0.5(D_P \operatorname{tg}\alpha_P + i_P - w_K - D_K \operatorname{tg}\alpha_K - i_K + w_P) \tag{8}$$

It should be noticed that regardless utilisation of the formula (5), (8) for calculations, the difference in the level of the instrument I and the target E may be calculated for one of directions of bi-directional observations, e.g. P-K:

$$\Delta h_{IE} = H_K + w_K - (H_P + i_P) = \Delta H_{PK} + w_K - i_P \tag{9}$$

and the obtained value will be calculated without utilisation of the refraction factor k. Thus, it may be used for calculation of this factor basing on one of known formulas [1], [2]:

$$k = 1 - \frac{2R}{S_P^2}(\Delta h_{IE} - S_P \sin\alpha_P) \tag{10}$$

or

$$k = 1 - \frac{2R}{D_P^2}(\Delta h_{IE} - D_P \operatorname{tg}\alpha_P) \tag{11}$$

We will educe the formula for calculation of the mean error m_K of the refraction factor k, for the case when Δh_{IE} is calculated from synchronic and bi-directional observations. It will be used then to calculate the lower limit of the length of sight lines, which enables to rationally calculate of the value of this factor. We will assume for that reason, that mean errors $m_{\alpha P}$ and $m_{\alpha K}$ of both measured vertical angles, as well as mean errors m_{SP} and m_{SK} of both measured inclined distances, and values of m_{iP} and m_{iK}, as well as m_{wP} and m_{wK}, i.e. mean errors of the height of the distance meter and prisms over the survey points P, K are known. Thus we consider a more general instance, when, besides vertical angles, inclined distances are also measured. Interesting conclusions may be drawn from the obtained formula.

In order to present k as the function of observations, we will introduce the right side of the relation (5) into the formula (9):

$$\Delta h_{IE} = 0.5(S_P \sin\alpha_P + i_P - w_K - S_K \sin\alpha_K - i_K + w_P) + w_K - i_P$$

what gives after reduction of similar terms:

$$\Delta h_{IE} = 0.5(S_P \sin\alpha_P - i_P + w_K - S_K \sin\alpha_K - i_K + w_P)$$

or

$$\Delta h_{IE} = 0.5(S_P \sin\alpha_P - S_K \sin\alpha_K - i_P - i_K + w_P + w_K)$$

Since

$$\Delta h_{IE} - S_P \sin\alpha_P = 0.5(-S_P \sin\alpha_P - S_K \sin\alpha_K - i_P - i_K + w_P + w_K) \tag{12}$$

so after substitution in (10) we obtain:

$$k = 1 - \frac{R}{S_P^2}(-S_P \sin\alpha_P - S_K \sin\alpha_K - i_P - i_K + w_P + w_K)$$

i.e.

$$k = 1 + \frac{R}{S_P^2}(S_P \sin\alpha_P + S_K \sin\alpha_K + i_P + i_K - w_P - w_K) \tag{13}$$

Now we will calculate partial derivatives:

$$\frac{\partial k}{\partial S_P} = \frac{S_P^2 R \sin\alpha_P - R(S_P \sin\alpha_P + S_K \sin\alpha_K + i_P + i_K - w_P - w_K)2S_P}{S_P^4} =$$

$$= \frac{S_P^2 R \sin\alpha_P - 2RS_P^2 \sin\alpha_P - 2RS_P(S_K \sin\alpha_K + i_P + i_K - w_P - w_K)}{S_p^4} =$$

$$= \frac{-RS_P^2 \sin\alpha_P - 2RS_p(S_K \sin\alpha_K + i_P + i_K - w_P - w_K)}{S_P^4} =$$

$$= \frac{-RS_P \sin\alpha_P - 2R(S_K \sin\alpha_K + i_P + i_K - w_P - w_K)}{S_P^2}$$

Assuming equal values of inclined distances

$$S_P = S_K = S \tag{14}$$

expression for $\partial k / \partial S_p$ may be further transformed and the following relation may be obtained

$$\frac{\partial k}{\partial S_P} = \frac{-R \sin\alpha_P - L_k 2R \sin\alpha_P}{S^2} - \frac{2R(i_P + i_K - w_P - w_K)}{S^3}$$

what, after letting

$$\sum i = i_P + i_K \quad ; \qquad \sum w = w_P + w_K \tag{15}$$

gives

$$\frac{\partial k}{\partial S_P} = -\left\{\sin\alpha_P + 2\sin\alpha_K) + \frac{2}{S}(\sum i + \sum w)\right\}\frac{R}{S^2} \tag{16}$$

