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Robust estimation of variance coefficient (VR-estimation) for dependent observations*

The paper presents the method of robust estimation of variance coefficient. The concept of VR-estimation presented in [6] is generalised in case of dependent observations. The basis of the method is usage of reinforcement matrix which guarantees the robustness of the estimate. The reinforcement matrix which is closely connected with the weight function of M-estimation, gives a possibility to perform robust adjustment. Thus such a method is also presented. At last, an example is shown too.

1. *Introduction*

Robust estimation has been investigated in the field of geodesy. Studies have concentrated on robust estimation of the parameter vector of the following model of geodetic network:

$$\boldsymbol{x} = \mathbf{A}\boldsymbol{X} + \varepsilon \tag{1}$$

where: $\boldsymbol{x}$ – vector of observations, $\mathbf{A}$ – known rectangular matrix, $\boldsymbol{X}$ – vector of parameters, ε - vector of observation errors with covariance matrix $\mathbf{C}_\varepsilon = \sigma_0^2\mathbf{Q} = \sigma_0^2\mathbf{P}^{-1}$, $\boldsymbol{Q}$ – cofactor matrix, $\mathbf{P}$ – weight matrix.

Up to now, proposed methods of robust estimation [1, 2] based on M-estimation, lead to the equivalent adjustment problem [6]:

$$\left.\begin{aligned} \varepsilon_M &= \boldsymbol{x} - \mathbf{A}\boldsymbol{X} \\ \mathbf{C}_{\varepsilon_M} &= \sigma_0^2\overline{\mathbf{P}}^{-1} \\ \min_{\boldsymbol{x}} &\, \varepsilon_M^T\overline{\mathbf{P}}^{-1}\varepsilon_M \end{aligned}\right\} \tag{2}$$

where: $\mathbf{P}$- equivalent weight matrix, ε_M – robust error vector with covariance matrix $\mathbf{C}_{\varepsilon_M}$

Resolving (2) one can obtain:

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$$\hat{\varepsilon}_M = \left(\mathbf{I} - \mathbf{A}\left(\mathbf{A}^T\overline{\mathbf{P}}\mathbf{A}\right)^{-1}\mathbf{A}\overline{\mathbf{P}}\right)x = \mathbf{M}_M \tag{3}$$

where: $\mathbf{I}$ – identity matrix.
Thus the estimator of variance coefficient can be determined as [7]:

$$\hat{\sigma}_0^2 = \frac{1}{n-r}\hat{\varepsilon}_M^T\overline{\mathbf{P}}\hat{\varepsilon}_M \tag{4}$$

The above-mentioned estimator is invariant, unbiased and the most efficient (if the vector x is normally distributed) but it is hard to say if it is robust. Considering robustness problem, a concept of a new method of estimation has been formulated [6] VR-estimator, obtained using the new method, is a quadratic estimate of the following type:

$$\hat{\sigma}_0^2 = x^T\Omega x = \varepsilon^T\Omega\varepsilon \tag{5}$$

where: Ω – symmetric matrix.

This estimate is assumed to have the same properties as (4) but also it is robust. Robustness is the major purpose of this estimation. It is obtained using reinforcement matrix $\mathbf{R}$ which is calculated on the basis of reinforcement function $r(\varepsilon)$.

The principles of VR-estimation were formulated in [6]. This paper presents generalisation of this method in case of dependent observations.

2. *VR estimation*

As it was mentioned, the VR-estimator is assumed to be:

i) invariant if $\Omega\,\mathbf{A} = \mathbf{0}$

ii) unbiased if $Tr(\Omega\,\mathbf{Q}) = 1$

iii) the most efficient if $Var(\hat{\sigma}_0^2) = \min$

2.1. The most efficient estimate

Let all matrices Ω, which fulfil the conditions i) and ii) form a set denoted as φ. Then the matrix defining the most efficient estimate should be determined as a solution of the following optimization problem:

$$\min_{\Omega\in\varphi} Var(\varepsilon^T\boldsymbol{\Omega}\varepsilon) \tag{6}$$

This problem can be resolved using the indefinite Lagrange multipliers method [3, 5]. Thus, if we assume that the excess for every variable ε_i is equal to zero, the unknown matrix can be determined by minimising the auxiliary Lagrange function [5]:

$$\varphi(\Omega) = Tr(\Omega\,\mathbf{Q}\,\Omega\,\mathbf{Q}) - 2\lambda[Tr(\Omega\,\mathbf{Q}) - 1] - 2Tr(\chi^T\,\Omega\,\mathbf{A}) \tag{7}$$

where: λ - the Lagrange multiplier corresponding to the condition i), χ - matrix of the Lagrange multipliers corresponding to the condition ii).

