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Existence of optimal control for multi-order fractional optimal control problems

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In this article we focus on optimal control problems involving a nonlinear fractional control system of different orders with Caputo derivatives, associated to a Lagrange cost functional. Based on a lower closure theorem for orientor fields combined with Filippov's approach, we derive an existence result for at least one optimal solution for such a problem.

Key words: existence of optimal control, multi-order fractional control system, lower closure theorem

1. Introduction

In our paper, we consider the following optimal control problem

$$\text{minimize} \quad H(x(\cdot), u(\cdot)) = \int_0^T f_0(t, x(t), u(t))\,\mathrm{d}t, \tag{1}$$

subject to

$$\begin{cases} \left({}^C D^{\alpha_1}_{0+} x_1\right)(t) = f_1(t, x(t), u(t)) & \\ \vdots & t \in [0,T] \;\; a.e. \\ \left({}^C D^{\alpha_n}_{0+} x_n\right)(t) = f_n(t, x(t), u(t)), & \\ x_1(0) = x_{10}, \quad \ldots, \quad x_n(0) = x_{n0}, & \\ u(t) \in M \subset \mathbb{R}^m, & t \in [0,T] \;\; a.e., \end{cases} \tag{2}$$

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where $x = (x_1, \ldots, x_n)$, $x_i : [0, T] \to \mathbb{R}^{r_i}$, $f_i : [0, T] \times \mathbb{R}^{r_1} \times \cdots \times \mathbb{R}^{r_n} \times M \to \mathbb{R}^{r_i}$, $i = 1, \ldots, n$, $f_0 : [0, T] \times \mathbb{R}^{r_1} \times \cdots \times \mathbb{R}^{r_n} \times M \to \mathbb{R}$ (here ${}^C D^{\alpha}_{0+}$ denotes the left-sided fractional derivative operator of order $\alpha \in (0, 1]$ in the Caputo sense).

The linear control system (2) was introduced by Kaczorek in paper [13]. It can be applied to study the linear electrical circuits composed of resistors, supercondensators, coils and voltage sources. In mentioned paper [13], a positivity of system (2) has been investigated. In paper [27], the necessary and sufficient conditions for state controllability and state observability of such a system (in a case of two orders) have been included. Since fractional derivatives are non-local operators therefore fractional-order models own better description memory and hereditary properties of various processes than classical models with integer order derivatives [1, 5, 8, 25, 26]. Recently, problem (1)–(2) (in a case $n = 2$) has been used to describe a nonlinear fractional Cucker-Smale optimal control problem under the interplay of memory effect (more details can be found in [2, 3]).

The aim of this paper is to derive conditions for the existence of an optimal solution to problem (1)–(2). The result of such a type in a case $n = 1$, where control system (2) is linear, has been obtained in [17]. The proof of this fact is based on an analogous existence result for the optimal control problem involving the Riemann-Liouville derivative obtained in [16]. In cited paper [16], existence of optimal solutions has been proved due to a theorem on the weak lower semicontinuity of integral functionals. In [11], a nonlinear control system ($n = 1$) with the Riemann-Liouville derivative and cost (1) has been studied. The existence result has been obtained there under convexity assumption of the so called extended velocities set (this is a stronger condition than (H_3)) based on the implicit function theorem for multivalued mappings. In [21], the Lagrange fractional optimal control problems with the Caputo distributed–order fractional derivatives have been considered. The sufficient optimality conditions of a Mangasarian-type have been obtained there. In [28, 29], authors proved the existence of optimal pairs for the Lagrange problem, described by semilinear fractional differential [29] and integro-differential [28] systems in Banach spaces with the help of the Filippov and Mazur theorems. In [22], a Bolza type fractional order optimal control problems in which the dynamic control system involves integer and fractional order derivatives have been studied. Using an appriopriate convexity assumptions sufficient optimality conditions have been proved.

In order to prove the main result of this paper, the lower closure theorem for orientor fields ([7, Theorem 10.7.i]) and a measurable selection theorem of Filippov type ([23, Theorem 2J]) have been used. A such approach was also used in [14] and [9]. In [14] a control system is described by a partial nonlinear differential equation with the fractional Dirichlet-Laplacian, while in [9] a Bolza problem described by a nonlinear integro-differential system of Volterra type is considered. The necessary optimality conditions for problem (1)–(2) by using a smooth-convex extremum principle have been derived in [2].

In the first part of this work, we study problem (2) with zero initial conditions. We formulate and prove a theorem on the existence of optimal solutions. In the second part, we obtain an analogous result for system (2) with nonzero initial conditions. To the best knowledge of the autor, all results proved in this work have not been obtained yet.

The paper is organized as follows. Section 2 is devoted some basic definitions and facts concerning fractional calculus. The main results of the work (Theorems 1 and 2) are stated in Sections 3 and 4. A theoretical illustrative example is presented in Section 5 as well as conclusions – in Section 6. Finally, Appendix 7 contains some necessary facts concerning multifunctions, as well as a some version of a fractional multi–term Gronwall lemma.

2. Preliminaries

This section is devoted to some necessary notions and properties concerning fractional derivatives and integrals (for more details we refer the readers to monographs [19, 24]).

Let $[a, b] \subset \mathbb{R}$ be any bounded interval.

For $f \in L^1([a, b], \mathbb{R}^n)$ we define the left-sided and the right-sided Riemann-Liouville integral of the function f of order $\alpha > 0$ as follows:

$$(I^{\alpha}_{a+} f)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} \mathrm{d}\tau, \quad t \in [a, b] \ a.e.,$$

$$(I^{\alpha}_{b-} f)(t) := \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(\tau)}{(\tau-t)^{1-\alpha}} \mathrm{d}\tau, \quad t \in [a, b] \ a.e.,$$

respectively.

Let $1 \leqslant p < \infty$. By $I^{\alpha}_{a+}(L^p([a, b], \mathbb{R}^n))$ (briefly $I^{\alpha}_{a+}(L^p)$) we denote the set of all functions $f : [a, b] \to \mathbb{R}^n$ that have the integral representation

$$f(t) = (I^{\alpha}_{a+} g)(t), \quad t \in [a, b] \ a.e.,$$

where $g \in L^p([a, b], \mathbb{R}^n)$. We identify functions belonging to the space $I^{\alpha}_{a+}(L^p)$ and equal almost everywhere on $[a, b]$.

Now, let $\alpha \in (0, 1]$ and $f \in L^1([a, b], \mathbb{R}^n)$. We say that the function f possesses the left-sided Riemann-Liouville derivative $D^{\alpha}_{a+} f$ of order α if the function $I^{1-\alpha}_{a+} f$ (f in the case of $\alpha = 1$) is absolutely continuous on $[a, b]$.

In such a case

$$(D^{\alpha}_{a+}f)(t) := \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(I^{1-\alpha}_{a+}f\right)(t) & \text{if } \alpha \in (0,1) \\ \dfrac{\mathrm{d}}{\mathrm{d}t}f(t) & \text{if } \alpha = 1 \end{cases}, \qquad t \in [a,b] \ \ a.e.$$

Similarly, we define the right-sided Riemann-Liouville derivative $D^{\alpha}_{b-}f$ of order $\alpha \in (0,1]$. More precisely,

$$(D^{\alpha}_{b-}f)(t) := \begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}t}\left(I^{1-\alpha}_{b-}f\right)(t) & \text{if } \alpha \in (0,1) \\ -\dfrac{\mathrm{d}}{\mathrm{d}t}f(t) & \text{if } \alpha = 1 \end{cases}, \quad t \in [a,b] \ \ a.e.,$$

provided that the function $I^{1-\alpha}_{b-}f$ (f in the case of $\alpha = 1$) is absolutely continuous on $[a,b]$. The set of all functions possessing the left-sided (the right-sided) Riemann-Liouville derivative is denoted by AC^{α}_{a+} (AC^{α}_{b-}).

We have the following useful property

Proposition 1 *Let $\alpha \in (0,1]$ and $1 \leqslant p < \infty$. Then $I^{\alpha}_{a+}(L^p)$ with the norm*

$$\|f\|_{I^{\alpha}_{a+}(L^p)} := \|D^{\alpha}_{a+}f\|_{L^p}$$

is a Banach space. Furthermore, if $p > 1$ and $0 < \dfrac{1}{p} < \alpha \leqslant 1$ then the compact embedding $I^{\alpha}_{a+}(L^p) \hookrightarrow C_a([a,b],\mathbb{R}^n)$ holds (here $C_a([a,b],\mathbb{R}^n)$ denotes the set of all continuous functions $f : [a,b] \longrightarrow \mathbb{R}^n$ such that $f(a) = 0$).

