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MECHANICAL AND AERONAUTICAL ENGINEERING, THERMODYNAMICS

Some applications of the generalized Laplace transform and the representation of a solution to Sobolev-type evolution equations with the generalized Caputo derivative

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Abstract. We introduce the Sobolev-type multi-term μ-fractional evolution with generalized fractional orders with respect to another function. We make some applications of the generalized Laplace transform. In the sequel, we propose a novel type of Mittag-Leffler function generated by noncommutative linear bounded operators with respect to the given function and give a few of its properties. We look for the mild solution formula of the Sobolev-type evolution equation by building on the aforementioned Mittag-Leffler-type function with the aid of two different approaches. We share new special cases of the obtained findings.

Keywords: degenerate evolution equation; Sobolev; generalized Laplace transform; fractional derivative with respect to another function; Mittag-Leffler type function.

1. INTRODUCTION

Fractional differential equations are regarded as a generalization of the traditional partial or ordinary differential equations because the aforesaid fractional order can be selected from an arbitrary real number or complex number as well as an arbitrary natural number. This facilitates the expansion of the present studies related to ordinary or partial differential equations in the literature to more general works [1–4] with fractional-order derivatives which are handed real-world problems and discovering some new findings in order to describe more complex physical systems. There is no doubt that one of them is Sobolev-type (degenerate) evolution equation which accepts abstract representations in the frame of an explicit continuous time-variable operator-differential equation with the multiplication of the highest derivative with any operator coefficient on a Banach space.

It is, in many ranges of real-world physical problems, used like thermodynamics, flow of fluid through rock fracture, propagation of long waves having small amplitude and so forth [5, 6]. There are several questions to be answered about degenerate evolution equation obtained from supra-physical process: the existence of particular types of solutions, such as time-periodic solutions, self-similar solutions, travelling waves, equilibrium solutions; the dynamic stability of these solutions; the asymptotic behavior of solutions; complete integrability, chaotic dynamics, singular perturbation expressions for solutions; convergence of numerical schemes; and so forth. Naturally, it has attracted many researchers' attention. Most of them have studied existence [7], controllability [8–11], approximate controllability [12], the asymptotic behavior of resolvent operators of Sobolev-type [13], stability of solutions [14] about it and its fractional version depending only one fractional (ordinary) order since mostly the last two decades.

While the notion of a fractional derivative of a function with respect to another function [15] is involved in many sorts of real-world problems, simultaneously it has been developed as in the studies [16, 17].

As far as we look for, there are so very few works in the literature about the Sobolev-type evolution equation with two independent generalized fractional orders. Nazim *et al.* [18] introduce a new approach for settling out the Sobolev-type fractional evolution equation with multi-orders, which are not generalized, in a Banach space by giving a new type of Mittag-Leffler function produced by linear bounded operators.

Taking inspiration of the studies [15, 16, 18], we will consider the following Sobolev-type multi-term μ-fractional evolution equations with generalized fractional orders with respect to another function μ

$$\begin{cases} \left({}^{C}_{0^+}\mathfrak{D}^{\alpha}_{\mu}Ex\right)(t) - A_0\left({}^{C}_{0^+}\mathfrak{D}^{\beta}_{\mu}x\right)(t) - B_0x(t) = g(t), \;\; t>0, \\ Ex(0)=\phi(0), \;\; (Ex(0))^{(k)} = \phi^{(k)}(0), \;\; k=1,2,\ldots,n-1, \end{cases} \tag{1}$$

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${}^{C}_{0^+}\mathfrak{D}^{\alpha}_{\mu}$ and ${}^{C}_{0^+}\mathfrak{D}^{\beta}_{\mu}$ are such Caputo fractional derivatives of orders α and β, respectively that $n-1<\alpha\le n$, $n-2<\beta\le n-1$, $n\ge 2$, and $\alpha-\beta>1$. The function μ is so increasing that $\mu'(t)\neq 0$ for every t. $B_0 : D(B_0)\subseteq X\to Y$, $A_0 : D(A_0)\subseteq X\to Y$ and $E : D(E)\subseteq X\to Y$ are linear operators where X and Y are Banach spaces and x is a X-valued function. $D(E)$ is a Banach space with regard to $\|x\|_{D(E)}=\|Ex\|_Y$, $x\in D(E)$. $\phi:\mathbb{R}^+\to Y$ is an arbitrary $(n-1)$-times continuously differentiable Y-valued function, $g:\mathbb{R}^+\to Y$ is continuous. Moreover, we impose the following conditions on equation (1)

H_1: A_0 is closed, B_0 is bounded,

H_2: $D(E)\subseteq D(A_0)$ and E is bijective,

H_3: $E^{-1}:Y\to D(E)\subseteq X$ is compact.

The principle contribution of the present paper:

- We introduce Sobolev-type multi-term μ-fractional evolution equation with general distinct fractional orders with respect to another function μ in (1).
- We offer novelly evolutional μ-Mittag-Leffler type functions.
- We share a little more novel features of evolutional μ-Mittag-Leffler type functions.
- We investigate, thanks to two different approaches which are the generalized Laplace integral technique and variation of constants' technique together with the substitution, an analytical solution to multi-term general μ-fractional evolution equation with bounded linear operators (7), an explicit solution to semi-linear version to (7), and a mild solution formula to (7) which is converted from evolution equation (1) with commutative and noncommutative coefficient matrices in terms of newly evolutional μ-Mittag-Leffler type functions.
- We reach to the mild closed solution formula to the Sobolev type multi-term μ-fractional evolution equation with general independent μ-fractional orders and commutative and noncommutative matrices.