An easier solution can be obtained when we replace above function with function of vectors formed with the elements of Ω :

$$vec(\Omega) = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \vdots \\ \Omega_n \end{bmatrix} \tag{8}$$

$$vetd(\Omega) = [\Omega_{1,1} \quad \Omega_{1,2} \quad \dots \quad \Omega_{1,n} \ \vdots \ \Omega_{2,2} \quad \Omega_{2,3} \quad \dots \quad \Omega_{2,n} \ \vdots \ \dots \ \vdots \ \Omega_{n,n}]^T$$

where: Ω_i – i-th column of the matrix.

Because Ω is symmetric, one can write [4]:

$$vec(\Omega) = \mathrm{S}_0 \cdot vetd(\Omega) \tag{9}$$

where: $\mathbf{S}_0$ – transformation vector.

To make the replacement we should additionally define the following transformation:

$$\mathcal{A}(\Omega^T) = \begin{bmatrix} \mathcal{A}_1(\Omega^T) \\ \mathcal{A}_2(\Omega^T) \\ \vdots \\ \mathcal{A}_n(\Omega^T) \end{bmatrix}; \quad \mathcal{A}_i(\Omega^T) = \begin{bmatrix} \Omega_i^T & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \Omega_i^T & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \dots & \Omega_i^T \end{bmatrix}$$

where: $\mathbf{0}$ – null vector.

Now, we can write (if $\mathbf{Q}$ is diagonal):

$$\varphi[vetd(\Omega)] = vetd(\Omega)^T \mathrm{W} vetd(\Omega) - 2\lambda[vetd(\mathbf{Q})^T vetd(\Omega) - 1] - 2\boldsymbol{k}\mathcal{A}_S(\mathrm{A}^T)\, vetd(\Omega) \tag{10}$$

where: $\mathbf{W} = \mathbf{S}_0^T(\mathbf{Q} \otimes \mathbf{Q})\mathbf{S}_0$, $\mathcal{A}_S(\mathbf{A}^T) = \mathcal{A}(\mathbf{A}^T)\mathbf{S}_0$, $\boldsymbol{k}$ – the Lagrange multipliers vector replacing χ, $\otimes$ - Kronecker product.

$\mathcal{A}_S(\mathbf{A}^T)$ is not always full rank, therefore a new matrix $\mathcal{A}_{S0}(\mathbf{A}^T)$ (a full rank one) should be formed. It can be made by removing right number of rows from the initial matrix [4]. Finally, the vector minimising (10) can be written in the form:

$$vetd(\Omega) = \mathrm{W}^{-1}\Theta_0^T\left(\Theta \mathrm{W} \Theta_0^T\right)^{-1}\Delta \tag{11}$$

where: $\Theta_0 = \begin{bmatrix} \mathcal{A}_{S0}(\mathbf{A}^T) \\ vetd(\mathbf{Q}) \end{bmatrix}; \ \Delta = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$

The presented method gives rather simple solution and, what is more, is very convenient in other applications, for example in VR-estimation [6].

2.2. Robust estimate

The main purpose of VR-estimation is to determine Ω which makes the estimate be robust (this is very similar to the purpose of M-estimation where we are searching for robust parameter vector). M-estimation is based on influence functions, thus in VR-estimation one can analogously define an influence matrix:

$$\mathbf{J} = \frac{\partial}{\partial \Omega} c_{\mathrm{J}} \times Var(\varepsilon^T \Omega \varepsilon) \tag{12}$$

where: c_{J} – positive constant.

Because $Var\ (\varepsilon^T \Omega\ \varepsilon) = \sigma_0^2 Tr\ (\Omega\ \mathbf{Q}\ \Omega\ \mathrm{Q})$, thus:

$$\mathrm{J} = 4\mathbf{Q}\Omega\mathbf{Q} - 2Diag(\mathbf{Q}\Omega\mathbf{Q}) \tag{13}$$

where: $c_{\mathbf{J}}^{-1} = 2\sigma_0^2$

J should be adequately modified to obtain robust estimator of variance coefficient. At first, it is possible to define the relationship between diagonal elements of the cofactor matrix **Q** and equivalent cofactor matrix $\overline{\mathbf{Q}}$ as follows:

$$\overline{q}_{ii} = r(\varepsilon_i) q_{ii} \tag{14}$$

where: $r(\varepsilon_i)$ – reinforcement function.

