GEODEZJA I KARTOGRAFIA
t. XLIX, z. 2, s. 107–117 (2000)

Andrzej Kobryń

Białystok University of Technology
Construction and Environmental Engineering Department
(15-351 Białystok, 45/E Wiejska St.)

Multinomial transition curves to design road grade lines*

The article presents a family of the so called multinomial transition curves. It is possible to conveniently form the curvatures of those curves through appropriate choice of minimum radius of curve R, tangent gradient at starting point $\mathrm{tg}u_0$, and the abscissa of ending point x_K, which affects the value of elementary design parameter $C = R\mathrm{tg}u_0/x_K$. The possibility to select C values from interval $<1/3;\ 2/3>$ and the unusual position of the curves within the Cartesian coordinate system make them remarkably useful for example at designing road grade lines in order to accomplish optimum compatibility with the relief. The foregoing being borne in mind, the article presents appropriate methods of designing road grade line with the use of the said curves.

1. *Introduction*

The need to develop and construct a road network which would meet the requirements the present times call for, alongside the progress of working techniques, these are the factors that account for the necessity to look for new geometrical solutions to be applied to the designing of the road grade line. While designing it, is advisable to make it comply with the relief, which is significant from the viewpoint of minimization of earth works. According to the accepted opinion, multinomial functions are by all means more advantageous in that respect than traditional elements in the form of straight lines and circular arcs; the former make it possible to drive the grade line in accordance with given direction points. Certain families of transition curves — which are characterized by remarkable freedom of curvature formation — have considerable merits in that line. Its withwhile to mention that such elements are more favourable also in the light of movement dynamics conditions. Such solutions, based upon the so called transition curves being immediately made compliant with the relief, have been presented in [3] and [5]. The so called multinominal transition curves, whose unusual location manner within the Cartesian coordinate system has been set forth in [2], are also useful in that respect. Multinominal transition curves combine the merits of curves given in the form of an explicit function of the form $y=f(x)$ with curves described by the curvature thereof,

* The issue has been accomplished at Białystok University of Technology within project W/IIB/7/96

for apart from clear dependencies expressing the coordinates, it is possible to conveniently form the curvature by means of an appropriate choice of parameters of the curve, which is considerably significant from the viewpoint of compliance with the relief; the issue will be referred to further on. A new family of curves of such a type, alongside tentative designing methods of a grade line with the use thereof, are just about to be presented.

2. *Multinomial transition curves and their designing conditions*

Let us consider a formula of the following form:

$$K=\frac{y''}{(1+y'^2)^{2/3}} \tag{1}$$

describing the curvature of explicit curve $y=f(x)$, let us assume maximum curvature at ending point K and let us locate the curve to be found within a Cartesian coordinate system (Fig. 1), formulating the following boundary conditions:

$$f(x_P)=0 \tag{2}$$

$$f'(x_P)=\mathrm{tg}u_0 \tag{3}$$

$$f''(x_P)=0 \tag{4}$$

$$f'(x_K)=0 \tag{5}$$

$$f''(x_K)=-\frac{1}{R} \tag{6}$$

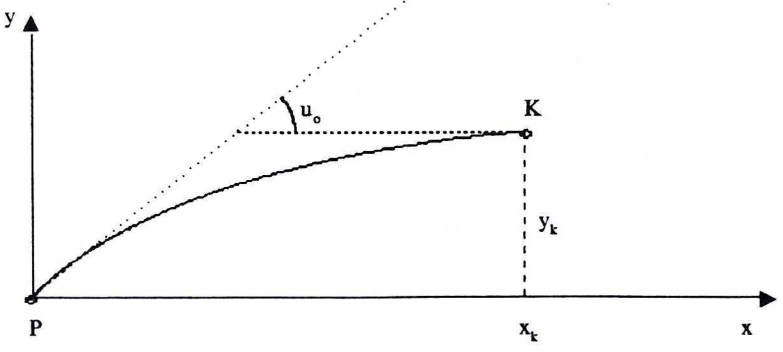

Fig. 1

In accordance with [4], deriving it from the general form of the multinomial to be found

$$y=f(x=\sum_{n=0}^{n=k} a_n x^n \tag{7}$$

at $k=4$, the following equation of the multinomial transition curve will be obtained:

$$y=\mathrm{tg}u_0\cdot x+\left(\frac{1}{3}\frac{1}{Rx_K}-\frac{\mathrm{tg}u_0}{x_K^2}\right)x^3-\left(\frac{1}{4}\frac{1}{Rx_K^2}-\frac{\mathrm{tg}u_0}{2x_K^3}\right)x^4 \tag{8}$$

which might be also expressed in the following form:

$$y=\frac{x_K^2}{R}B_1+x_K\mathrm{tg}u_0\cdot B_2 \tag{9}$$

where:

$$B_1=\frac{1}{3}t^3-\frac{1}{4}t^4$$

$$B_2=t-t^3+\frac{1}{2}t^4$$

Note that $t=\dfrac{x}{x_K}$, $x\in<0;\ x_k>$. If the following is introduced:

$$C=\frac{R\mathrm{tg}u_0}{x_K} \tag{10}$$

equation (9) will assume the following form:

$$y=\frac{x_K^2}{R}\left[Ct+\frac{1-3C}{3}t^3-\frac{1-2C}{4}t^4\right] \tag{11}$$

Since values x_K, R and $\mathrm{tg}u_0$ for $u_0\in\left(0;\ \dfrac{\Pi}{2}\right)$ are positive, as well as B_1' and B_2' for each $x\in(0,\ x_K)$ being greater than zero, function y is an increasing one in that interval. Practical use of function (9) as a transition curve will be possible if the graph of its curve within interval $<0;\ x_K>$ decreases from zero at the starting point down to the value of $-1/R$ at the end point. According to [4], in case of function (9), this calls for

$$\frac{1}{3}\leqslant\frac{R\mathrm{tg}u_0}{x_K}\leqslant\frac{2}{3} \tag{12}$$

Due to (10), dependence (12) determines at the same time the interval of admissible values of parameter C. The graphs of curvatures of curves (9) for selected parameters C have been presented in Fig. 2.

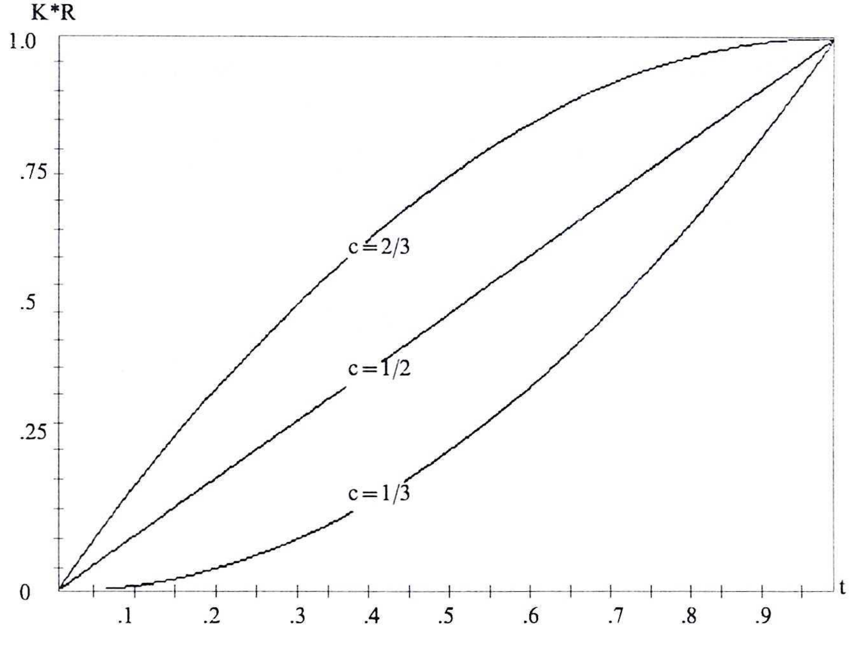

Fig. 2

3. *The use of curves at grade line designing*

Multinominal transition curves (9), just like general transition curves discussed in [3] and [5], do not provide a possibility of describing the entire curvilinear transition between two rectilinear directions by means of a single equation. However, the mathematical form thereof and the manner they have been located within a coordinate system, alongside the possibility of free formation of a curve through a change of parameter C described by dependence (10), determine their usability at grade line designing.