2. PRELIMINARIES

Let $\mathbb{R}$ which is the set of all real numbers, and let $\mathbb{R}^+=[0,\infty)$, and $\mathbb{N}=\{0,1,2,\ldots\}$. For $-\infty<a<b<\infty$, $J=[a,b]$ is the interval of $\mathbb{R}$. Let X and Y be Banach spaces. For $n\in\mathbb{N}$, let $C(\mathbb{R}^+,X)$, $C^n(\mathbb{R}^+,X)$ and $B(X,Y)$ be the Banach space of continuous functions from $\mathbb{R}^+$ to X, real-valued functions $f(x)$ which have continuous derivatives up to order n such that $f^{(n-1)}\in C(\mathbb{R}^+,X)$ and all bounded linear operators from X into Y, respectively. The maximum norm $\|.\|_C$ is $\|g\|_C=\max_{t\in\mathbb{R}^+}|g(t)|$, where $|.|$ is an arbitrary norm on $\mathbb{R}$. Let $AC[a,b]$ be the space of functions which are absolutely continuous on $[a,b]$. For $n\in\{1,2,3,\ldots\}$ we denote by $AC^n[a,b]$ the space of complex-valued functions $f(x)$ which have continuous derivatives up to order $n-1$ on $[a,b]$ such that $f^{(n-1)}(x)\in AC[a,b]$.

Definition 1. [15] Let g be an integrable function defined on J and $\mu\in C^1(I)$ such that μ is increasing and $\mu'(t)\neq 0$ for every $t\in J$. The μ-Riemann Liouville type fractional integral of order $\alpha>0$ is defined as

$$\left({}^{RL}_{0^+}\mathfrak{I}^{\alpha}_{\mu}g\right)(t):=\frac{1}{\Gamma(\alpha)}\int_0^t\mu'(s)\big(\mu(t)-\mu(s)\big)^{\alpha-1}g(s)\,\mathrm{d}s$$
$$=\frac{1}{\Gamma(\alpha)}\int_0^t\big(\mu(t)-\mu(s)\big)^{\alpha-1}g(s)\,\mathrm{d}\mu(s)$$

and the μ-Riemann Liouville type fractional derivative of order $\alpha>0$ is defined as

$$\left({}^{RL}_{0^+}\mathfrak{D}^{\alpha}_{\mu}g\right)(t):=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{\mu'(t)dt}\right)^n\int_0^t\mu'(s)\big(\mu(t)-\mu(s)\big)^{n-\alpha-1}g(s)\,\mathrm{d}s$$
$$=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{d\mu(t)}\right)^n\int_0^t\big(\mu(t)-\mu(s)\big)^{n-\alpha-1}g(s)\,\mathrm{d}\mu(s),$$

where $n=[\alpha]+1$, $[\alpha]$ means the integer part of α.

Definition 2. [15,16] Let $g\in AC^n(J)$ and $\mu\in C^1(J)$ such that μ is increasing and $\mu'(t)\neq 0$ for every $t\in J$. The μ-Caputo type fractional derivative of order $\alpha>0$ is defined as

$$\left({}^{C}_{0^+}\mathfrak{D}^{\alpha}_{\mu}g\right)(t):=\frac{1}{\Gamma(n-\alpha)}\int_0^t\mu'(s)\big(\mu(t)-\mu(s)\big)^{n-\alpha-1}\cdot\left(\frac{d}{\mu'(s)ds}\right)^n g(s)\,\mathrm{d}s$$
$$=\frac{1}{\Gamma(n-\alpha)}\int_0^t\big(\mu(t)-\mu(s)\big)^{n-\alpha-1}\cdot\left(\frac{d}{d\mu(s)}\right)^n g(s)\,\mathrm{d}\mu(s),$$

where $n=[\alpha]+1$. For $\alpha=m\in\mathbb{N}$, it is clear that

$$\left({}^{C}_{0^+}\mathfrak{D}^{\alpha}_{\mu}g\right)(t)=g^{[m]}_{\mu}(t),\quad g^{[m]}_{\mu}(t)=\left(\frac{d}{d\mu(t)}\right)^m g(t).\tag{2}$$

Remark 1. If we consider $\mu(t)=t$, we obtain the classical Riemann Liouville fractional integral $\left({}^{RL}_{0^+}\mathfrak{I}^{\alpha}_{t}g\right)(t)$, the classical Riemann Liouville fractional derivative $\left({}^{RL}_{0^+}\mathfrak{D}^{\alpha}_{t}g\right)(t)$, and the classical Caputo fractional derivative $\left({}^{C}_{0^+}\mathfrak{D}^{\alpha}_{t}g\right)(t)$. Furthermore, we can obtain other fractional derivatives like Hadamard fractional derivative [19,20], Erdélyi–Kober fractional derivative [21], etc.