Thus the modified influence matrix can be written in the form:

$$\overline{\mathrm{J}}(\varepsilon) = 4\mathrm{Q}(\mathrm{R}(\varepsilon) * \Omega)\mathrm{Q} - 2Diag(\mathrm{Q}(\mathrm{R}(\varepsilon) * \Omega)\mathrm{Q})$$

where: $\mathbf{R}(\varepsilon)$ – reinforcement matrix, $R_{i,j} = r(\varepsilon_i) r(\varepsilon_j)$, * - Hadamard product.

The estimator:

$$\hat{\sigma}_{0R}^2 = \varepsilon^T \Omega \varepsilon$$

with matrix $\Omega = \Omega_R$, such that:

$$\min_{\Omega \in \varphi} Var\left(\varepsilon^T (\mathbf{R} * \Omega)\varepsilon\right) = Var\left(\varepsilon^T (\mathbf{R} * \Omega_R)\varepsilon\right)$$

is the robust estimate of variance coefficient and can be called VR-estimator [6]. Unknown Ω_R itself can be determined using the method described in the previous chapter. Therefore:

$$vetd(\Omega_R) = \mathbf{W}_R^{-1} \Theta_0^T \left(\Theta_0 \mathbf{W}_R^{-1} \Theta_0^T\right)^{-1} \Delta \tag{15}$$

where: $\mathbf{W}_R = \mathbf{S}_0^T \mathbf{D}_{vec(\mathrm{R})} (\mathbf{Q} \otimes \mathbf{Q}) \mathbf{D}_{vec(\mathrm{R})} \mathbf{S}_0$ and $\mathbf{D}_{vec(\mathbf{R})}$ - diagonal matrix formed from the elements of the vector $vec(\mathbf{R})$.

The presented solution is correct for independent observations while generalised approach is shown in the next chapter.

3. *VR-estimation for dependent observations*

When the set of observations includes dependent ones, the cofactor matrix is not diagonal. This should be considered in estimation process. The influence matrix can still be determined according to the definition (12) and it is easy to prove that in considered case we also obtain (13) (the same result for dependent and independent observations). Taking into account assumption (14), relation between full matrices $\mathbf{Q}$ and $\overline{\mathbf{Q}}$ can be written in the following form:

$$\overline{\mathbf{Q}} = \mathbf{R}_Q * \mathbf{Q} = \mathbf{R}_D \mathbf{Q} \mathbf{R}_D \tag{16}$$

where: $R_{Q_{i,j}} = \sqrt{r(\varepsilon_i)}\sqrt{r(\varepsilon_i)}$, $\mathbf{R}_D = Diag\left(\sqrt{r(\varepsilon_i)}, \sqrt{r(\varepsilon_i)}, \ldots, \sqrt{r(\varepsilon_i)}\right)$

Therefore one can write:

$$\overline{\mathbf{J}} = 4\mathbf{R}_D\mathbf{Q}\mathbf{R}_D\Omega\mathbf{R}_D\mathbf{Q}\mathbf{R}_D - 2Diag(\mathbf{R}_D\mathbf{Q}\mathbf{R}_D\Omega\mathbf{R}_D\mathbf{Q}\mathbf{R}_D)$$

Modified influence matrix can also be expressed in more useful form. Replacing $\mathbf{R}_\Omega = \mathbf{Q}^{-1}\mathbf{R}_D\mathbf{Q}\mathbf{R}_D$ (if $\mathbf{Q}^{-1}$ exists) we finally obtain:

$$\overline{\mathbf{J}} = 4\mathbf{Q}\,\mathbf{R}_\Omega\Omega\,\mathbf{R}_\Omega^T\mathbf{Q} - 2Diag(\mathbf{R}_\Omega\Omega\,\mathbf{R}_\Omega^T\mathbf{Q}) \tag{17}$$