Remark 1 *The proof of completness of $I^{\alpha}_{a+}(L^p)$ is analogous to the proof of [18, Theorem 2.5]. The second part of the above proposition has been obtained in [2, Proposition 1].*

We say that $f \in C([a,b],\mathbb{R}^n)$ has the left-sided Caputo derivative ${}^{C}D^{\alpha}_{a+}f$ of order α on the interval $[a,b]$ if the function $f(\cdot) - f(a) \in AC^{\alpha}_{a+}$. In such a case

$$\left({}^{C}D^{\alpha}_{a+}f\right)(t) := D^{\alpha}_{a+}(f(\cdot) - f(a))(t), \ \ t \in [a,b] \ \ a.e.$$

Remark 2 *It is clear that for $\alpha = 1$ ${}^{C}D^{\alpha}_{a+}f = \dfrac{\mathrm{d}}{\mathrm{d}t}f$. Moreover, for $\alpha \in (0,1)$, if both derivatives $D^{\alpha}_{a+}f$ and ${}^{C}D^{\alpha}_{a+}f$ exist and $f(a) = 0$ then they coincide.*

Let $1 \leqslant \dfrac{1}{\alpha} < p < \infty$ and define the following set of functions:

$$
{}_{C}AC^{\alpha,p}_{a+} := \Big\{ f : [a,b] \to \mathbb{R}^n : \quad f(t) = c_a + (I^{\alpha}_{a+}\varphi)(t), \quad t \in [a,b] \ a.e., \\
c_a \in \mathbb{R}^n, \ \varphi \in L^p([a,b],\mathbb{R}^n) \Big\}.
$$

[4, Property 4] guarantees that each function $f \in {}_{C}AC^{\alpha,p}_{a+}$ is continuous on $[a,b]$ and $f(a) = c_a$. Consequently, f possesses a Caputo derivative ${}^{C}D^{\alpha}_{a+}f$ and (cf. [19, Lemma 2.4])

$$
\left({}^{C}D^{\alpha}_{a+}f\right)(t) = D^{\alpha}_{a+}(f - f(a))(t) = (D^{\alpha}_{a+}I^{\alpha}_{a+}\varphi)(t) = \varphi(t), \quad t \in [a,b] \ a.e.
$$

Furthermore, ${}_{C}AC^{1,p}_{a+} = AC^p$, where

$$
AC^p = AC^p([a,b],\mathbb{R}^n) = \{ f \in AC([a,b],\mathbb{R}^n) : \quad \dot{f} \in L^p([a,b],\mathbb{R}^n) \}
$$

and ${}_{C}AC^{\alpha,p}_{a+} = I^{\alpha}_{a+}(L^p)$ if and only if $f(a) = 0$.

Using [19, Lemmas 2.4 and 2.5] and Remark 2 we immediately obtain the following composition properties

Proposition 2 *Let* $0 < \alpha \leqslant 1$ *and* $1 \leqslant \frac{1}{\alpha} < p < \infty$.

(a) If $f \in L^p([a,b],\mathbb{R}^n)$ *then*

$$
\left({}^{C}D^{\alpha}_{a+}I^{\alpha}_{a+}f\right)(t) = f(t), \quad t \in [a,b] \ a.e.,
$$

(b) if $f \in {}_{C}AC^{\alpha,p}_{a+}$ *then*

$$
\left(I^{\alpha}_{a+}\,{}^{C}D^{\alpha}_{a+}f\right)(t) = f(t) - f(a), \quad t \in [a,b] \ a.e.
$$

Now, let $0 < \alpha_i \leqslant 1$, $i = 1,\ldots,n$ and $\alpha = (\alpha_1,\ldots,\alpha_n)$. We define a space $\mathbb{I}^{\alpha}_{0+}(L^p)([0,T],\mathbb{R}^{r_1} \times \cdots \times \mathbb{R}^{r_n})$ (shortly $\mathbb{I}^{\alpha}_{0+}(L^p)$) as follows:

$$
\mathbb{I}^{\alpha}_{0+}(L^p) := I^{\alpha_1}_{0+}\big(L^p([0,T],\mathbb{R}^{r_1})\big) \times \cdots \times I^{\alpha_n}_{0+}\big(L^p([0,T],\mathbb{R}^{r_n})\big).
$$

The space $\mathbb{I}^{\alpha}_{0+}(L^p)$ with the norm

$$
\|z\|_{\mathbb{I}^{\alpha}_{0+}(L^p)} = \left(\sum_{i=1}^{n} \|z_i\|^p_{I^{\alpha_i}_{0+}(L^p)} \right)^{\frac{1}{p}},
$$

where $z = (z_1,\ldots,z_n)$, is a Banach space as a Cartesian product of Banach spaces $I^{\alpha_i}_{0+}(L^p)$, $i = 1,\ldots,n$.

3. Optimal control problem with zero initial conditions

Let us consider the following optimal control problem:

$$\text{minimize} \quad J(y(\cdot),u(\cdot)) = \int_0^T g_0(t,y(t),u(t))\,\mathrm{d}t, \tag{3}$$

subject to

$$\begin{cases} \left({}^C D_{0+}^{\alpha_1} y_1\right)(t) = g_1(t,y(t),u(t)) & \\ \vdots & t \in [0,T] \ a.e. \\ \left({}^C D_{0+}^{\alpha_n} y_n\right)(t) = g_n(t,y(t),u(t)), & \\ y(0) = 0, & \\ u(t) \in M \subset \mathbb{R}^m, & t \in [0,T] \ a.e., \end{cases} \tag{4}$$

where $y = (y_1,\ldots,y_n)$, $y_i : [0,T] \to \mathbb{R}^{r_i}$, $g_i : [0,T]\times\mathbb{R}^{r_1}\times\cdots\times\mathbb{R}^{r_n}\times M \to \mathbb{R}^{r_i}$, $i = 1,\ldots,n$, $g_0 : [0,T]\times\mathbb{R}^{r_1}\times\cdots\times\mathbb{R}^{r_n}\times M \to \mathbb{R}$.

Let us define the following set of controls:

$$\mathcal{U}_M := \left\{u\colon [0,T] \to \mathbb{R}^m - \text{measurable on } [0,T]; \quad u(t) \in M,\ t \in [0,T] \ a.e.\right\}.$$

Now, let $0 < \dfrac{1}{p} < \alpha_i \leqslant 1$, $i = 1,\ldots,n$. By a solution of control system (4), corresponding to any fixed control $u \in \mathcal{U}_M$, we mean a function $y \in \mathbb{I}_{0+}^{\alpha}(L^p)$.

From Proposition 1 it follows that if $0 < \dfrac{1}{p} < \alpha_i \leqslant 1$, $i = 1,\ldots,n$ then the function $y \in \mathbb{I}_{0+}^{\alpha}(L^p)$ satisfies the condition $y(0) = 0$.

By an admissible control (strategy) we mean an element $u \in \mathcal{U}_M$. A function $y \in \mathbb{I}_{0+}^{\alpha}(L^p)$ is called an admissible trajectory for system (4) if it is a solution to (4), corresponding to an admissible control $u \in \mathcal{U}_M$. Any couple $(y,u) \in \mathbb{I}_{0+}^{\alpha}(L^p)\times\mathcal{U}_M$ is called admissible for system (4) if y is an admissible trajectory, corresponding to an admissible control $u \in \mathcal{U}_M$.

In what follows, we assume that system (4) is controllable in the sense that at least one admissible pair exists.

In the first part of this section we shall prove a some useful property of admissible trajectories.