Theorem 1. [16,22] If $g\in AC^n(J)$ and $\alpha>0$, then

$$\left({}^{C}_{0^+}\mathfrak{D}^{\alpha}_{\mu}g\right)(t)=\left({}^{RL}_{0^+}\mathfrak{D}^{\alpha}_{\mu}g\right)\left[g(t)-\sum_{k=0}^{n-1}\frac{(\mu(t)-\mu(0))^k}{k!}g^{[k]}_{\mu}(0)\right].$$

and

$$ {}^{C}_{0^+}\mathfrak{D}^{\alpha}_{\mu}\left({}^{RL}_{0^+}\mathfrak{I}^{\alpha}_{\mu}g\right)(t)=g(t),\ n-1<\alpha\leq n,\ t>0. $$

Lemma 1. [15, 16] Given $\beta\in\mathbb{R}$. For $\alpha>0$,

$$ \begin{aligned} &{}^{C}_{0^+}\mathfrak{D}^{\alpha}_{\mu}(\mu(t)-\mu(0))^{\beta-1}\\ &=\begin{cases}\dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\mu(t)-\mu(0))^{\beta-\alpha-1}, & \beta>[\alpha],\\ 0, & \beta=0,1,2,\dots,[\alpha],\\ \text{undefined}, & \text{otherwise}.\end{cases} \end{aligned} $$

Definition 3. [23] Let $g\colon[0,\infty)\to\mathbb{R}$ be a real-valued function and μ be a nonnegative increasing function such that $\mu(0)=0$. Then the Laplace transform of g with respect to μ is defined by

$$ \mathscr{L}_{\mu}\{g(t)\}(s)=\int_{0}^{\infty}e^{-s\mu(t)}g(t)\,\mathrm{d}\mu(t) $$

for all $s\in\mathbb{C}$, which is the set of all complex numbers, such that this integral converges. We call this transform the generalized Laplace or μ-Laplace transform.

It should be stressed that more general Laplace-type integral transform is defined in [24], where one can find the definition of the inverse Laplace transform of G with respect to μ and conditions for existence:

$$ \mathscr{L}^{-1}_{\mu}\{G(s)\}(t)=g(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{s\mu(t)}G(s)\,\mathrm{d}s. $$

for $s\in\mathbb{C}$ such that $\mathrm{Re}(s)=c$.

Lemma 2. [24] Let $G(s)$ be analytic function of s (assuming that $G(s)$ has not the branch point) except at finite number of poles and each of poles lies to the left of the vertical line $\mathrm{Re}(s)=c$. If $G(s)\to0$ as $s\to\infty$ through the left plane $\mathrm{Re}(s)\leq c$, and

$$ \mathscr{L}_{\mu}\{g(t)\}(s)=G(s)=\int_{0}^{\infty}e^{-s\mu(t)}g(t)\,\mathrm{d}\mu(t), $$

then

$$ \mathscr{L}^{-1}_{\mu}\{G(s)\}(t)=g(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{s\mu(t)}G(s)\,\mathrm{d}s. $$

Definition 4. [23] A function $g:[0,\infty)\to\mathbb{R}^n$ is said to be μ-exponentially bounded of order $c>0$ if there exist constants M and T such that for all $t>T$,

$$ \|g\|_{C^n}=\max_{1\leq i\leq n}\|g_i\|_C\leq Me^{c\mu(t)}. $$

Lemma 3. [25] If $g\colon[0,\infty)\to\mathbb{R}$ is a (piecewise) continuous function and is of μ-exponentially bounded of order $c>0$, where μ is a nonnegative increasing function such that $\mu(0)=0$, then the generalized Laplace transform of g exists for $\mathrm{Re}(s)>0$.

Definition 5. If the generalized Laplace transform of $f\colon[a,\infty)\to\mathbb{R}$ exists for $s>c$ and the generalized Laplace transform of $g\colon[a,\infty)\to\mathbb{R}$ exists for $s>d$. Then, for any constants a and b, the generalized Laplace transform of $af+bg$, where a and b are constant, exists and

$$ \mathscr{L}_{\mu}\{af(t)+bg(t)\}(s)=a\mathscr{L}_{\mu}\{f(t)\}(s)+b\mathscr{L}_{\mu}\{g(t)\}(s), $$

for $s>\max(c,d)$.

In the reference [23], the μ-Laplace transform of the Caputo's fractional differentiation operator of general order $n-1<\alpha\leq n$ is given by

$$ \mathscr{L}_{\mu}\left\{\left({}^{C}_{0^+}\mathfrak{D}^{\alpha}_{\mu}g\right)(t)\right\}(s)=s^{\alpha}\mathscr{L}_{\mu}\{g(t)\}(s)-\sum_{k=0}^{n-1}s^{\alpha-k-1}g^{[k]}_{\mu}(0), \tag{3} $$

where $n-1<\alpha\leq n$, $n\in\mathbb{N}-\{0\}$.