Thus the auxiliary Lagrange function (7) should be changed as follows:

$$\varphi(\Omega) = Tr\left[\mathbf{R}_\Omega\Omega\mathbf{R}_\Omega^T\mathbf{Q}\mathbf{R}_\Omega\Omega\mathbf{R}_\Omega^T\mathbf{Q}\right] - 2\lambda\left[Tr(\Omega\mathbf{Q}) - 1\right] - 2Tr\left(\chi^T\Omega\mathbf{A}\right) \tag{18}$$

To find a matrix which minimises above function, of course we apply the method based on the vector $vetd(\Omega)$. Taking into account that:

$$Tr\left[\mathbf{R}_\Omega\Omega\mathbf{R}_\Omega^T\mathbf{Q}\mathbf{R}_\Omega\Omega\mathbf{R}_\Omega^T\mathbf{Q}\right] = vec\left(\mathbf{R}_\Omega\Omega\mathbf{R}_\Omega^T\right)^T(\mathbf{Q}\otimes\mathbf{Q})vec\left(\mathbf{R}_\Omega\Omega\mathbf{R}_\Omega^T\right)$$

and because:

$$vec\left(\mathbf{R}_\Omega\Omega\mathbf{R}_\Omega^T\right) = (\mathbf{I}\otimes\mathbf{R}_\Omega)vec\left(\Omega\mathbf{R}_\Omega^T\right) = (\mathbf{I}\otimes\mathbf{R}_\Omega)\mathcal{A}(\mathbf{R}_\Omega)\mathbf{S}_0 vetd(\Omega)$$

thus finally:

$$Tr\left[\mathbf{R}_\Omega\Omega\mathbf{R}_\Omega^T\mathbf{Q}\mathbf{R}_\Omega\Omega\mathbf{R}_\Omega^T\mathbf{Q}\right] = vetd(\Omega)^T\mathbf{W}_R vetd(\Omega)$$

where: $\mathbf{W}_R = \mathbf{S}_0^T\mathcal{A}(\mathbf{R}_\Omega)^T(\mathbf{I}\otimes\mathbf{R}_\Omega)^T(\mathbf{Q}\otimes\mathbf{Q})(\mathbf{I}\otimes\mathbf{R}_\Omega)\mathcal{A}(\mathbf{R}_\Omega)\mathbf{S}_0$

Applying the above transformations, the function (18) can be written as follows:

$$\varphi\left[vetd(\Omega)\right]=vetd(\Omega)^T \mathrm{W}_R vetd(\Omega)-2\lambda\left[vec(\mathbf{Q})^T \mathrm{S}_0 vetd(\Omega)-1\right]-2\boldsymbol{k}\mathcal{A}_S\left(\mathbf{A}^T\right)vetd(\Omega)$$

According to the solution (11) the vector minimising this function has got the form:

$$vetd(\Omega_R)=\mathbf{W}_R^{-1}\Theta_0^T\left(\Theta\mathbf{W}_R\Theta_0^T\right)^{-1}\Delta \tag{19}$$

where: $\Theta_0=\begin{bmatrix}\mathcal{A}_{S0}\left(\mathbf{A}^T\right)\\ vec(\mathbf{Q})^T\mathrm{S}_0\end{bmatrix};\ \Delta=\begin{bmatrix}\mathbf{0}\\ 1\end{bmatrix}$

It can be proved that $\mathbf{W}_R$ depends on the vector ε, therefore the presented solution is an iterative process. New matrices, $\mathbf{W}_R$ and Ω_R, thereafter new VR-estimator should be determined for every approximation of vector ε.

3.1. Reinforcement function

Reinforcement matrix denoted as **R** plays the major role in VR-estimation. Determination of this matrix is based on a reinforcement function. Therefore essential properties of such a function should be given.

Function $r(\varepsilon_i)$ can be defined in several ways. This paper presents the most easy and natural one. The concept is based on the influence of outliers on the variance of ε_i. We can write that:

$$Var(\varepsilon_i+g_i)=E(\varepsilon_i+g_i)^2=E(\varepsilon_i)^2+E(g_i)^2=Var(\varepsilon_i)+g_i^2=\left(1+\frac{g_i^2}{Var(\varepsilon_i)}\right)Var(\varepsilon_i)$$

where: g_i – outlier of the i-th observation.