To this end, the following assumption is required:

(H_1) functions $g_1, \ldots, g_n$ are measurable on $[0, T]$, continuous on $\mathbb{R}^{r_1} \times \cdots \times \mathbb{R}^{r_n} \times \mathbb{R}^m$ and satisfy the following growth conditions: there exist $A_i > 0$, $a_i \in L^p([0, T], \mathbb{R}_0^+)$ such that

$$|g_i(t, y_1, \ldots, y_n, u)|_{\mathbb{R}^{r_i}} \leqslant A_i(|y_1|_{\mathbb{R}^{r_1}} + \cdots + |y_n|_{\mathbb{R}^{r_n}}) + a_i(t), \quad i = 1, \ldots, n \tag{5}$$

for a.e. $t \in [0, T]$ and all $(y_1, \ldots, y_n) \in \mathbb{R}^{r_1 + \cdots + r_n}$, $u \in M$ (here $|\cdot|_{\mathbb{R}^n}$ denotes an Euclidean norm in $\mathbb{R}^n$),

Lemma 1 *Let* $0 < \dfrac{1}{p} < \alpha_i \leqslant 1$, $i = 1, \ldots, n$. *If assumption* (H_1) *is satisfied then the set of admissible trajectories is bounded on* $\mathbb{I}^{\alpha}_{0+}(L^p)$.

Proof. Let us fix any control $u \in \mathcal{U}_M$ and assume that $y^u = (y_1^u, \ldots, y_n^u)$ is an admissible trajectory for system (4), corresponding to u. Without loss of generality we can assume that $0 < \alpha_1 < \cdots < \alpha_n \leqslant 1$. Then, using growth conditions (5) and Proposition 2 (b), we assert that

$$\begin{aligned}
\left|y_1^u(t)\right|_{\mathbb{R}^{r_1}} &\leqslant I_{0+}^{\alpha_1} \left|g_1(t, y_1^u(t), \ldots, y_n^u(t), u(t)\right|_{\mathbb{R}^{r_1}} \\
&\leqslant A_1 I_{0+}^{\alpha_1}\left(\left|y_1^u(t)\right|_{\mathbb{R}^{r_1}} + \cdots + \left|y_n^u(t)\right|_{\mathbb{R}^{r_n}}\right) + \left(I_{0+}^{\alpha_1} a_1\right)(t), \\
&\vdots \\
\left|y_n^u(t)\right|_{\mathbb{R}^{r_n}} &\leqslant I_{0+}^{\alpha_1} \left|g_n(t, y_1^u(t), \ldots, y_n^u(t), u(t)\right|_{\mathbb{R}^{r_n}} \\
&\leqslant A_n I_{0+}^{\alpha_n}\left(\left|y_1^u(t)\right|_{\mathbb{R}^{r_1}} + \cdots + \left|y_n^u(t)\right|_{\mathbb{R}^{r_n}}\right) + \left(I_{0+}^{\alpha_n} a_n\right)(t),
\end{aligned}$$

for all $t \in [0, T]$. Hence,

$$\begin{aligned}
&\left|y_1^u(t)\right|_{\mathbb{R}^{r_1}} + \cdots + |y_n^u(t)|_{\mathbb{R}^{r_n}} \\
&\qquad \leqslant \sum_{i=1}^{n} A_i I_{0+}^{\alpha_i}\left(\left|y_1^u(t)\right|_{\mathbb{R}^{r_1}} + \cdots + \left|y_n^u(t)\right|_{\mathbb{R}^{r_n}}\right) + \sum_{i=1}^{n}\left(I_{0+}^{\alpha_i} a_i\right)(t) \\
&\qquad \leqslant \sum_{i=1}^{n} A_i I_{0+}^{\alpha_i}\left(\left|y_1^u(t)\right|_{\mathbb{R}^{r_1}} + \cdots + \left|y_n^u(t)\right|_{\mathbb{R}^{r_n}}\right) + A_0, \quad t \in [0, T],
\end{aligned}$$

where $A_0 = \max_{t \in [0,T]} I_{0+}^{\alpha_1} a_1(t) + \cdots + \max_{t \in [0,T]} I_{0+}^{\alpha_n} a_n(t)$ (due to [4, Property 4] functions $I_{0+}^{\alpha_i} a_i$, $i = 1, \ldots, n$ are continuous on $[0, T]$). From Corollary 1 it follows that there exists $C > 0$ such that

$$|y_1^u(t)|_{\mathbb{R}^{r_1}} + \cdots + |y_n^u(t)|_{\mathbb{R}^{r_n}} \leqslant C, \quad t \in [0, T],$$

so

$$\begin{aligned}\left|D_{0+}^{\alpha_i} y_i^u(t)\right|_{\mathbb{R}^{r_i}} &= \left|g_i(t, y_1^u(t), \ldots, y_n^u(t), u(t)\right|_{\mathbb{R}^{r_i}} \\ &\leqslant A_i\left(\left|y_1^u\right|_{\mathbb{R}^{r_1}} + \cdots + \left|y_n^u\right|_{\mathbb{R}^{r_n}}\right) + a_i(t) \\ &\leqslant A_i C + a_i(t), \quad t \in [0, T], \quad i = 1, \ldots, n.\end{aligned}$$

Finally,

$$\begin{aligned}\|y^u\|_{\mathbb{I}_{0+}^{\alpha}(L^p)} &= \left(\|D_{0+}^{\alpha_1} y_1^u\|_{L^p}^p + \cdots + \|D_{0+}^{\alpha_n} y_n^u\|_{L^p}^p\right)^{\frac{1}{p}} \\ &= \left(\int_0^T \left(\left|(D_{0+}^{\alpha_1} y_1^u)(t)\right|_{\mathbb{R}^{r_1}}^p + \cdots + \left|(D_{0+}^{\alpha_n} y_n^u)(t)\right|_{\mathbb{R}^{r_n}}^p\right) \mathrm{d}t\right)^{\frac{1}{p}} \\ &\leqslant 2^{\frac{p-1}{p}}\left(\|a_1\|_{L^p}^p + \cdots + \|a_n\|_{L^p}^p + (A_1^p + \cdots + A_n^p) C^p T\right)^{\frac{1}{p}}.\end{aligned}$$

The proof is completed. □

3.1. Existence of an optimal solution

In this section we formulate and prove one of the main results of this paper, namely a theorem on the existence of optimal solutions to problem (3)–(4).

We assume that:

(H_2) the function g_0 is measurable on $[0, T]$ and continuous on $\mathbb{R}^{r_1} \times \cdots \times \mathbb{R}^{r_n} \times \mathbb{R}^m$,

(H_3) the sets

$$Q(t, y) := \Big\{(\mu_0, \mu) \in \mathbb{R} \times \mathbb{R}^{r_1} \times \ldots \times \mathbb{R}^{r_n};\ \exists_{u \in M}\ \ \mu_0 \geqslant g_0(t, y_1, \ldots, y_n, u) \\ \mu = g(t, y_1, \ldots, y_n, u)\Big\} \tag{6}$$

for a.e. $t \in [0, T]$ and all $y \in \mathbb{R}^{r_1} \times \cdots \times \mathbb{R}^{r_n}$, where $\mu = (\mu_1, \ldots, \mu_n)$, $g = (g_1, \ldots, g_n)$, are convex.

We start with the following useful result.

Proposition 3 *If assumptions (H_1)–(H_3) are satisfied and the set M is compact then the multifunction $Q(t, \cdot) : \mathbb{R}^{r_1} \times \cdots \times \mathbb{R}^{r_n} \ni y \longrightarrow Q(t, y) \in \mathbb{R} \times \mathbb{R}^{r_1} \times \cdots \times \mathbb{R}^{r_n}$ given by (6) has property (K)*[1].

[1] A definition of property (K) can be found in Appendix.

Proof. The proof is analogous to the proof of [14, Proposition 3]. □

In what follows, we assume that for any admissible pair (y, u) integral (3) is finite. The set of all such pairs will be denoted by $\mathcal{A}$. Since control system (4) is controllable, therefore $\mathcal{A} \neq \emptyset$.

A couple $(y^*, u^*) \in \mathcal{A}$ is called an optimal solution to problem (3)–(4) if it minimizes cost (3) among all couples $(y, u) \in \mathcal{A}$.

In order to prove the main result of this section the following additional hypothesis is required:

(H_4) there exists a function $\lambda \in L^1([0,T],\mathbb{R})$ such that for any pair $(y,u) \in \mathcal{A}$

$$g_0(t, y_1(t), \ldots, y_n(t), u(t)) \geqslant \lambda(t), \quad t \in [0,T] \ a.e.$$

We have

Theorem 1 *Assume that* $\alpha = (\alpha_1, \ldots, \alpha_n)$, $0 < \dfrac{1}{p} < \alpha_i \leqslant 1$, $i = 1, \ldots, n$ *and* M *is a compact set. If assumptions* $(H_1) - (H_4)$ *are satisfied then problem (3)–(4) has an optimal solution* $(y^*, u^*) \in \mathbb{I}^{\alpha,p}_{0+}(L^p) \times \mathcal{U}_M$.