Definition 6. [23] Let f and g be of μ-exponential order, piecewise continuous functions over each finite interval $[0,T]$. Then, the μ-convolution of f and g is the function $f*_{\mu}g$ defined by

$$ (f*_{\mu}g)(t)=\int_{0}^{t}f\left(\mu^{-1}(\mu(t)-\mu(s))\right)g(s)\mu'(s)\,\mathrm{d}s, $$

which has the commutativity property

$$ f*_{\mu}g=g*_{\mu}f. $$

Theorem 2. [25] Suppose that f and g are (piecewise) continuous functions over each finite interval $[0,T]$ and are of μ-exponential order $c>0$. Then,

$$ \mathscr{L}_{\mu}\{f*_{\mu}g\}(s)=\mathscr{L}_{\mu}\{f\}(s)\mathscr{L}_{\mu}\{g\}(s). $$

In the present literature, Neumann series of an operator is known as a generalization of the geometric series. Under the conditions such that A is a linear bounded operator on a Banach space and $\|A\|<1$, $(I-A)^{-1}$ is so linear and bounded that the Neumann series of A is convergent and

$$ (I-A)^{-1}=\sum_{k=0}^{\infty}A^k, \tag{4} $$

where I is the unit operator. It is inferred from the reference [26] that

$$ \sum_{i=0}^{\infty}\frac{(\delta)_i}{i!}\frac{A^i}{s^{i\alpha+\beta}}=s^{-\beta}(I-As^{-\alpha})^{-\delta}. \tag{5} $$

which converges when $\mathrm{Re}(s)>\|A\|^{\frac{1}{\alpha}}$ where $s\in\mathbb{C}$, A is linear and bounded operator, $(\delta)_i$ is the Pochhammer symbol, i

$$ (\delta)_0=1,\quad(\delta)_i=\delta(\delta+1)\dots(\delta+i-1),\ \ i=0,1,2,\dots. $$

Theorem 3. [18] A linear operator $\mathscr{Q}_{m,n}^{A,B} \in B(Y,Y)$ for $m,n \in \mathbb{N}$ has the below properties for $A,B \in B(Y,Y)$:

$$\mathscr{Q}_{m,n}^{A,B} = A\mathscr{Q}_{m-1,n}^{A,B} + B\mathscr{Q}_{m,n-1}^{A,B}, \quad m,n \in \mathbb{N},$$

which is called as Pascal's rule.

$$\mathscr{Q}_{m,n}^{A,B} = \binom{m+n}{n} A^m B^n, \quad m,n \in \mathbb{N}-\{0\},$$

where $AB = BA$. Here $\mathscr{Q}_{m,0}^{A,B} = A^m$ and $\mathscr{Q}_{0,n}^{A,B} = B^n$ for $m,n \in \mathbb{N}$ and

$$\mathscr{Q}_{m,n}^{A,B} = \sum_{k=0}^{m} A^{m-k} B \mathscr{Q}_{k,n-1}^{A,B}.$$

3. SOME APPLICATIONS OF THE GENERALIZED LAPLACE TRANSFORMS

In this section, we will examine some of well-known special functions which are expressed by means of a linear bounded matrix and its generalized Laplace transform.

Lemma 4. For $\alpha > 0$, the generalized Laplace transform of the nonnegative increasing function μ^α with $\mu(0) = 0$ is

$$\mathscr{L}_\mu \{\mu^\alpha(t)\}(s) = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}.$$

Proof. Let μ be the nonnegative increasing function μ with $\mu(0) = 0$ and $\alpha > 0$. With the help of the substitution $u = s\mu(t)$, one can easily calculate

$$\mathscr{L}_\mu \{\mu^\alpha(t)\}(s) = \int_0^\infty \mu^\alpha(t) e^{-s\mu(t)} \,\mathrm{d}\mu(t) = \int_0^\infty \frac{u^\alpha}{s^\alpha} e^{-u} \frac{\mathrm{d}u}{s}$$
$$= \frac{1}{s^{\alpha+1}} \int_0^\infty u^{\alpha+1-1} e^{-u} \,\mathrm{d}u = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}},$$
$$\mathrm{Re}(s) > 0. \qquad \square$$

Now, we need to generalise Mittag-Leffler type matrix function of two parameters using a linear bounded matrix and calculate its generalized Laplace transform.

Definition 7. Let $A \in B(Y,Y)$. The general version of two-parameter Mittag-Leffler type matrix function $\mathfrak{E}_{\alpha,\beta}(A,t) : \mathbb{R} \to Y$ is defined by

$$\mathfrak{E}_{\alpha,\beta}(A,t) = t^{\beta-1} \sum_{i=0}^{\infty} A^i \frac{t^{i\alpha}}{\Gamma(i\alpha+\beta)} = t^{\beta-1} E_{\alpha,\beta}(At^\alpha), \ \alpha,\beta \in \mathbb{R}^+.$$

Lemma 5. For any $\alpha,\beta > 0$, $A \in B(Y,Y)$, a nonnegative increasing μ, the Laplace transform of the general version of Mittag-Leffler type function of two parameters $\mathfrak{E}_{\alpha,\beta}(A,t)$ is

$$\mathscr{L}_\mu \{\mathfrak{E}_{\alpha,\beta}(A,\mu(t))\}(s) = s^{-\beta}(I - As^{-\alpha})^{-1}, \tag{6}$$

which holds for $\mathrm{Re}(s) > \|A\|^{\frac{1}{\alpha}}$, where $I \in B(Y,Y)$ is the identity operator.