The sugested applications of curves (9) at grade line designing will be based upon the economically justified need to make the course of grade line compliant with the relief. The volumes of excavations and embankments will be roughly identified with the depths and heights thereof. A function of the following form will be assumed as the criterion of curves (9) being compliant with the relief:

$$F = \sum_{i=1}^{n} (y_i - Y_i)^2 \tag{13}$$

where Y_i — working ordinate of the i-th point a profile of abscissa X_i, y_i — ordinate of the i-th point of the curve calculated for $t=X_i/x_K$. We will also assume that the starting point of the curve, i.e. the one that corresponds to zero curvature, each time will be marked as P, whereas the end point — curvature $1/R$ — will be marked as K. Designing process calls for altitude ordinates H and kilometrage L of particular point of longitudinal profile to be known. Due to advisable continuity of changes of curvature to be maintained, it will be assumed that the curvature of particular curves at their contact point should amount to the same value.

Applying the mathematical describing form of multinomial transition curves and the position thereof within a coordinate system, it will be generally taken for granted that tangent at the point that corresponds to maximum curvature should assume a horizontal position. As it, is the defining manner of multinomial transition curves has some merits that make it possible to considerably simplify the designing process, due to which it is not purposeful to apply arcs whose tangent is situated differntly at the point. This will make it possible to avoid the labour consuming transformation of the ordinates of the curve to a direction compliant with that of the vertical line, which is advisable while approximating the lingitudinal profile with the use of the applied curve.

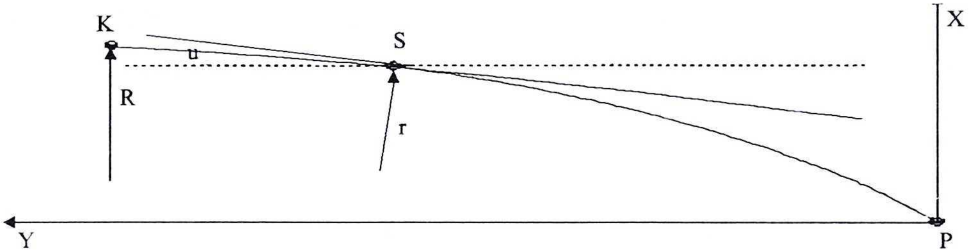

Fig. 3

It is only if it is necessary to bridge the designed grade line with the already existing segments of a given tangent direction (tg$u$) and certain value of curvature radius r at contact point ($r<R$), that the situation presented in Figure 3 should be taken into consideration. If a certain point S of the curve ($0<t<1$) turns out to be the contact point that is being looked for, the following dependences should come about:

$$y'_S = |\, \mathrm{tg} u \,| \tag{14}$$

$$y''_S = -\frac{1}{r} \tag{15}$$

Values y'_S and y''_S, on the basis of equation (11), being expressed as

$$y'=\frac{x_K}{R}[C+(1-3C)t^2-(1-C)t^3] \tag{16}$$

$$y''=\frac{1}{R}[2(1-3C)t-3(1-2C)t^2] \tag{17}$$

according to [4] the following will be obtained:

$$y_S+r\mathrm{tg}^2u\frac{[2(1-3C)t_S-3(1-C)t_S^2]\left[Ct_S+\frac{1-3C}{3}t_S^3-\frac{1-2C}{4}t_S^4\right]}{[C+(1-3C)t_S^2-(1-2C)t_S^3]^2}=0 \tag{18}$$

Since due to the choince of a certain point of the profile as the potential end point of a current segment of the grade line, the values of abscissa x_S ($x_S=L_P-L_S$) and ordinate y_S ($y_S=H_S-H_P$ for convex arcs or $y_S=H_P-H_S$ for concave arcs) have been determined, therefore after a certain value of C has been assumed, dependence (18) becomes a non-linear equation of unknown t_S. After it has been determined, the following will be calculated: the radius of curvature R at poin K of the curve (on the basis of (15) and (17)), abscissa x_K as $x_K=\frac{x_S}{t_S}$, and the value of $\mathrm{tg}u_0$ (on the basis of dependence (10)). This set of values characterizing a certain curve makes it possible to determine at length the appropriate value of expression (13) converted to the number of components. After all those calculations have been done for all the points taken into consideration as a potential starting point P of the current curve (identified with the end of the current segment), each time different values of C being possible taken into account, the point whose foregoing value of expression (13) assumes its minimum becomes the beginning of the curve.