Proof. Let us denote

$$s := \inf_{(y,u)\in\mathcal{A}} J(y,u).$$

Assumption (H_4) guarantees that s is finite. Let $\{(y^l, u^l)\}_{l\in\mathbb{N}} \subset \mathcal{A}$ be a minimizing sequence of J, i.e.

$$\lim_{l\to\infty} J(y^l, u^l) = s.$$

From Lemma 1 and [6, Theorem 3.18] it follows that the sequence of trajectories $(y^l)_{l\in\mathbb{N}} = \{(y^l_1, \ldots, y^l_n)\}_{l\in\mathbb{N}} \subset \mathbb{I}^{\alpha,p}_{0+}(L^p)$ contains a subsequence (still denoted by $(y^l)_{l\in\mathbb{N}}$) weakly convergent in $\mathbb{I}^{\alpha,p}_{0+}(L^p)$ to a some function $y^* = (y^*_1, \ldots, y^*_n) \in \mathbb{I}^{\alpha,p}_{0+}(L^p)$. Let us denote:

$$G = [0,T], \quad A(t) = \mathbb{R}^{r_1} \times \cdots \times \mathbb{R}^{r_n}, \quad A = [0,T] \times \mathbb{R}^{r_1} \times \cdots \times \mathbb{R}^{r_n},$$

$$\eta^l(t) = g_0\left(t, y^l_1(t), \ldots, y^l_n(t), u^l(t)\right),$$

$$\xi^l(t) = \left(\xi^l_1(t), \ldots, \xi^l_n(t)\right) = \left(({}^C D^{\alpha_1}_{0+} y^l_1)(t), \ldots, ({}^C D^{\alpha_n}_{0+} y^l_n)(t)\right) = ({}^C D^{\alpha}_{0+} y^l)(t),$$

$$\xi(t) = (\xi_1(t), \ldots, \xi_n(t)) = \left(({}^C D^{\alpha_1}_{0+} y^*_1)(t), \ldots, ({}^C D^{\alpha_n}_{0+} y^*_n)(t)\right) = ({}^C D^{\alpha}_{0+} y^*)(t),$$

$$\lambda^l(t) = \lambda(t), \quad y(t) = y^*(t), \quad \widetilde{Q}(t,y) = Q(t,y)$$

for a.e. $t \in [0,T]$ and all $l \in \mathbb{N}$.

Of course, functions $y, y^l, \lambda^l, \lambda$, where $l \in \mathbb{N}$, are measurable on $[0,T]$, $\xi, \xi^l \in L^p([0,T],\mathbb{R}^{r_1}) \times \cdots \times L^p([0,T],\mathbb{R}^{r_n}) \subset L^1([0,T],\mathbb{R}^{r_1}) \times \cdots \times L^1([0,T],\mathbb{R}^{r_n})$ and $\eta^l \in L^1([0,T],\mathbb{R})$. The weak convergence $\xi^l \rightharpoonup \xi$ in L^1 is a consequence of linearity and continuity of the mapping:

$$\mathbb{I}^{\alpha}_{0+}(L^p) \ni y \longrightarrow {}^C D^{\alpha}_{0+} y \in L^p([0,T],\mathbb{R}^{r_1}) \times \cdots \times L^p([0,T],\mathbb{R}^{r_n}).$$

Hence and from compactness of the operator $I^{\alpha_i}_{0+}$ (cf. [20]) we conclude that $y^l \longrightarrow y^*$ strongly in L^1, so also in measure on $[0,T]$. Furthermore, it is clear that for all $t \in [0,T]$ the set $A(t)$ is closed, the sets $\widetilde{Q}(t,y)$ are convex (assumption (H_3)), have property (K) with respect to $y \in \mathbb{R}^{r_1} \times \cdots \times \mathbb{R}^{r_n}$ (Proposition 3), so are also closed (Remark 3). We have also

$$y^l \in A(t), \quad (\eta^l, \xi^l) \in \widetilde{Q}(t, y^l), \quad t \in [0.T] \ a.e., \ l \in \mathbb{N},$$

$$\liminf_{l\to\infty} \int_0^T \eta^l(t)\,\mathrm{d}t = \lim_{l\to\infty} J(y^l, u^l) = s \in (-\infty, +\infty),$$

$$\eta^l(t) = g_0(t, y_1^l(t), \ldots, y_n^l(t), u^l(t)) \geqslant \lambda(t) = \lambda^l(t), \quad t \in [0.T] \ a.e., \ l \in \mathbb{N}$$

and

$$\lambda^l = \lambda \rightharpoonup \lambda \ weakly \ in \ L^1([0,T],\mathbb{R}).$$

So, all assumptions of a Lower Closure Theorem (Appendix, Theorem 5) are satisfied. Consequently, there exists a function $\eta \in L^1([0,T],\mathbb{R})$ such that

$$(\eta(t), \xi(t) \in \widetilde{Q}(t, y^*(t)), \quad t \in [0.T] \ a.e.$$

and

$$\int_0^T \eta(t)\,\mathrm{d}t \leqslant s. \tag{7}$$

Now, let us consider the multifunction $\Phi : [0,T] \ni t \longrightarrow \Phi(t) \subset \mathbb{R}^m$ given by

$$\Phi(t) = \left\{ u \in M : \ \eta(t) \geqslant g_0(t, y_1^*(t), \ldots, y_n^*(t), u), \ \ \xi(t) = g(t, y_1^*(t), \ldots, y_n^*(t), u) \right\}.$$

From Theorem 3 it follows that Φ is a closed-valued, measurable multifunction and there exists a measurable function $u^* : [0,T] \longrightarrow \mathbb{R}^m$ such that $u^*(t) \in \Phi(t)$ for a.e. $t \in [0,T]$. This means that

$$\eta(t) \geqslant g_0(t, y_1^*(t), \ldots, y_n^*(t), u^*(t)), \quad t \in [0,T] \ a.e. \tag{8}$$

and

$$\begin{cases} \left({}^C D_{0+}^{\alpha_1} y_1^*\right)(t) = g_1(t, y_1^*(t), \ldots, y_n^*(t), u^*(t)) \\ \quad \vdots \\ \left({}^C D_{0+}^{\alpha_n} y_n^*\right)(t) = g_n(t, y_1^*(t), \ldots, y_n^*(t), u^*(t)), \quad t \in [0, T] \ a.e. \\ y_1^*(0) = 0, \quad \ldots, \quad y_n^*(0) = 0, \\ u^*(t) \in M, \qquad\qquad t \in [0, T] \ a.e. \end{cases}$$

(equalities $y_i^*(0) = 0$, $i = 1, \ldots, n$ follow from the fact that $I_{0+}^{\alpha_i}(L^p) \subset C_0$, $i = 1, \ldots, n$). Consequently, the pair (y^*, u^*) satisfies constraints (4). Finally, using (7) and (8), we assert that

$$s \leqslant \int_0^T g_0(t, y_1^*(t), \ldots, y_n^*(t), u^*(t))\,\mathrm{d}t \leqslant \int_0^T \eta(t)\,\mathrm{d}t \leqslant s,$$

so (y^*, u^*) is an optimal solution to problem (3)–(4).

The proof is completed. □

4. Optimal control problem with nonzero initial conditions

In this section we shall work on problem (1)–(2).

Let $0 < \dfrac{1}{p} < \alpha_i \leqslant 1$, $i = 1, \ldots, n$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$. By a solution of control system (2), corresponding to any fixed control $u \in \mathcal{U}_M$, we mean a function $x \in {}_C\mathbb{AC}_{0+}^{\alpha,p}$, where

$${}_C\mathbb{AC}_{0+}^{\alpha,p} := {}_C AC_{0+}^{\alpha_1,p}([0, T], \mathbb{R}^{r_1}) \times \cdots \times {}_C AC_{0+}^{\alpha_n,p}([0, T], \mathbb{R}^{r_n}).$$

A function $x \in {}_C\mathbb{AC}_{0+}^{\alpha,p}$ is called an admissible trajectory for system (2) if it is a solution to (2), corresponding to an admissible control $u \in \mathcal{U}_M$. Any couple $(x, u) \in {}_C\mathbb{AC}_{0+}^{\alpha,p} \times \mathcal{U}_M$ is called admissible for system (2) if x is an admissible trajectory, corresponding to an admissible control $u \in \mathcal{U}_M$. By $\hat{\mathcal{A}}$ we shall denote the set of all admissible pairs (x, u) for which integral (1) is finite. A pair $(x^*, u^*) \in \hat{\mathcal{A}}$ is called an optimal solution to problem (1)–(2) if it minimizes cost (1) among all pairs $(x, u) \in \hat{\mathcal{A}}$.