Definition 8. Let $A \in B(Y,Y)$. The general version of three-parameter Mittag-Leffler type matrix function $\mathfrak{E}_{\alpha,\beta}^{\delta}(A,t) : \mathbb{R} \to Y$ is defined by

$$\mathfrak{E}_{\alpha,\beta}^{\delta}(A,t) = t^{\beta-1} \sum_{i=0}^{\infty} A^i \frac{(\delta)_i}{\Gamma(i\alpha+\beta)} \frac{t^{i\alpha}}{i!} = t^{\beta-1} E_{\alpha,\beta}^{\delta}(At^\alpha),$$

for $\alpha,\beta,\delta \in \mathbb{R}^+$.

Lemma 6. For any $\alpha,\beta,\delta > 0$, $A \in B(Y,Y)$, a nonnegative increasing μ, the Laplace transform of the general version of Mittag-Leffler type function of three parameters $\mathfrak{E}_{\alpha,\beta}^{\delta}(A,t)$ is

$$\mathscr{L}_\mu \left\{\mathfrak{E}_{\alpha,\beta}^{\delta}(A,\mu(t))\right\}(s) = s^{-\beta}(I - As^{-\alpha})^{-\delta},$$

which holds for $\mathrm{Re}(s) > \|A\|^{\frac{1}{\alpha}}$, where $I \in B(Y,Y)$ is the identity operator.

4. A NOVEL EVOLUTIONAL μ-MITTAG-LEFFLER TYPE OPERATOR

In this section, we introduce a novel evolutional μ-Mittag-Leffler type operator which plays a fundamental role for a solution of Sobolev-type μ-fractional evolution equations (1).

Definition 9. Let $A,B \in B(Y,Y)$ be noncommutative. We define an evolutional μ-Mittag-Leffler type operator $\mathscr{X}_{\alpha,\beta,\gamma}^{A,B,h} : \mathbb{R} \to Y$ is given by

$$\mathscr{X}_{\alpha,\beta,\gamma}^{A,B,\mu}(t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \mathscr{Q}_{m,n}^{A,B} \frac{[\mu(t)]^{m\alpha+n\beta+\gamma-1}}{\Gamma(m\alpha+n\beta+\gamma)}, \quad \alpha,\beta,\ \gamma \in \mathbb{R}.$$

In the sequent lemma, we give special cases of matrix equation $\mathscr{Q}_{m,n}^{A,B}$.

Lemma 7. Let $\mathscr{Q}_{m,n}^{A,B}$ be given in Theorem 3.

1. $\mathscr{Q}_{m,n}^{A,B} = B^n$ for $A = \Theta$, $\quad \mathscr{Q}_{m,n}^{A,B} = A^m$ for $B = \Theta$,
2. $\mathscr{Q}_{m,n}^{A,B} = \binom{m+n}{n} B^n$ for $A = I$, $\quad \mathscr{Q}_{m,n}^{A,B} = \binom{m+n}{n} A^m$ for $B = I$,

where Θ is the zero operator and I is the identity operator.

Proof. The proof clearly follows from the definition of $\mathscr{Q}_{m,n}^{A,B}$. $\square$

Now, we will share relationships of the evolutional μ-Mittag-Leffler type operator with known operators(functions) in the present literature, for $\mu(t) = t$.

Lemma 8. [18] The following relations hold true.

1. $\mathscr{X}_{\alpha,\beta,\gamma}^{A,B,\mu}(t) = t^{\gamma-1} \mathscr{E}_{\alpha,\beta,\gamma}^{A,B}(t)$, where the Mittag-Leffler function $\mathscr{E}_{\alpha,\beta,\gamma}^{A,B}(t)$ is given in the reference [18]:

$$\mathscr{E}_{\alpha,\beta,\gamma}^{A,B}(t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \mathscr{Q}_{m,n}^{A,B} \frac{t^{m\alpha+n\beta}}{\Gamma(m\alpha+n\beta+\gamma)}, \quad \alpha,\beta,\ \gamma \in \mathbb{R}.$$

2. $\mathscr{X}^{A,B,\mu}_{\alpha,\beta,\gamma}(t) = t^{\gamma-1}E_{\alpha,\beta,\gamma}\left(At^{\alpha}, Bt^{\beta}\right)$ for $X = Y = \mathbb{R}$, where $E_{\alpha,\beta,\gamma}\left(At^{\alpha}, Bt^{\beta}\right)$ is a bivariate Mittag-Leffler-type function [27]:

$$E_{\alpha,\beta,\gamma}\left(At^{\alpha}, Bt^{\beta}\right) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\binom{k+m}{m}\frac{A^k B^m t^{k\alpha+m\beta}}{\Gamma(k\alpha+m\beta+\gamma)}.$$

Theorem 4. For $k,m \in \mathbb{N}$, the below inequality holds true

$$\left\|\mathscr{Q}^{A,B}_{k,m}\right\| \leq \binom{k+m}{m}\|A\|^k\|B\|^m.$$

Proof. Theorem can be easily proved by mathematical induction. □

5. TRANSFORMATION OF OUR SYSTEM TO A NEW EQUATION

Our aim is to turn Sobolev-type fractional multi-term μ-evolution equation (1) into fractional differential equation (7) with multi-orders whose coefficients are linear bounded operators under some of conditions.

Now, we share three following hypotheses on the linear operators

H_1**:** A_0 is closed, B_0 is bounded,
H_2**:** $D(E) \subseteq D(A_0)$ and E is bijective,
H_3**:** $E^{-1} : Y \to D(E) \subseteq X$ is compact.