This formula describes relation between variances of the same observation with and without an outlier. On this basis, one can consider a more general case and express relation between variances of ε_i in successive iteration steps in the form:

$$Var^{i+1}(\varepsilon_i+g_i)=\left(1+\frac{g_i^2}{Var^i(\varepsilon_i)}\right)Var^i(\varepsilon_i) \tag{20}$$

Thereafter it is possible to assume a proportionality between outliers and error estimator after standardisation:

$$g_i \leftrightarrow c\frac{\hat{\varepsilon}_i}{m_{\hat{\varepsilon}_i}}$$

where: c – constant, $m_{\hat{\varepsilon}_i}$ — mean error of estimator $\hat{\varepsilon}_i$.

Thus the formula (20) can be changed:

$$Var^{i+1}(\varepsilon_i + g_i) = \left(1 + \frac{c^2 \left(\frac{\hat{\varepsilon}_i}{m_{\hat{\varepsilon}_i}} \right)^2}{Var^i(\varepsilon_i)} \right) Var^i(\varepsilon_i) \tag{21}$$

The above relation presents the way of changing the variance of observation influenced by outliers. In turn, variances of observations without gross errors should not be modified at all. Thus the reinforcement function can be defined as follows [6]:

$$\left. \begin{array}{l} r(\varepsilon_i) = 1 \quad \text{if} \quad |\varepsilon_i| \le \Delta\varepsilon \\ r(\varepsilon_i) > 1 \quad \text{if} \quad |\varepsilon_i| > \Delta\varepsilon \end{array} \right\} \tag{22}$$

where: $\Delta\varepsilon$ - acceptable standardised value of random errors of measurements.
Taking into account the general form of reinforcement function (22) and the derived relation (21), we can write:

$$r(\hat{\varepsilon}_i) = \begin{cases} 1 & \text{if} \quad \left| \frac{\hat{\varepsilon}_i}{m_{\hat{\varepsilon}_i}} \right| \le \Delta\varepsilon \\ \left(1 + \frac{c^2 \left(\frac{\hat{\varepsilon}_i}{m_{\hat{\varepsilon}_i}} \right)^2}{Var^i(\varepsilon_i)} \right) & \text{if} \quad \left| \frac{\hat{\varepsilon}_i}{m_{\hat{\varepsilon}_i}} \right| > \Delta\varepsilon \end{cases} \tag{23}$$

The graph of the proposed function is shown in Fig.1

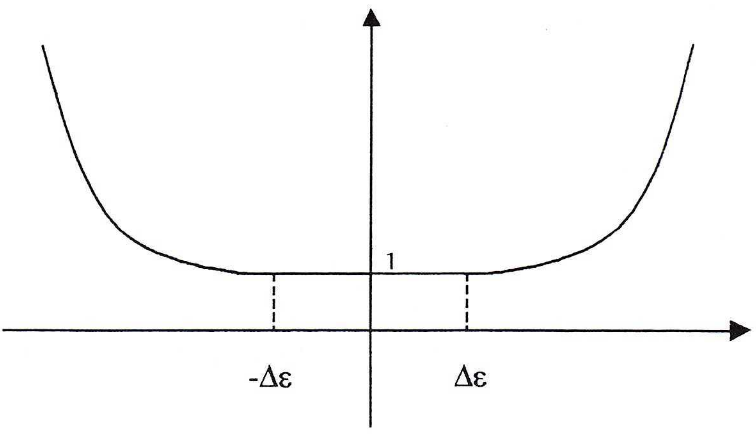

Fig. 1. The reinforcement function

4. *Iterative method of adjustment based on VR-estimation*

Usage of the reinforcement function gives a possibility for robust adjustment, not only for estimation of variance coefficient. Combining VR-estimation with M-estimation creates a new method of adjustment which results are robust estimates of vector of parameters and of variance coefficient. The idea of this method is presented in the following scheme:

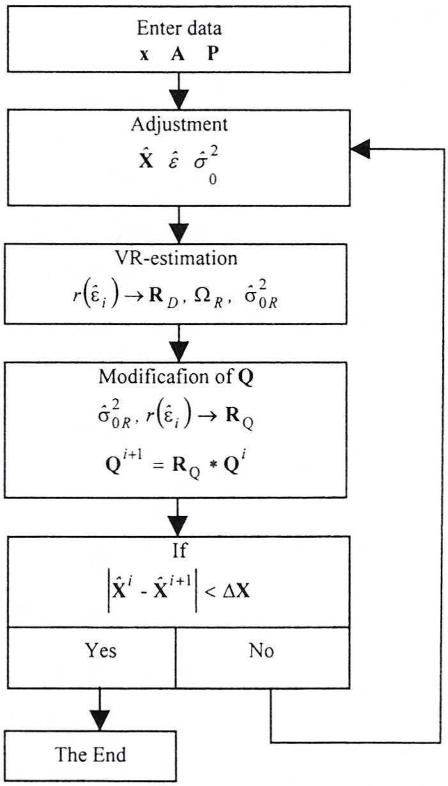

Fig. 2. The general scheme of iterative adjustment

As it is shown every iteration consists of some steps:

- LS adjustment
- VR-estimation
- modification of **Q** matrix
- checking the convergence condition.