Assume that system (2) is controllable and

$(\overline{H}_1)$ functions $f_1, \ldots, f_n$ are measurable on $[0, T]$, continuous on $\mathbb{R}^{r_1} \times \cdots \times \mathbb{R}^{r_n} \times \mathbb{R}^m$ and satisfy the following growth conditions: there exist $\overline{A}_i > 0$,

$\overline{a}_i \in L^p([0,T],\mathbb{R}_0^+)$ such that

$$|f_i(t,x_1,\ldots,x_n,u)|_{\mathbb{R}^{r_i}} \leqslant \overline{A}_i(|x_1|_{\mathbb{R}^{r_1}} + \cdots + |x_n|_{\mathbb{R}^{r_n}}) + \overline{a}_i(t), \quad i = 1,\ldots,n$$

for a.e. $t \in [0,T]$ and all $(x_1,\ldots,x_n) \in \mathbb{R}^{r_1+\cdots+r_n}$, $u \in M$,

$(\overline{H}_2)$ the function f_0 is measurable on $[0,T]$ and continuous on $\mathbb{R}^{r_1}\times\cdots\times\mathbb{R}^{r_n}\times\mathbb{R}^m$,

$(\overline{H}_3)$ the sets

$$Q(t,x) := \Big\{(\mu_0,\mu) \in \mathbb{R}\times\mathbb{R}^{r_1}\times\cdots\times\mathbb{R}^{r_n};\ \exists_{u\in M}\ \mu_0 \geqslant f_0(t,x_1,\ldots,x_n,u) \\ \mu = f(t,x_1,\ldots,x_n,u)\Big\}$$

for a.e. $t \in [0,T]$ and all $x \in \mathbb{R}^{r_1}\times\cdots\times\mathbb{R}^{r_n}$, where $\mu = (\mu_1,\ldots,\mu_n)$, $f = (f_1,\ldots,f_n)$, are convex.

$(\overline{H}_4)$ there exists a function $\overline{\lambda} \in L^1([0,T],\mathbb{R})$ such that for any pair $(x,u) \in \hat{\mathcal{A}}$

$$f_0(t,x_1(t),\ldots,x_n(t),u(t)) \geqslant \overline{\lambda}(t), \quad t \in [0,T]\ a.e.$$

We have

Theorem 2 *Assume that* $\alpha = (\alpha_1,\ldots,\alpha_n)$, $0 < \dfrac{1}{p} < \alpha_i \leqslant 1$, $i = 1,\ldots,n$ *and* M *is a compact set. If assumptions* $(\overline{H}_1) - (\overline{H}_4)$ *are satisfied then problem (1)–(2) has an optimal solution* $(x^*,u^*) \in {}_C\mathbb{AC}^{\alpha,p}_{0+}\times\mathcal{U}_M$.

Proof. Let us consider problem (3)–(4) with functions

$$g_i(t,y,u) = f_i(t,y+x_0,u), \quad i = 0,1,\ldots,n, \tag{9}$$

where $x_0 = (x_{10},\ldots,x_{n0})$. It is easy to verify that if a pair $(y^*(\cdot),u^*(\cdot)) \in \mathbb{I}^{\alpha,p}_{0+}(L^p)\times\mathcal{U}_M$ is an optimal solution to problem (3)–(4) with functions g_i, $i = 0,1,\ldots,n$ given by (9) then the pair

$$(x^*(\cdot),u^*(\cdot)) = (y^*(\cdot)+x_0,u^*(\cdot)) \in {}_C\mathbb{AC}^{\alpha,p}_{0+}\times\mathcal{U}_M$$

is an optimal solution to problem (1)–(2). Consequently, it is sufficient to show that if functions $f_i, i = 0,1,\ldots,n$ satisfy assumptions $(\overline{H}_1) - (\overline{H}_4)$ then functions g_i given by (9) satisfy conditions (H_1)–(H_4). First, let us note that controllability of system (2) guarantees controllability of system (4) with functions g_i given by (9). Moreover, for $i = 1,\ldots,n$ we have

$$\begin{aligned}|g_i(t,y_1,\ldots,y_n,u)|_{\mathbb{R}^{r_i}} &= |f_i(t,y_1+x_{10},\ldots,y_n+x_{n0},u)|_{\mathbb{R}^{r_i}} \\ &\leqslant \overline{A}_i(|y_1+x_{10}|_{\mathbb{R}^{r_1}}+\cdots+|y_n+x_{n0}|_{\mathbb{R}^{r_n}}) + \overline{a}_i(t) \\ &\leqslant \overline{A}_i(|y_1|_{\mathbb{R}^{r_1}}+\cdots+|y_n|_{\mathbb{R}^{r_n}}) + \overline{A}_i(|x_{10}|_{\mathbb{R}^{r_1}}+\cdots+|x_{n0}|_{\mathbb{R}^{r_n}}) + \overline{a}_i(t)\end{aligned}$$

for a.e. $t \in [0,T]$ and all $(y_1,\dots,y_n) \in \mathbb{R}^{r_1+\dots+r_n}$, $u \in M$, so condition (H_1) is satisfied. Of course, conditions $(\overline{H}_2)$ and $(\overline{H}_3)$ imply conditions (H_2) and (H_3), respectively. Now, let $(y(\cdot),u(\cdot)) \in \mathcal{A}$. Then $(y(\cdot)+x_0,u(\cdot)) \in \overline{\mathcal{A}}$. Consequently, from $(\overline{H}_4)$ we obtain

$$\begin{aligned} g_i(t,y_1(t),\dots,y_n(t),u(t)) &= f_i(t,y_1(t)+x_{10},\dots,y_n(t)+x_{n0},u(t)) \\ &\geqslant \overline{\lambda}(t), \quad t \in [0.T] \ a.e., \end{aligned}$$

so condition (H_4) is fulfilled.

The proof is completed. $\square$

5. Theoretical example

Example 1 Let us consider the following optimal control problem

$$\begin{cases} \left({}^C D^{\alpha_1}_{0+} x_1\right)(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1 u^3(t) \\ \left({}^C D^{\alpha_2}_{0+} x_2\right)(t) = A_{21}x_1(t) + A_{22}x_2(t) + B_2 u^3(t), \quad t \in [0,2] \ a.e. \\ x_1(0) = x_{10}, \qquad x_2(0) = x_{20}, \\ u(t) \in [-1,1], \quad t \in [0,2] \ a.e. \end{cases} \tag{10}$$

$$H((x_1,x_2),u) = \int_0^2 f_0(t,x_1(t),x_2(t),u(t))\,\mathrm{d}t \longrightarrow \min, \tag{11}$$

where $\alpha_1 = \dfrac{2}{3}$, $\alpha_2 = \dfrac{1}{2}$, $p = 3$,

$x_i = \begin{bmatrix} x_{i1} \\ x_{i2} \end{bmatrix} : [0,T] \to \mathbb{R}^2$, $i = 1,2$, $x_{10} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $x_{20} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$,

$$A_{11} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad A_{22} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad A_{12} = A_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -2\Gamma\left(\frac{7}{3}\right) \\ -\Gamma\left(\frac{7}{3}\right) \end{bmatrix}, \qquad B_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

$$f_0\colon [0,2] \times \mathbb{R}^2 \times \mathbb{R}^2 \times [-1,1] \to \mathbb{R},$$