Because of H_3, E^{-1} is bounded. Since E^{-1} is bounded and Y is closed (A_0 is closed and Y is complete), by Lemma (Closed operator) [28], E^{-1} is closed ($D(A_0)$ is a closed subset of X). Due to H_2, E^{-1} is bijective and so it has the inverse. Since the inverse of a closed linear operator E^{-1} exists, E is a closed linear operator obtained from the reference [28]. The closed graph theorem guarantees that A_0 is bounded and so $A := A_0E^{-1} : Y \to Y$ is bounded. $B := B_0E^{-1} : Y \to Y$ is also bounded because E^{-1} and B are bounded. Now by applying the substitution $x(t) = E^{-1}z(t)$ which is equivalent to $z(t) = Ex(t)$, we transform Sobolev-type fractional multi-term μ-evolution equation to the following fractional differential equation with linear bounded operators A and B under the conditions H_1–H_3:

$$\begin{cases} \left({}^{C}_{0^+}\mathfrak{D}^{\alpha}_{\mu}z\right)(t) - A\left({}^{C}_{0^+}\mathfrak{D}^{\beta}_{\mu}z\right)(t) - Bz(t) = g(t), & t > 0, \\ z^{(k)}(0) = \phi^{(k)}(0), & k = 0,1,2,\ldots,n-1. \end{cases} \tag{7}$$

One can easily realize that a mild solution of Sobolev-type multi-term μ-fractional evolution equation (1) is the multiplication of E^{-1} and the solution of an initial value problem for fractional μ-differential equation involving linear bounded operators and multi-orders (7). In the similar approach, an exact analytical solution of a linear inhomogeneous version of Sobolev-type multi-term μ-fractional evolution equation (1) is the multiplication of E^{-1} and the explicit solution of a linear inhomogeneous version of an initial value problem for fractional μ-differential equation involving linear bounded operators and multi-orders (7).

It is time to investigate the explicit solution of a linear inhomogeneous version of an initial value problem for fractional μ-differential equation involving linear bounded operators and multi-orders (7).

6. AN EXPLICIT SOLUTION OF THE LINEAR SOBOLEV-TYPE FRACTIONAL NONHOMOGENEOUS EVOLUTION SYSTEM

In this section, we offer two main theorems which provide the mentioned solution to system (7). In each proof of the theorems, different techniques will be exploited. Before giving main theorem, we need some lemmas. Let us start with the following lemma related to exponentially bounded notion.

Theorem 5. Suppose that equation (7) has a unique continuous solution $z(t)$ on $\mathbb{R}^+$. If g and ${}^{C}_{0^+}\mathfrak{D}^{\beta}_{\mu}z$ for $n-2 < \beta \leq n-1$ are μ-exponentially bounded of order c, then the Laplace transformations of $z(t)$ and ${}^{C}_{0^+}\mathfrak{D}^{\alpha}_{\mu}z$ for $n-1 < \alpha \leq n$ exist.

Proof. According to Lemma 3, the generalized Laplace transform of ${}^{C}_{0^+}\mathfrak{D}^{\beta}_{\mu}z$ for $n-2 < \beta \leq n-1$ exists. When we have a look at equation 3, the second term on its right hand-side also exists when considered $z^{(k)}(0)$ for $k = 0,1,\ldots,n$ together with equation (2). Due to equation 3, the generalized Laplace transform of $z(t)$ is available. Since ${}^{C}_{0^+}\mathfrak{D}^{\alpha}_{\mu}z$ for $n-1 < \alpha \leq n$ is the linear combination of the functions ${}^{C}_{0^+}\mathfrak{D}^{\beta}_{\mu}z$, $z(t)$ and g which have their generalized Laplace transforms, the generalized Laplace transform of ${}^{C}_{0^+}\mathfrak{D}^{\alpha}_{\mu}z$ exists due to the linearity of the generalized Laplace integral transformation. The proof is complete. □

Theorem 6. Suppose that equation (7) has a unique continuous solution $z(t)$ on $\mathbb{R}^+$. If g and ${}^{C}_{0^+}\mathfrak{D}^{\alpha}_{\mu}z$ for $n-1 < \alpha \leq n$ are exponentially bounded, then the Laplace transformations of $z(t)$ and ${}^{C}_{0^+}\mathfrak{D}^{\beta}_{\mu}z$ for $n-2 < \beta \leq n-1$ exist.

Proof. The proof is the same as that one of Theorem 5. □

Now we will present two lemmas related to the generalized Laplace transforms of special expressions to make the proof of the first one of the main theorems clearer and shorter.