Calculation of remaining derivatives is considerably simpler:

$$\frac{\partial k}{\partial S_K} = \frac{R\sin\alpha_K}{S_P^2} = \frac{R\sin\alpha_K}{S} \quad ; \quad \frac{\partial k}{\partial \alpha_P} = \frac{R\cos\alpha_P}{S_P} = \frac{R\cos\alpha_P}{S}$$

$$\frac{\partial k}{\partial \alpha_K} = \frac{RS_K\cos\alpha_K}{S_P^2} = \frac{R\cos\alpha_K}{S} \tag{17}$$

$$\frac{\partial k}{\partial i_P} = \frac{\partial k}{\partial i_K} = \frac{R}{S_P^2} = \frac{R}{S^2} \quad ; \quad \frac{\partial k}{\partial w_P} = \frac{\partial k}{\partial w_K} = -\frac{R}{S_P^2} = -\frac{R}{S^2}$$

Therefore we have the following exact formula for calculation of the mean error of the refraction factor k

$$\begin{aligned} m_K^2 = {} & \left(\frac{R}{S}\cos\alpha_P m_{\alpha P}\right)^2 + \left(\frac{R}{S}\cos\alpha_K m_{\alpha K}\right)^2 + \left(\frac{R}{S^2} m_{iP}\right)^2 + \left(\frac{E}{S^2} m_{iK}\right)^2 + \\ & + \left(\frac{R}{S^2} m_{wP}\right)^2 + \left(\frac{R}{S^2} m_{wK}\right)^2 + \\ & + \left[\frac{R}{S^2}\left\{(\sin\alpha_P + 2\sin\alpha_K) + \frac{2}{S}(\sum i - \sum w)\right\} m_{SP}\right]^2 + \left(\frac{R}{S^2}\sin\alpha_K m_{SK}\right)^2 \end{aligned} \tag{18}$$

If equal mean errors of heights of instruments:

$$m_{iP} = m_{iK} = m_{wP} = m_{wK} = m_i \tag{19}$$

and measured angles:

$$m_{\alpha P} = m_{\alpha K} = m_\alpha \tag{20}$$

as well as inclined distances:

$$m_{SP} = m_{SK} = m_S \tag{21}$$

are assumed and if it is assumed that values of measured angles are small, i.e.

$$\cos\alpha_P = \cos\alpha_K \approx 1 \tag{22}$$

the formula (18) will be considerably simplified:

$$m_k^2 = 2\left(\frac{R}{S}m_\alpha\right)^2 + 4\left(\frac{R}{S^2}m_i\right)^2 + \left[\frac{R}{S^2}\left\{\sin\alpha_P + 2\sin\alpha_K + \frac{2}{S}\left(\sum i - \sum w\right)\right\}m_S\right]^2 + \\ + \left(\frac{R}{S^2}\sin\alpha_K m_S\right)^2 \tag{23}$$

It may be easily checked, that mean errors of inclined distances do not significantly influence the results. Therefore calculations of the mean error m_K may be performed basing on the following working formula:

$$m_k^2 = 2\left(\frac{R}{S}m_\alpha\right)^2 + 4\left(\frac{R}{S^2}m_i\right)^2 \tag{24}$$

which – besides mean errors of the angle m_α and heights of instruments m_i, contains only two variables: the radius of the reference sphere R and the inclined distance S. It is obvious, that m_α should be specified as an arc measure.

Similar relation is presented by A.Wróbel in his Doctor's Thesis [3].

Basing on (24) the values of the mean error m_k of the refraction factor will be calculated, listing values of components in the same order as they appear in the formula. Therefore the first component will represent the influence of the mean error of the vertical angle and the second component will represent the influence of heights of instruments. We assume that m_α = 5", m_i = 0.02 m. We will obtain for changing values of S:

S = 500 m $m_k = (0.1915 + 1.0427)^{1/2} = 1.11_1$
S = 1000 m $m_k = (0.0479 + 0.0652)^{1/2} = 0.33_6$
S = 1167 m $m_k = (0.0351 + 0.0351)^{1/2} = 0.26_5$
S = 1500 m $m_k = (0.0213 + 0.0129)^{1/2} = 0.18_5$
S = 2000 m $m_k = (0.0120 + 0.0041)^{1/2} = 0.12_7$
S = 2500 m $m_k = (0.0077 + 0.0017)^{1/2} = 0.09_7$
S = 3000 m $m_k = (0.0053 + 0.0008)^{1/2} = 0.07_8$
S = 4000 m $m_k = (0.0030 + 0.0003)^{1/2} = 0.05_7$
S = 5000 m $m_k = (0.0019 + 0.0001)^{1/2} = 0.04_5$

The following conclusions may be drawn from the above list, which apply to assumed values of mean errors; it should be stressed that they do not overstate the accuracy of measurements.