$$f_0(t,x_1,x_2,u) = x_{11} - 2x_{12} + x_{21} + x_{22} + 2u^3.$$

From Theorem [2, Theorem 1] it follows that if the pair $((x_1^*(\cdot), x_2^*(\cdot)), u_*(\cdot)) \in \mathbb{AC}_{0+}^{(\frac{2}{3},\frac{1}{2}),3} \times \mathcal{U}_M$ is a locally optimal solution to problem (10)–(11) then there exists a function $\lambda(\cdot) = (\lambda_1(\cdot), \lambda_2(\cdot)) \in \mathbb{I}_{2-}^{(\frac{2}{3},\frac{1}{2})}(L^{\frac{3}{2}}) := I_{2-}^{\frac{2}{3}}(L^{\frac{3}{2}}([0,2],\mathbb{R}^2)) \times I_{2-}^{\frac{1}{2}}(L^{\frac{3}{2}}([0,2],\mathbb{R}^2))$ such that

$$\begin{cases} \left(D_{2-}^{\frac{2}{3}}\lambda_1\right)(t) = A_{11}^T\lambda_1(t) + \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\ \left(D_{2-}^{\frac{1}{2}}\lambda_2\right)(t) = A_{22}^T\lambda_2(t) + \begin{bmatrix} -1 \\ -1 \end{bmatrix} \end{cases}, \quad t \in [0,2]\ a.e. \tag{12}$$

and

$$\left(I_{2-}^{\frac{1}{3}}\lambda_1\right)(2) = 0, \qquad \left(I_{2-}^{\frac{1}{2}}\lambda_2\right)(2) = 0. \tag{13}$$

Furthermore,

$$\begin{aligned} &2u_*^3(t) - \lambda_1(t)B_1u_*^3(t) - \lambda_2(t)B_2u_*^3(t) \\ &\qquad = \min_{u\in[-1,1]}\left\{2u^3 - \lambda_1(t)B_1u^3 - \lambda_2(t)B_2u^3\right\} \end{aligned} \tag{14}$$

for a.e. $t \in [0,2]$.

From [15, Theorem 11] it follows that a solution $\lambda(\cdot) = (\lambda_1(\cdot), \lambda_2(\cdot))$ to system (12)–(13) is given by

$$\begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{(2-t)^{\frac{2}{3}}}{\Gamma\left(\frac{5}{3}\right)} & \dfrac{2(2-t)^{\frac{2}{3}}}{\Gamma\left(\frac{5}{3}\right)} - \dfrac{(2-t)^{\frac{4}{3}}}{\Gamma\left(\frac{7}{3}\right)} \\ t-2-\dfrac{(2-t)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)} & -\dfrac{(2-t)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)} \end{bmatrix}, \quad t \in [0,2]\ \ a.e.$$

Consequently, condition (14) is equivalent to the following one:

$$\begin{aligned} \left(t-(2-t)^{\frac{4}{3}}\right)u_*^3(t) &= \min_{u\in[-1,1]}\left\{\left(t-(2-t)^{\frac{4}{3}}\right)u^3\right\} \\ &= \begin{cases} t-(2-t)^{\frac{4}{3}}, & \text{if } t \in [0,1]\ \ a.e. \\ (2-t)^{\frac{4}{3}} - t, & \text{if } t \in (1,2]\ \ a.e. \end{cases} \end{aligned}$$

Hence

$$u_*(t) = \begin{cases} 1, & \text{if } t \in [0,1] \ a.e. \\ -1, & \text{if } t \in (1,2] \ a.e. \end{cases} \tag{15}$$

Finally, using [12, Theorem 1] we conclude that the solution $(x_1^*(\cdot), x_2^*(\cdot))$ to system (10)–(11), corresponding to $u_*(\cdot)$ is given by

$$\begin{aligned} x_i^*(t) &= \Phi_{i0}(t)x_{i0} + \int_0^t \Phi_i(t-s)B_i u_*^3(s)\,\mathrm{d}s \\ &= \Phi_{i0}(t)x_{i0} + \begin{cases} \displaystyle\int_0^t \Phi_i(t-s)B_i\,\mathrm{d}s & t \in [0,1] \ a.e. \\ \displaystyle\int_0^1 \Phi_i(t-s)B_i\,\mathrm{d}s - \int_1^t \Phi_i(t-s)B_i\,\mathrm{d}s & t \in [1,2] \ a.e., \end{cases} \end{aligned}$$

where

$$\Phi_{i0}(t) = \sum_{k=0}^{\infty} \frac{A_{ii}^k t^{k\alpha}}{\Gamma(k\alpha+1)} \qquad \text{and} \qquad \Phi_i(t) = \sum_{k=0}^{\infty} \frac{A_{ii}^k t^{\alpha(k+1)-1}}{\Gamma(\alpha(k+1))}, \qquad i = 1,2.$$

Hence,

$$x_*^1(t) = \begin{cases} \begin{bmatrix} \dfrac{2t^{\frac{2}{3}}}{\Gamma\left(\frac{5}{3}\right)}\left(1-\Gamma\left(\frac{7}{3}\right)\right)-t^{\frac{4}{3}}+1 \\ 2-\dfrac{\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(\frac{5}{3}\right)}t^{\frac{2}{3}} \end{bmatrix}, & t \in [0,1] \\ \begin{bmatrix} 2(t-1)^{\frac{4}{3}}-t^{\frac{4}{3}}+4\dfrac{\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(\frac{5}{3}\right)}(t-1)^{\frac{2}{3}}+\dfrac{2t^{\frac{2}{3}}}{\Gamma\left(\frac{5}{3}\right)}\left(1-\Gamma\left(\frac{7}{3}\right)\right)+1 \\ 2+\dfrac{\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(\frac{5}{3}\right)}\left(2(t-1)^{\frac{2}{3}}-t^{\frac{2}{3}}\right) \end{bmatrix}, & t \in (1,2] \end{cases} \tag{16}$$

and

$$x_*^2(t) = \begin{cases} \begin{bmatrix} -1 - \dfrac{t^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)} \\ 1 - t \end{bmatrix}, & t\in[0,1] \\ \begin{bmatrix} \dfrac{2(t-1)^{\frac{1}{2}} - t^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)} - 1 \\ \dfrac{-2(t-1)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)} + t - 1 \end{bmatrix}, & t\in(1,2]. \end{cases} \tag{17}$$

This means that the pair

$$(x_*, u_*) = \left(\left(x_*^1, x_*^2\right), u_*\right)$$

given by (16), (17) and (15) is the only pair which can be a locally optimal solution to problem (10)–(11). Now, we check that all assumptions of Theorem 2 are satisfied. It is clear that f_0 satisfies assumption $(\overline{H}_2)$. Moreover, functions

$$f_i(t, x_1, x_2, u) = A_{i1}x_1 + A_{i2}x_2 + B_i u^3, \quad i = 1, 2$$

are measurable on $[0,2]$, continuous on $\mathbb{R}^2 \times \mathbb{R}^2 \times [-1,1]$ and

$$|f_i(t, x_1, x_2, u)|_{\mathbb{R}^2} \leqslant \max\{|A_{i1}|, |A_{i2}|\}(|x_1|_{\mathbb{R}^2} + |x_2|_{\mathbb{R}^2}) + |B_i|, \quad i = 1, 2,$$

for a.e. $t \in [0,2]$ and all $(x_1, x_2) \in \mathbb{R}^2 \times \mathbb{R}^2$, $u \in [-1,1]$, so $(\overline{H}_1)$ holds. Using similar arguments as in the proof of Lemma 1 we conclude that x_1 and x_2 are bounded on $[0,2]$. Consequently, there exists a constant $C > 0$ such that for all pairs $((x_1, x_2), u) \in \mathbb{AC}_{0+}^{(\frac{2}{3},\frac{1}{2}),3} \times \mathcal{U}_M$

$$f_0(t, x_1(t), x_2(t), u(t)) \geqslant C, \quad t \in [0,2] \ a.e.,$$

so $(\overline{H}_4)$ is satisfied. Now, we show that sets $Q(t,x)$ are convex for a.e. $t \in [0,2]$ and all $x = (x_1, x_2) \in \mathbb{R}^2 \times \mathbb{R}^2$. Let us fix such a $t \in [0,2]$ and $x = (x_1, x_2) \in \mathbb{R}^2 \times \mathbb{R}^2$ and let $\gamma \in [0,1]$, $(\mu_0, (\mu_1, \mu_2))$, $(\beta_0, (\beta_1, \beta_2)) \in Q(t,x)$. This means that there exist $u_1, u_2 \in [-1,1]$ such that

$$\mu_0 \geqslant f_0(t, x_1, x_2, u_1),\ \ \mu_1 = f_1(t, x_1, x_2, u_1),\ \ \mu_2 = f_2(t, x_1, x_2, u_1)$$

and

$$\beta_0 \geqslant f_0(t, x_1, x_2, u_1),\ \ \beta_1 = f_1(t, x_1, x_2, u_1),\ \ \beta_2 = f_2(t, x_1, x_2, u_1).$$