In all the other situations, at horizontal position of the tangent at curve point (9) corresponding to maximum curvature, the following will come about:

— the value of curvature at the starting point of a segment equals to zero and an optional or given direction of the starting tangent,

— the value of curvature at the starting point equals to $1/R$, where R is the assumed minimum radius of the curvature.

According to the foregoing, it is necessary to appropriately change the position of the axes of the coordinate system within which the curve has been expressed, i.e. to mirrorreverse them (Fig. 4). This is also the working system that the coordinates of particular points of the longitudinal profile should be expressed in (Table 1), with particular regard to the kind of arc (convex/concave).

a)

b)

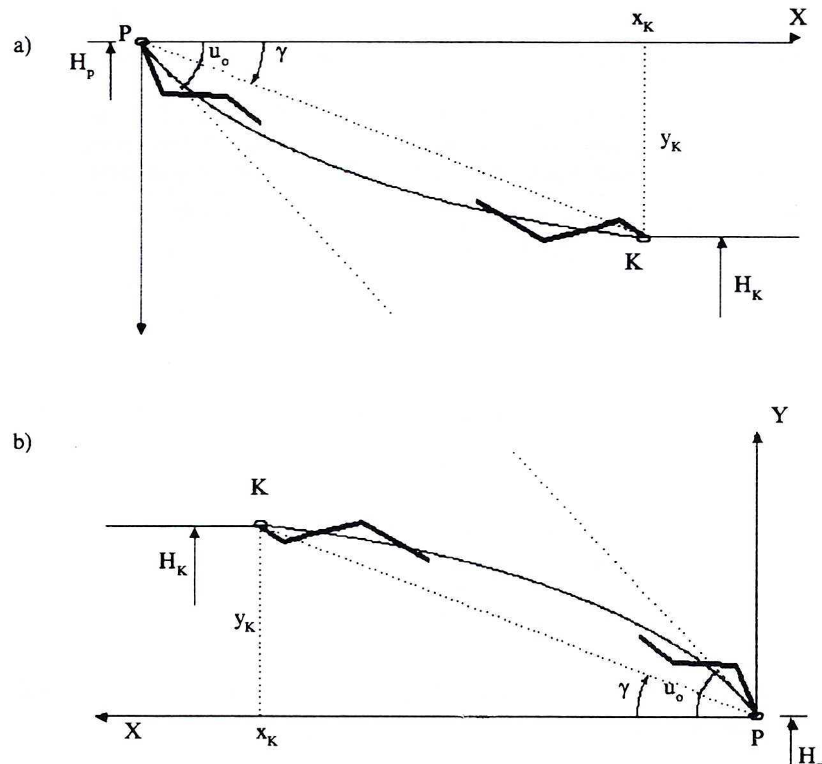

Fig. 4

Curve (9) to be closely matched with the relief will call for minimization of the value of function (13). This will be done according to C, which means that

$$\frac{\partial F}{\partial C}=0 \tag{19}$$

According to [4] the following will be obtained:

$$C=\frac{\sum_{i=1}^{n} Y_i\left(t_i-t_i^3+\frac{1}{2}t_i^4\right)-\frac{x_K^2}{R}\sum_{i=1}^{n}\left(t_i-t_i^3+\frac{1}{2}t_i^4\right)\left(\frac{1}{3}t_i^3-\frac{1}{4}t_i^4\right)}{\frac{x_K^2}{R}\sum_{i=1}^{n}\left(t_i-t_i^3+\frac{1}{2}t_i^4\right)^2} \tag{20}$$

values t_i being calculated as $t_i=X_i/x_K$, where X_i is defined by an appropriate formula included in Table 1.