1. For sight lines, which are shorter than 1167 m (S<1167), the mean error of heights of instruments over the survey points has stronger influence on the mean error of the factor of refraction than the mean error of an angle.
2. For S = 1167 influence of both errors is the same.
3. For S>1167 errors of angles have stronger influence on the mean error of the refraction factor, however the second component of the formula (24) should not be neglected until S = 4000 m.
4. For a sight line S = 2000 m the mean error of the factor $m_k = 0.12_7$, so it is almost equal to the commonly applied value k = 0.13.
5. Only for the sight line S = 3000 m the mean error $m_k = 0.07_8$, what equals to about 60% of the value k = 0.13, which is used in calculations.

So it turns out that the refraction factor should be calculated basing on sight lines, which are not shorter than 3000 m. In the case of such distances, they are "the high" sight lines; for shorter sight lines it is unreasonably to calculated the refraction factor since its mean error is equal or bigger than the value of this factor.

The above comments apply to climatic and geographic conditions, which are similar to the Polish ones.

It may be checked that utilisation of special, precise methods of measurements of heights of instruments, which decrease the mean error m_i even to parts of a millimetre, will not change the conclusion concerning the lower limit of the length of sight lines. Thus, the condition S>3000 m for measurements used for determination of the refraction factor k basing on synchronic and bi-directional observations, is still valid.

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Właściwa długość celowej przy wyznaczaniu współczynnika refrakcji z obserwacji synchronicznych i dwustronnych

S t r e s z c z e n i e

W artykule wyprowadzono wzór na błąd średni m_k współczynnika refrakcji wyznaczonego na podstawie synchronicznych i dwustronnych obserwacji kątów pionowych dokonanych na przęśle niwelacji trygonometrycznej. Z analizy wzoru wynika, że na błąd średni tego współczynnika największy wpływ mają błędy średnie kątów pionowych oraz wysokości dalmierzy i luster nad punktami geodezyjnymi. Nieistotny jest natomiast błąd odległości, którego pominięcie prowadzi do prostej formuły roboczej.

$$m_k^2 = 2\left(\frac{R}{S} m_\alpha\right)^2 + 4\left(\frac{R}{S^2} m_i\right)^2$$

Dla przyjętych wartości błędów średnich: kąta pionowgo m_α = 5" i wysokości przyrządu m_i = 0,02 m oraz zmieniających się odległości S policzono wartości m_k. Z uzyskanych liczb wynika, że współczynnik refrakcji k powinno się wyznaczać przy celowych przekraczających 3000 m, gdyż przy odległościach krótszych błędy średnie współczynnika są tego samego rzędu co powszechnie stosowana jego wartość tj. $k = 0{,}13$ lub znacznie ją przewyższają.

Zastosowanie precyzyjnych metod pomiaru wysokości przyrządów nie zmienia wypowiedzianego tu wniosku i warunek $S > 3000$ nadal pozostaje w mocy.

Александер Скурчыньски

Правильная длина визирной линии для определения коэффициента рефракции на основе синхронических и двусторонных наблюдений

Р е з ю м е

В статье выведена формула для средней квадратической ошибки m_k коэффициента рефракции, определённого на основе синхронических и двусторонних наблюдений вертикальных углов, проведенных на секции тригонометрического нивелированния. С анализа формулы вытекает, что на среднюю квадратическую ошибку этого коэффициента самое большое влияние имеют среднее квадратические ошибки вертикальных углов, а также высоты дальномеров и зеркал над геодезическими пунктами. На это место второстепенным является средняя квадратическая ошибка расстояния, которого опускание ведёт к простой рабочей формуле

$$m_k^2 = 2\left(\frac{R}{S} m_\alpha\right)^2 + 4\left(\frac{R}{S^2} m_i\right)^2$$

Для принятых величин средних квадратических ошибок: вертикаляного угла m_α = 5" и высоты прибора m_i = 0,02 м, а также изменяющихся расстояний S были вычислены величины m_k. С полученных чисел вытекает, что коэффициент рефракции k должен определяться с визирными линиями больше чем 3000 m, так как при более коротких расстояниях среднее квадратические ошибки коэффициента получаются того-же ряда, что обычно принимаемая его величина, т. е. $k = 0{,}13$ или даже на много её превосходят.

Применение точных методов измерений высоты прибора не изменяет высказанного здесь вывода и условие $S > 3000$ сохраняет свою силу.