Putting

$$u_3 = \sqrt[3]{\gamma u_1^3 + (1-\gamma)u_2^3} \in [-1,1]$$

we obtain

$$\gamma\mu_0 + (1-\gamma)\beta_0 \geqslant x_{11} - 2x_{12} + x_{21} + x_{22} + 2(\gamma u_1^3 + (1-\gamma)u_2^3) = f_0(t, x_1, x_2, u_3),$$

$$\gamma\mu_1 + (1-\gamma)\beta_1 = A_{11}x_1 + A_{12}x_2 + B_1(\gamma u_1^3 + (1-\gamma)u_2^3) = f_1(t, x_1, x_2, u_3),$$

$$\gamma\mu_2 + (1-\gamma)\beta_2 = A_{21}x_1 + A_{22}x_2 + B_2(\gamma u_1^3 + (1-\gamma)u_2^3) = f_2(t, x_1, x_2, u_3).$$

Consequently, condition $(\overline{H}_3)$ holds.

Using Theorem 2, we assert that the pair (x_*, u_*) is the optimal solution of (10)–(11).

6. Conclusions

In the paper, a nonlinear control systems, involving Caputo derivatives of different orders, associated to an integral performance index have been studied. In the first part, using a lower closure theorem and measurable selection theorem of Filippov type for multivalued mappings, the existence of optimal solutions for a problem with zero initial conditions has been proved. Next, due to obtained existence result for problem with zero initial conditions, the result of such a type for problem (1)–(2) has been derived (problem (1)–(2) is replaced with an equivalent problem with zero initial conditions by substitution (9)). In conclusion, we presented one illustrative example.

Appendix

In the first part of this section, we provide a some useful corollary of the following fractional version of Gronwall's lemma ([10, Theorem 3])

Lemma 2 (fractional multi–term Gronwall lemma) *Let* $0 < \gamma_1 < \cdots < \gamma_p$, $a(t)$ *be a nonnegative function integrable on* $J := [a, b]$, $g_1(t), \ldots, g_p(t)$ *be nonnegative, measurable, essentially bounded functions on* J *and* $v(t)$ *be a nonnegative function integrable on* J *with*

$$v(t) \leqslant a(t) + g_1(t) \int\limits_a^t \frac{v(s)}{(t-s)^{1-\gamma_1}} \mathrm{d}s + \cdots + g_p(t) \int\limits_a^t \frac{v(s)}{(t-s)^{1-\gamma_p}} \mathrm{d}s, \quad a.e.\ on\ J.$$

Then

$$v(t) \leqslant a(t) + \sum_{n<\frac{1}{\gamma_1}} \frac{G^n p^n}{\Gamma(n\gamma_1)} \int\limits_a^t \frac{\Psi_a(s)}{(t-s)^{1-n\gamma_1}} \mathrm{d}s + \sum_{n\geqslant\frac{1}{\gamma_1}} \frac{G^n p^n (b-a)^{n\gamma_1-1}}{\Gamma(n\gamma_1)} \int\limits_a^t \Psi_a(s)\mathrm{d}s$$

$$+ E \sum_{n=1}^{\infty} \frac{G^n p^n (b-a)^{\gamma_1 n} (1+(b-a)^{(\gamma_2-\gamma_1)n+\dots+(\gamma_p-\gamma_1)n-1})}{\Gamma(n\gamma_1+1)} \int\limits_a^t a(s)\mathrm{d}s, \quad a.e. \ on \ J, \tag{18}$$

where

$$G = essup\{G(t): \quad t \in J\}, \quad G(t) = \max\left\{\Gamma(\gamma_1)g_1(t), \dots, \Gamma(\gamma_p)g_p(t)\right\},$$

$$E = \frac{1}{\min\{\Gamma(s); \ s \geqslant 1\}}$$

and

$$\Psi_z(t) = \max\Big\{I_{a+}^{(\gamma_2-\gamma_1)j_2+\dots+(\gamma_p-\gamma_1)j_p} z(t); \quad j_2, \dots, j_p \in \mathbb{N}_0,$$
$$(\gamma_2-\gamma_1)j_2 + \dots + (\gamma_p-\gamma_1)j_p < 1\Big\}$$

for z integrable on J.

Corollary 1 *If all assumptions of Lemma 2 are satisfied, whereby*

$$a(t) \equiv c_0, \qquad g_i(t) \equiv c_i, \quad i = 1, \dots, p,$$

($c_i > 0, i = 0, \dots, p$) then there exists $C > 0$ such that

$$v(t) \leqslant C, \quad a.e. \ on \ J. \tag{19}$$

Proof. First, let us note that

$$I_{a+}^{(\gamma_2-\gamma_1)j_2+\dots+(\gamma_p-\gamma_1)j_p} a(t)$$
$$= \frac{c_0}{\Gamma((\gamma_2-\gamma_1)j_2 + \dots + (\gamma_p-\gamma_1)j_p + 1)} (t-a)^{(\gamma_2-\gamma_1)j_2+\dots+(\gamma_p-\gamma_1)j_p}$$
$$\leqslant Ec_0(b-a)^{(\gamma_2-\gamma_1)j_2+\dots+(\gamma_p-\gamma_1)j_p} \leqslant Ec_0 \max\{1, (b-a)\}.$$

Consequently,

$$\Psi_a(t) \leqslant Ec_0 \max\{1, (b-a)\} =: D.$$

Moreover,

$$G = \max\{\Gamma(\gamma_1)c_1, \ldots, \Gamma(\gamma_p)c_p\}.$$

Condition (18) can be written as follows:

$$\begin{aligned}
v(t) \leqslant c_0 + &\sum_{n<\frac{1}{\gamma_1}} \frac{G^n p^n}{\Gamma(n\gamma_1)} \int_a^t \frac{\Psi_a(s)}{(t-s)^{1-n\gamma_1}} \mathrm{d}s + \sum_{n\geqslant\frac{1}{\gamma_1}} \frac{G^n p^n (b-a)^{n\gamma_1-1}}{\Gamma(n\gamma_1)} \int_a^t \Psi_a(s)\,\mathrm{d}s \\
&+E \sum_{n=1}^{\infty} \frac{G^n p^n (b-a)^{\gamma_1 n}(1+(b-a)^{(\gamma_2-\gamma_1)n+\cdots+(\gamma_p-\gamma_1)n-1})}{\Gamma(n\gamma_1+1)} c_0(b-a) \\
&= c_0 + S_1 + S_2 + S_3, \quad a.e. \ on \ J.
\end{aligned}$$

We have

$$S_1 = \sum_{n<\frac{1}{\gamma_1}} \frac{G^n p^n}{\Gamma(n\gamma_1)} \int_a^t \frac{\Psi_a(s)}{(t-s)^{1-n\gamma_1}} \mathrm{d}s \leqslant D \sum_{n<\frac{1}{\gamma_1}} \frac{(Gp(b-a)^{\gamma_1})^n}{\Gamma(n\gamma_1+1)} =: C_1.$$

Furthermore, for sufficiently large n, using the following Gauss-Legandre formula (cf. [24, formula (1.62)])

$$\Gamma(mz) = \frac{m^{mz-\frac{1}{2}}}{(2\pi)^{\frac{m-1}{2}}} \prod_{k=0}^{m-1} \Gamma\left(z + \frac{k}{m}\right), \quad m = 2, 3, \ldots, \quad z > 0,$$

we get

$$\begin{aligned}
S_2 = &\sum_{n\geqslant\frac{1}{\gamma_1}} \frac{G^n p^n (b-a)^{n\gamma_1-1}}{\Gamma(n\gamma_1)} \int_a^t \Psi_a(s)\,\mathrm{d}s \leqslant D \sum_{n\geqslant\frac{1}{\gamma_1}} \frac{(Gp(b-a)^{\gamma_1})^n}{\Gamma(n\gamma_1)} \\
&\leqslant D \sum_{n\geqslant\frac{1}{\gamma_1}} \frac{(2\pi EGp(b-a)^{\gamma_1})^n}{n^{n\gamma_1-\frac{1}{2}}}.
\end{aligned}$$

From the Cauchy root test it follows that there exists a constant $C_2 > 0$ such that $S_2 \leqslant C_2$.