The first three steps do not need any comments, in the fourth step we should assume a criterion of convergence. Analogously to other robust methods [7] the convergence of parameters has been chosen as the mentioned criterion.

5. *Numerical tests*

The proposed method of estimation has been applied to adjustment of the model geodetic network presented in the paper [7]. This network consists of four fixed points called A, B, C and D and two new points – I and II. Sixteen vertical angles has been taken to the adjustment (Fig. 2). According to the mentioned paper the following cofactor matrices have been applied:

$$\left.\begin{array}{l} \mathbf{Q}_A = \mathbf{Q}_C = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \\ \mathbf{Q}_B = \mathbf{Q}_D = \mathbf{Q}_I = \mathbf{Q}_{II} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \end{array}\right\} \Rightarrow \mathbf{Q} = \begin{bmatrix} \mathbf{Q}_A & & & & & \\ & \mathbf{Q}_B & & & & \\ & & \mathbf{Q}_C & & & \\ & & & \mathbf{Q}_D & & \\ & & & & \mathbf{Q}_I & \\ & & & & & \mathbf{Q}_{II} \end{bmatrix}$$

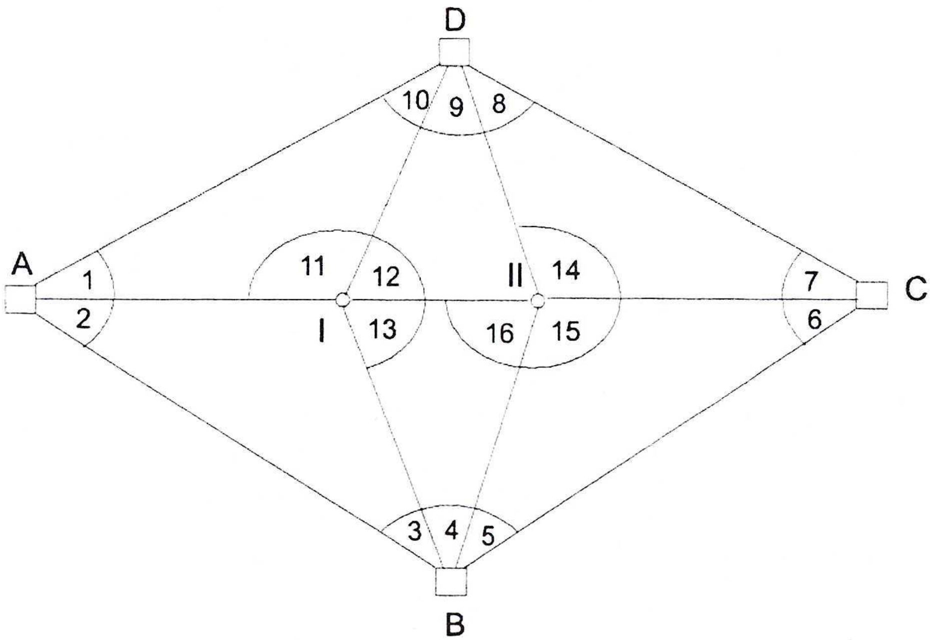

Fig. 3. The tested network

The first test was a comparison of results obtained from LS-adjustment with results of the iterative method with several values of the constant c (see 23). It was assumed that: $\Delta\varepsilon = 1.5$, $g_{10} = -40^{cc}$ and $g_{12} = 60^{cc}$. The convergence condition was as follows:

$$\max_j \left| \hat{X}_j^{k+1} - \hat{X}_j^k \right| < 1.0 \text{ mm} \tag{24}$$

The outcomes are given in Table 1. The first column presents the results of the LS adjustment, the second one - also LS but with the outliers, next five – the results of the proposed method with several values of c.