Lemma 9. If A, B are linear bounded operators which are noncommutative and if $\alpha_1 > \alpha_2 > 0, \alpha_1 > \alpha_3 > 0$, then we have

$$\mathscr{L}^{-1}_{\mu}\left\{s^{\alpha_3}\left[(Is^{\alpha_1} - As^{\alpha_2})^{-1}B\right]^n(Is^{\alpha_1} - As^{\alpha_2})^{-1}\right\} = \sum_{m=0}^{\infty}\mathscr{Q}^{A,B}_{m,n}\frac{[\mu(t)]^{m(\alpha_1-\alpha_2)+n\alpha_1+\alpha_1-\alpha_3-1}}{\Gamma(m(\alpha_1-\alpha_2)+n\alpha_1+\alpha_1-\alpha_3)}.$$

Lemma 10. If noncommutative operators A, B belong to $B(Y,Y)$ and if $\alpha_1 > \alpha_2 > 0, \alpha_1 > \alpha_3 > 0$, then we have

$$\mathscr{L}^{-1}_{\mu}\left\{s^{\alpha_3}\left(Is^{\alpha_1} - As^{\alpha_2} - B\right)^{-1}\right\} = \mathscr{X}^{A,B,\mu}_{\alpha_1-\alpha_2,\alpha_1,\alpha_1-\alpha_3}(t).$$

From now on, unless indicated otherwise we assume that A and B are two matrices which do not have to be commutative, that is, can be commutative or not. It is time to give the first main theorem as follows:

Theorem 7. $A, B \in B(Y,Y)$. Suppose that g and ${}^C_{0^+}\mathfrak{D}^{\beta}_{\mu} z$ for $n-2 < \beta \le n-1$ or ${}^C_{0^+}\mathfrak{D}^{\alpha}_{\mu} z$ for $n-1 < \alpha \le n$ are exponentially bounded and that $\mu \in C^1([0,T])$ such that μ is increasing and $\mu'(t) \neq 0$ for every $t \in [0,T]$. An exact solution formula $z \in AC^n([0,T],Y)$ of Cauchy type system (7) can be given as noted below:

$$z(t) = \sum_{i=0}^{n-2} \left(\frac{\mu^i(t)}{\Gamma(i+1)} + \mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,\alpha+i+1}(t) B \right) \phi^{[i]}_{\mu}(0) \tag{8}$$

$$+ \mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,n}(t) \phi^{[n-1]}_{\mu}(0) + \int_0^t \mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,\alpha} \left(\mu^{-1}(\mu(t)-\mu(s)) \right) g(s)\, \mathrm{d}\mu(s). \tag{9}$$

Proof. By keeping (3) in mind, we put into practice the Laplace integral equation for (7) and so we face the below equality

$$\left(Is^{\alpha} - As^{\beta} - B \right) Z_{\mu}(s) = \sum_{k=0}^{n-2} \left(s^{\alpha-k-1} - As^{\beta-k-1} \right) \phi^{[k]}_{\mu}(0) + s^{\alpha-n} \phi^{[n-1]}_{\mu}(0) + \mathscr{L}_{\mu}\{g(t)\}(s),$$

where $Z_{\mu}(s) = \mathscr{L}_{\mu}\{z(t)\}(s)$. If we arrange the above equality, we get

$$\begin{aligned} Z_{\mu}(s) = &\sum_{k=0}^{n-2} s^{-k-1} \left(1 + (Is^{\alpha} - As^{\beta} - B)^{-1} B \right) \phi^{[k]}_{\mu}(0) \\ &+ s^{\alpha-n} (Is^{\alpha} - As^{\beta} - B)^{-1} \phi^{[n-1]}_{\mu}(0) \\ &+ (Is^{\alpha} - As^{\beta} - B)^{-1} \mathscr{L}_{\mu}\{g(t)\}(s). \end{aligned}$$

From the inverse of μ-Laplace transform [25] to the both sides and Lemmas 4 and 10, one can easily obtain

$$\begin{aligned} z(t) = &\sum_{k=0}^{n-2} \Big(\mathscr{L}^{-1}_{\mu}\{s^{-k-1}\}(t) \\ &+ \mathscr{L}^{-1}_{\mu}\{s^{-k-1}(Is^{\alpha} - As^{\beta} - B)^{-1} B\}(t) \Big) \phi^{[k]}_{\mu}(0) \\ &+ \mathscr{L}^{-1}_{\mu}\{s^{\alpha-n}(Is^{\alpha} - As^{\beta} - B)^{-1}\}(t) \phi^{[n-1]}_{\mu}(0) \\ &+ \mathscr{L}^{-1}_{\mu}\{(Is^{\alpha} - As^{\beta} - B)^{-1} \mathscr{L}_{\mu}\{g(t)\}(s)\}(t), \end{aligned}$$

which is equal to

$$\begin{aligned} z(t) = &\sum_{i=0}^{n-2} \left(\frac{\mu^i(t)}{\Gamma(i+1)} + \mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,\alpha+i+1}(t) B \right) \phi^{[i]}_{\mu}(0) \\ &+ \mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,n}(t) \phi^{[n-1]}_{\mu}(0) + \left(\mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,\alpha} * \mu g \right)(t), \end{aligned}$$

which provides the desired result. □

Note that as stated in Theorem 7, we have assumed that g and ${}^C_{0^+}\mathfrak{D}^{\beta}_{\mu} z$ for $n-2 < \beta \le n-1$ or ${}^C_{0^+}\mathfrak{D}^{\alpha}_{\mu} z$ for $n-1 < \alpha \le n$ are exponentially bounded. They can be regarded as strict conditions, but in the following theorem just after the below lemma we will remove them.