Now, let us note that for sufficiently large n (such that $(\gamma_2-\gamma_1)n+\cdots+(\gamma_p-\gamma_1)n > 1$ and $n\gamma_1 \geqslant 1$), using inequality (cf. [30, Lemma 3.1])

$$\Gamma(x+1) > \left(\frac{x}{e}\right)^x \sqrt{2\pi x}\left(1+\frac{1}{12x}\right) > \left(\frac{x}{e}\right)^x, \quad x \geqslant 1,$$

we obtain

$$
\begin{aligned}
S_3 =& E\sum_{n=1}^{\infty}\frac{G^n p^n (b-a)^{\gamma_1 n}(1+(b-a)^{(\gamma_2-\gamma_1)n+\dots+(\gamma_p-\gamma_1)n-1})}{\Gamma(n\gamma_1+1)}c_0(b-a)\\
=& c_0E\sum_{n=1}^{\infty}\frac{(Gp(b-a)^{\gamma_1})^n((b-a)+(b-a)^{(\gamma_2-\gamma_1)n+\dots+(\gamma_p-\gamma_1)n})}{\Gamma(n\gamma_1+1)} \leqslant 2c_0E\sum_{n=1}^{\infty}\frac{(KGp(b-a)^{\gamma_1})^n}{\Gamma(n\gamma_1+1)}\\
\leqslant& 2c_0E\sum_{n=1}^{\infty}\frac{(KGp(e(b-a))^{\gamma_1})^n}{n\gamma_1^{n\gamma_1}},
\end{aligned}
$$

where $K = \max\{1, (b-a)^{(\gamma_2-\gamma_1)+\dots+(\gamma_p-\gamma_1)}\}$. Using the Cauchy root test once again we assert that there exists a constant $C_3 > 0$ such that $S_3 \leqslant C_3$.

Putting $C = c_0 + C_1 + C_2 + C_3$ we get condition (19).

The proof is completed. □

Now, for a convenience of the reader, we present some necessary facts regarding multifunctions (cf. [7, 23]) that are required in this paper.

Let S be an arbitrary nonempty set equipped with a σ – algebra $\mathcal{B}$ and $\Lambda : S \ni s \longrightarrow \Lambda(s) \subset \mathbb{R}^r$ be a closed-valued multifunction.

We shall say that Λ is measurable if for each closed set $C \subset \mathbb{R}^r$ the set $\Lambda^{-1}(C)$ given by

$$\Lambda^{-1}(C) := \{s \in S : \quad \Lambda(s) \cap C \neq \emptyset\}$$

is measurable (i.e. $\Lambda^{-1}(C) \in \mathcal{B}$).

Let us define the set:

$$\operatorname{dom}\Lambda := \{s \in S : \quad \Lambda(s) \neq \emptyset\}.$$

A function $\lambda\colon \operatorname{dom}\Lambda \to \mathbb{R}^r$ such that $\lambda(s) \in \Lambda(s)$ for all $s \in \operatorname{dom}\Lambda$, is called a selection of the multifunction Λ.

We shall say that a function $f : S \times \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ is a normal integrand on $S \times \mathbb{R}^n$ if f is lower semicontinuous on $\mathbb{R}^n$ for all $s \in S$ and the epigraph

$$E_f(s) := \operatorname{epi} f(s,\cdot) = \{(w,v) \in \mathbb{R}^{n+1} : \quad v \geqslant f(s,w)\}$$

is a measurable multifunction.

In the proof of the main result of this paper we apply the following version of Filippov's lemma (cf. [23, Theorem 2J]):

Theorem 3 (Measurable selection theorem) *Let $\Lambda : S \ni s \longrightarrow \Lambda(s) \subset \mathbb{R}^r$ be a multifunction of the form*

$$\Lambda(s) := \{w \in C(s) : \quad F(s,w) = a(s) \quad \text{and} \quad f_i(s,w) \leqslant \kappa_i, \quad i \in J\},$$

where $C : S \ni s \longrightarrow \Lambda(s) \subset \mathbb{R}^r$ is a measurable (closed-valued) multifunction, $F : S \times \mathbb{R}^r \to \mathbb{R}^k$ is a Carathéodory mapping, $(f_i : \ i \in J)$ is a countable collection of normal integrands on $S \times \mathbb{R}^r$ and $a : S \to \mathbb{R}^k$, $\kappa_i : S \to \mathbb{R} \cup \{\pm\infty\}$ are measurable. Then Λ is the measurable (closed-valued) multifunction and hence Λ has a measurable selection $\lambda : \mathrm{dom}\,\Lambda \to \mathbb{R}^r$.

Now, let us assume that (S,ρ) is a metric space and $\Lambda : S \ni s \longrightarrow \Lambda(s) \subset \mathbb{R}^r$ is an arbitrary multifunction.

We say that $\Lambda : S \ni s \longrightarrow \Lambda(s) \subset \mathbb{R}^r$ has property (K) at the point $s_0 \in S$ iff

$$\Lambda(s_0) = \bigcap_{\delta>0} \mathrm{cl}\left(\bigcup\{\Lambda(s) : \quad \rho(s,s_0) < \delta\}\right),$$

where $\mathrm{cl}Z$ denotes the closure of the set Z.

We say that Λ has property (K) in S if it has property (K) at every point $s \in S$.

We have (cf. [7, Theorem 8.5.iii])

Theorem 4 *Let $\Lambda : S \ni s \longrightarrow \Lambda(s) \subset \mathbb{R}^r$ be a multifunction. Then Λ has property (K) if and only if the graph of Λ given by*

$$\mathrm{Gr}\Lambda := \{(s,w) : \quad s \in S, \quad w \in \Lambda(s)\},$$

is closed in the product space $S \times \mathbb{R}^r$.

Remark 3 *From the above theorem it follows that if Λ has property (K) then its values are closed.*

In conclusion, we formulate a key result in our study, namely, a lower closure theorem ([7, Theorem 10.7.i]). First, we give the necessary notation.

Let $G \subset \mathbb{R}^\nu$ be a measurable set of finite measure, for every $x = (x^1, \ldots, x^\nu) \in G$ let $A(x)$ be a given nonempty subset of $\mathbb{R}^n$ and let

$$A = \{(x,z) : \quad x \in G, \quad z \in A(x)\},$$

whereby $z = (z^1, \ldots, z^n)$. For every $(x,z) \in A$ let $\widetilde{Q}(x,z)$ be a given subset of the space $\mathbb{R}^{r+1}$.

Theorem 5 *Let us assume that for almost all $x \in G$, the set $A(x)$ is closed, the sets $\widetilde{Q}(x,z)$ are closed, convex and have property (K) with respect to $z \in A(x)$. Let $\xi, \xi_k : G \to \mathbb{R}^r$, $z, z_k : G \to \mathbb{R}^n$, $\lambda, \eta, \lambda_k, \eta_k : G \to \mathbb{R}$, $k = 1, 2, \ldots,$ be measurable functions, $\xi, \xi_k \in (L^1(G))^r$, $\eta_k \in L^1(G)$, with $z_k \to z$ in measure on G, $\xi_k \rightharpoonup \xi$ weakly in $(L^1(G))^r$ as $k \to \infty$,*

$$z_k(x) \in A(x), \quad (\eta_k(x), \xi_k(x)) \in \widetilde{Q}(x, z_k(x)), \quad x \in G, \quad k = 1, 2, \ldots,$$

$$-\infty < i = \liminf_{k \to \infty} \int_G \eta_k(x)\,\mathrm{d}x < +\infty,$$

$$\eta_k(x) \geqslant \lambda_k(x), \quad \lambda, \lambda_k \in L^1(G), \quad \lambda_k \to \lambda \text{ weakly in } L^1(G).$$

Then there exists a function $\eta \in L^1(G)$ such that

$$x(x) \in A(x), \quad (\eta(x), \xi(x)) \in \widetilde{Q}(x, z(x)), \quad x \in G, \quad \int_G \eta(x)\,\mathrm{d}x \leqslant i.$$

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