Table 1

Results of the iterative adjustments

	LS	LS with outliers	c^2					c^2
			1	2	5	10	20	2
$\hat{\sigma}_0^2$	10.128	251.279	100.330	59.683	17.800	7.800	4.021	27.778
$d\mathbf{X}$ [m]	−0.0102	−0.0016	−0.0048	−0.0062	−0.0086	−0.0099	−0.0104	−0.0083
	−0.0109	−0.0041	−0.0075	−0.0083	−0.0098	−0.0112	−0.0117	−0.0099
	−0.0002	0.0060	0.0019	0.0009	−0.0002	−0.0007	−0.0011	−0.0003
	0.0101	0.0100	0.0096	0.0096	0.0100	0.0100	0.0100	0.0097
$\hat{\varepsilon}$ [^{cc}]	−6.89	−13.80	−11.88	−10.66	−8.46	−7.62	−7.32	−8.74
	−3.26	3.66	1.72	0.48	−1.68	−2.52	−2.82	−1.41
	0.42	24.35	13.99	10.66	7.64	0.48	−1.12	4.82
	3.75	−25.05	−12.45	−8.28	−0.58	4.10	5.95	−1.42
	2.79	7.66	5.42	4.58	2.90	2.38	2.13	3.56
	1.89	9.87	5.15	3.79	1.96	1.19	0.73	2.24
	1.15	−6.83	−2.11	−0.74	1.08	1.85	2.31	0.80
	−3.90	−10.43	−5.26	−4.14	−3.81	−3.27	−2.81	−3.03
	1.45	3.84	1.63	0.78	1.03	1.60	1.61	0.46
	−3.41	40.73	37.76	38.00	36.92	35.80	35.34	36.71
	3.21	10.15	9.99	9.45	6.72	4.46	3.92	7.00
	0.65	−38.22	−51.38	54.43	−58.31	−61.31	−62.85	−58.61
	−0.69	2.07	3.11	2.13	0.53	0.82	0.98	1.02
	−1.25	13.26	3.37	0.88	−1.26	−2.58	−3.50	−1.78
	−0.68	−13.53	−6.58	4.37	−0.86	0.42	1.14	−1.79
	−0.40	7.27	7.83	6.72	2.89	1.21	0.89	4.03

It is obvious, that results of the robust adjustment depend on c^2. Among taken to the test, the best results were obtained for $c^2 = 10$. For the other, the results seem to be worse. It is caused by the role of c^2 in adjustment process (see (23)). If the constant is too big, e. g. 20, changes of $\mathbf{Q}$ are big too, and some element of this matrix may be overstated. On the other hand, a small value of c^2 should influence the convergence of the method and increase the number of iteration steps. However, the results for $c^2 = 1$ or 2 are not satisfactory at all. It was caused by the unsuitable convergence condition. In case of a small value of the constant, differences between successive iterations would be small too, they could be even smaller than 1 mm assumed in the criterion of convergence (24), and that caused, the iterative process was finished too early. To verify such a hypothesis the new condition was assumed:

$$\max_j \left| \hat{X}_j^{k+1} - \hat{X}_j^k \right| < 0.01 \text{ mm}$$

The results for this condition and $c_2 = 2$ are shown in the last column of the table. They are more similar to the values of the LS adjustment presented in the first column. It confirms the hypothesis and the correctness of the method of adjustment.

The second test was a comparison of the proposed method with other robust methods of adjustment. Two methods were chosen to the test: Huber's and IGG Scheme. Both of them are iterative and based on application of proper weight functions. The following modified Huber's weight was assumed [7]:

$$\bar{p}_{i,j} = \begin{cases} p_{i,j} & \left|\hat{\varepsilon}'_j\right| < a_0 \\ p_{i,j} \cdot \dfrac{a_0}{\left|\hat{\varepsilon}'_j\right|} & \left|\hat{\varepsilon}'_j\right| \geq a_0 \end{cases}$$

where: $\hat{\varepsilon}'_j$ – standardised $\hat{\varepsilon}_j$, $a_0 = 1.5$

In IGG Scheme the elements of the weight matrix were modified according to the following relation [7]:

$$\bar{p}_{i,j} = \begin{cases} p_{i,j} & \left|\hat{\varepsilon}'_j\right| \leq k_0 \\ p_{i,j} \cdot w_j & k_0 < \left|\hat{\varepsilon}'_j\right| \leq k_1 \\ 0 & \left|\hat{\varepsilon}'_j\right| > k_1 \end{cases}$$

where: $w_j = \dfrac{k_0}{\left|\hat{\varepsilon}'_j\right|} \cdot \dfrac{k_1 - \left|\hat{\varepsilon}'_j\right|}{\Delta k}$, $\Delta k = k_1 - k_0$

The values of the used constant were assumed as 1.5 and 2.5 [7].