Table 1

Working values of abscissa X_i and ordinate Y_i of a longitudinal profile

	Beginning of designed segment overlapping with beginning of curve	Beginning of designed segment overlapping beginning of curve
convex arc	$Y_i = H_i - H_P$ $X_i = L_i - L_P$	$Y_i = H_i - H_P$ $X_i = L_P - L_i$
concave arc	$Y_i = H_P - H_i$ $X_i = L_i - L_P$	$Y_i = H_P - H_i$ $X_i = H_P - H_i$

Marking expressions included in brackets by C_1 and C_2, dependence (20) will be written in its general form:

$$C = \frac{\sum_{i=1}^{n} Y_i C_1^{(i)} - \frac{x_K^2}{R} \sum_{i=1}^{n} C_1^{(i)} C_2^{(i)}}{\frac{x_K^2}{R} \sum_{i=1}^{n} {C_1^{(i)}}^2} \tag{21}$$

Applying (10) in equation (11), after appropriate transformations with respect to function (13) it can be found:

$$C = \frac{x_K \mathrm{tg} u_0 \sum_{i=1}^{n} {C_1^{(i)}}^2 {C_2^{(i)}}^2}{\sum_{i=1}^{n} Y_i C_1^{(i)} - x_K \mathrm{tg} u_0 \sum_{i=1}^{n} C_1^{(i)} C_2^{(i)}} \tag{22}$$

While designing it is necessary to be careful whether maximum gradients and minimum curvature radii do not exceed their admissible values; on the basis of (10) they might be defined as

$$\mathrm{tg} u_0 = \frac{C x_K}{R} \tag{23}$$

and

$$R = \frac{C x_K}{\mathrm{tg} u_0} \tag{24}$$

As it is possible that not for all points value C calculated according to (21) or (22) on the basis of the position of points of the profile, will be contained in the admissible interval, in accordance with [4] application of a simplified method should be taken into account.

Also in this case we will assume that tangent at a point corresponding to maximum curvature should be horizontal (Fig. 4). Ordinate y_K of the ending point of the curve (i.e. for $t=1$) implied by equation (11) will amount to

$$y_K=\frac{x_K^2}{R}\left(\frac{1}{2}C+\frac{1}{12}\right) \tag{25}$$

The foregoing equation implice value C corresponding to the assumed minimum radius of curvature R:

$$C=\frac{2R}{x_K^2}\left(y_K-\frac{1}{12}\frac{x_K^2}{R}\right) \tag{26}$$

Formula (10) being borne in mind, dependence (26) will be written as:

$$C=\frac{1}{6}\frac{x_K \mathrm{tg} u_0}{2y_K-x_K \mathrm{tg} u_0} \tag{27}$$

Designing of particular curves which make up the grade line follows the course presented in Table 2. Besides, the grade line being optimally matched with the relief

Table 2

Designing course of particular curves making up a grade line

Conditions		Proceedings		
Curvature value at starting point of designed segment	Position of starting tangent	Determination of parameter C on the basis of dependences within a method		Controls
		compact m.	simplified m.	
0	optional	(21)	(26)	• C not to exceed interval <1/3; 2/3> • admissible gradient not to be exceeded by $\mathrm{tg}u_0$ determined in accord with (23)
	given	(22)	(27)	• C not to exceed interval <1/3; 2/3> • admissible radius not to be exceeded by R determined in accord with (24)
$1/R$	horizontal	(21)	(26)	• C not to exceed interval <1/3; 2/3> • admissible gradient not to be exceeded $\mathrm{tg}u_0$ determined in accord with (23)

requires the compliance of the kind of arc (convex/concave) implied by the kind of the preceding one with the admissible kind of arc resulting from the position of the ending point. If the angle of inclination of the chord based upon extreme points is marked as $\gamma(+/-)$, at $\gamma \in (-90°; 90°)$, and clockwise calculation direction is negative, anticlocwise being position, then control follows the course given in Table 3.

Table 3

Dependence of the kind of arc on the kind of preceding arc and the sign of angle γ

	Preceding arc	Angle γ	Current arc
beginning of segment overlaps with beginning of curve	convex	$\gamma > 0$	design impossible
		$\gamma < 0$	concave if $\lvert \gamma \rvert < u_0$
	concave	$\gamma > 0$	convex if $\gamma < u_0$
		$\gamma < 0$	design impossible
beginning of segment overlaps with end of curve	convex	$\gamma > 0$	concave*
		$\gamma < 0$	convex
	concave	$\gamma > 0$	concave
		$\gamma < 0$	convex*

* design possible providing that radii are sufficiently big

After the foregoing calculations and control for all the points taken into consideration as a potential ending one have been done, such a point will be finally selected as the end of the current curve which the minimum value of expression (13) converted to the number of components corresponds to. After the grade line for the entire length of a profile has been designed, its coordinates should be converted to the supreme system within which a given longitudinal profile is given.

4. *Final remarks*

The family of multinomial transition curves described by equation (9) is a valuable designing tool of a grade line. Apart from the quality referring to the whole of transition curves, i.e. satisfaction of the dynamic condition [2] that secures the safety and comfort of travelling (certain values of temporary changes of centripetal acceleration not to be exceeded), the merit of curves (9) is the possibility to form the curvature by means of selection of parameter $C = \frac{R\,\mathrm{tg}\,u_0}{x_K}$ from interval $<1/3>; 2/3>$, which is remarkably significant from the viewpoint of the course of the grade line to be matched with the relief. The suggested designing methods with the use of curves (9), albeit they are relatively more labour consuming than the traditional ones in respect of the necessary calculations, are characterized by considerable simplicity, which adds to their practical usability.

REFERENCES

[1] Grabowski R. J., *Gładkie przejścia krzywoliniowe w drogach kołowych i kolejowych*, Zeszyty Naukowe AG-H, Geodezja No. 82, Kraków 1984.

[2] Herring H., *Die exakte Herleitung der kinematischen Gleichungen fuer Geschwindigkeit*, Beschleunigung und Ruck Strasse und Autobahn 2/1981, Bonn.

[3] Kobryń A., *Design of the Grade Line of a Route by Means of General Transition Curves*, Geodezja i Kartografia, v. XLVI, fasc. 1, p. 83 – 101 (1997).

[4] Kobryń A., *Geometryczne kształtowanie niwelety tras drogowych*, Wydawnictwo Politechniki Białostockiej, Rozprawy No. 60, Białystok 1999.

[5] Kobryń A., *Zmodyfikowane ogólne wielomianowe krzywe przejściowe w projektowaniu niwelet*, Geodezja i Karografia (in print).

Received March 8.2000
Accepted May 15.2000

Andrzej Kobryń

Wielomianowe krzywe przejściowe do projektowania niwelety tras drogowych

Streszczenie

W pracy zaprezentowano rodzinę tzw. wielomianowych krzywych przejściowych. Istnieje możliwość wygodnego kształtowania krzywizny tych krzywych przez odpowiedni dobór minimalnego promienia krzywizny R, pochylenia stycznej w punkcie początkowym $\mathrm{tg}u_0$ oraz odciętej punktu końcowego x_K, co ma wpływ na wartość podstawowego parametru projektowego $C=\frac{R\mathrm{tg}u_0}{x_K}$. Możliwość doboru wartości C z przedziału $<1/3, 2/3>$ oraz oryginalny sposób usytuowania tych krzywych w układzie współrzędnych prostokątnych czyni je bardzo przydatnymi np. do projektowania niwelety tras drogowych w celu osiągnięcia optymalnego dopasowania do rzeźby terenu. Uwzględniając powyższe, w pracy przedstawiono odpowiednie metody projektowania niwelety tras drogowych przy użyciu tych krzywych.

Анджей Кобрынь

Многочленные переходные кривые для проектирования проектной линий дорожных трасс

Резюме

В работе представлено семейство переходных кривых. Существует возможность удобного формирования кривизны этих кривых через соответственный подбор минимального радиуса кривизны R, наклона касательной в начальной точке $\mathrm{tg}u_0$, а также абсциссы конечной точки x_K, что влияет на величину основного проектного параметра $C=(R\mathrm{tg}u_0)/x_K$. Возможность подбора величины C в пределах $<1/3; 2/3>$, а также оригинальный способ расположения этих кривых в системе прямоугольных координат, делает эти кривые очень пригодными нп. для проектирования проектной линии дорожных трасс с целью достижения оптимального подбора к рельефу местности. Учитывая вышеуказанное, в работе предсталены соответственные методы проектирования проектной линии дорожных трасс с использованием этих кривых.