Table 2

Comparison of robust adjustment methods

	LS	LS with outliers	$c^2 = 101$	IGG Scheme	Huber's method
$\hat{\sigma}_0^2$	10.128	251.279	7.800	58.191	387.736
$d\mathbf{X}$ [m]	−0.0102	−0.0016	−0.0099	−0.0095	−0.0083
	−0.0109	−0.0041	−0.0112	−0.0107	−0.0099
	−0.0002	0.0060	−0.0007	0.0003	−0.0003
	0.0101	0.0100	0.0100	0.0100	0.0096
$\hat{\varepsilon}$ [cc]	−6.89	−13.80	−7.62	−7.78	−8.87
	−3.26	3.66	−2.52	−2.36	−1.28
	0.42	24.35	0.48	1.86	4.97
	3.75	−25.05	4.10	1.77	−1.77
	2.79	7.66	2.38	3.33	3.77
	1.89	9.87	1.19	2.63	2.33
	1.15	−6.83	1.85	0.41	0.70
	−3.90	−10.43	−3.27	−4.32	−2.80
	1.45	3.84	1.60	2.16	0.30
	−3.41	40.73	35.80	36.30	36.64
	3.21	10.15	4.46	3.30	7.13
	0.65	−38.22	−61.31	−58.29	−58.66
	−0.69	2.07	0.82	0.48	1.22
	−1.25	13.26	−2.58	−0.09	−1.90
	−0.68	−13.53	0.42	−1.96	−2.11
	−0.40	7.27	1.21	0.51	4.44

Modifications of weight matrices were performed for the columns and for the rows (changing $\hat{\varepsilon}'_j \rightarrow \hat{\varepsilon}'_i$), so that the modified matrices were still symmetric.

The convergence criterion (24) was also applied in that test. The results of adjustments are listed in Table 2.

It is easy to see that all the three methods give almost the same results except the estimator of variance coefficient. The estimates of parameter vector and the estimates of ε are robust. This test confirms that using VR-estimation it is possible to obtain the robust estimate of variance coefficient as well as robust vectors $\hat{\mathbf{X}}$ and $\hat{\varepsilon}$.

The results of the tests lead us to the conclusion that the method of adjustment proposed in this paper is just an alternative for the other ones. It should be especially applied in case we want to know the value of the variance coefficient without the influence of gross errors. However to obtain better results, this method should be still developed especially about the reinforcement function.

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Odporna estymacja współczynnika wariacji (VR-estymacja) w przypadku zmiennych zależnych

S t r e s z c z e n i e

W pracy przedstawiono odporną metodę estymacji współczynnika wariacji. Koncepcję takiej estymacji (VR-estymacji), przestawiono w pracy [6], uogólniono na przypadek zmiennych zależnych. Podstawą VR-estymacji jest użycie macierzy wzmocnienia, która zapewnia szukanemu estymatorowi odporność na błędy grube. Macierz wzmocnienia, podobnie jak funkcja wzmocnienia, jest ściśle powiązana z funkcją wagową stosowaną w M-estymacji, co daje możliwość przeprowadzenia odpornego wyrównania. W pracy zaproponowano odporną metodę wyrównania bazującą na VR-estymacji oraz przestawiono przykład jej zastosowania do wyrównania sieci geodezyjnej.

Роберт Духновски

Устойчивая оценка вариационного коэффициента (оценка VR) в случае зависимых переменных

Р е с ю м е

В работе представлен устойчивый метод оценки вариационного коэффициента. Концепция такой оценки (оценка VR), представленная в работе [6], обобщена для случая зависимых переменных. Основой оценки VR является применение матрицы усиления, которая обеспечивает искомой оценке устойчивости к промахам. Матрица усиления, подобно тому как функция усиления, близко связана с весовой функцией применяемой в оценке M, что даёт возможность проведения устойчивого выравнивания. В работе предложен устойчивый метод выравнивания, основан на оценке VR. а также представлен пример его применения для выравнивания геодезической сети.