Lemma 11. Let $\mu \in C^1([-h,T])$ such that μ is increasing and $\mu'(t) \neq 0$ for every $t \in [-h,T]$. For $\gamma, \beta > 0$

$$\begin{aligned} &\int_0^t [\mu(t)-\mu(x)]^{\gamma-1} \left(\int_0^x [\mu(x)-\mu(s)]^{\beta-1} g(s)\, \mathrm{d}\mu(s) \right) \mathrm{d}\mu(x) \\ &= \int_0^t [\mu(t)-\mu(s)]^{\gamma+\beta-1} g(s)\, \mathrm{d}\mu(s) B(\gamma,\beta). \end{aligned}$$

Theorem 8. Assume that $\mu \in C^1([-h,T])$ such that μ is increasing and $\mu'(t) \neq 0$ for every $t \in [-h,T]$ and that $A, B \in B(Y,Y)$. An explicit closed solution $z \in AC^n([-h,T],Y)$ of Cauchy type system (7) can be given as noted below:

$$\begin{aligned} z(t) = &\sum_{i=0}^{n-2} \left(\frac{\mu^i(t)}{\Gamma(i+1)} + \mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,\alpha+i+1}(t) B \right) \phi^{[i]}_{\mu}(0) \\ &+ \mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,n}(t) \phi^{[n-1]}_{\mu}(0) \\ &+ \int_0^t \mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,\alpha} \left(\mu^{-1}(\mu(t)-\mu(s)) \right) g(s)\, \mathrm{d}\mu(s), \end{aligned}$$

Proof. To make the proof clearer, we look for a solution both to Cauchy type system (7) with $g = 0$ and to Cauchy type system (7) with the initial condition $z^{(k)}(0) = \phi^{(k)}(0) = 0$, $0 \le k \le n-1$, separately. Now, to make the solution of the linear homogeneous Cauchy type system simple, we set

$$\begin{aligned} z_1(t) &= \sum_{i=0}^{n-2} \left(\frac{\mu^i(t)}{\Gamma(i+1)} + \mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,\alpha+i+1}(t) B \right) \phi^{[i]}_{\mu}(0), \\ z_2(t) &= \mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,n}(t) \phi^{[n-1]}_{\mu}(0), \end{aligned}$$

and calculate the derivatives of them individually.

$$\begin{aligned} &\left({}^C\mathfrak{D}^{\alpha}_{0^+} z_1 \right)(t) \\ &= \sum_{i=0}^{n-2} {}^C_{0^+}\mathfrak{D}^{\alpha}_{\mu} \mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,\alpha+i+1}(t) B \phi^{[i]}_{\mu}(0) \\ &= \sum_{i=0}^{n-2} \mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,i+1}(t) B \phi^{[i]}_{\mu}(0) \\ &= \sum_{i=0}^{n-2} \left(\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \mathscr{Q}^{A,B}_{m,n} \frac{[\mu(t)]^{m(\alpha-\beta)+n\alpha+i}}{\Gamma(m(\alpha-\beta)+n\alpha+i+1)} \right) B \phi^{[i]}_{\mu}(0) \\ &= \sum_{i=0}^{n-2} \left(\frac{\mu^i(t)}{\Gamma(i+1)} + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} A \mathscr{Q}^{A,B}_{m-1,n} \frac{[\mu(t)]^{m(\alpha-\beta)+n\alpha+i}}{\Gamma(m(\alpha-\beta)+n\alpha+i+1)} \right. \\ &\quad \left. + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} B \mathscr{Q}^{A,B}_{m,n-1} \frac{[\mu(t)]^{m(\alpha-\beta)+n\alpha+i}}{\Gamma(m(\alpha-\beta)+n\alpha+i+1)} \right) B \phi^{[i]}_{\mu}(0) \\ &= \sum_{i=0}^{n-2} \left(B \frac{\mu^i(t)}{\Gamma(i+1)} + A \mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,\alpha-\beta+i+1}(t) B \right) \phi^{[i]}_{\mu}(0) \\ &\quad + B \sum_{i=0}^{n-2} \mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,\alpha+i+1}(t) B \phi^{[i]}_{\mu}(0). \end{aligned}$$

$$
\begin{aligned}
&\left({}^{C}\mathfrak{D}^{\alpha}_{0^+} z_2\right)(t)\\
&= {}^{C}\mathfrak{D}^{\alpha}_{0^+} \mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,n}(t)\phi^{[n-1]}_{\mu}(0)\\
&= {}^{C}\mathfrak{D}^{\alpha}_{0^+}\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathscr{Q}^{A,B}_{i,j}\frac{[\mu(t)]^{i(\alpha-\beta)+j\alpha+n-1}}{\Gamma(i(\alpha-\beta)+j\alpha+n)}\right)\phi^{[n-1]}_{\mu}(0)\\
&= {}^{C}\mathfrak{D}^{\alpha}_{0^+}\left(\frac{\mu^{n-1}(t)}{\Gamma(n)}+\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}A\mathscr{Q}^{A,B}_{i-1,j}\frac{[\mu(t)]^{i(\alpha-\beta)+j\alpha+n-1}}{\Gamma(i(\alpha-\beta)+j\alpha+n)}\right.\\
&\quad\left.+\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}B\mathscr{Q}^{A,B}_{i,j-1}\frac{[\mu(t)]^{i(\alpha-\beta)+j\alpha+n-1}}{\Gamma(i(\alpha-\beta)+j\alpha+n)}\right)\phi^{[n-1]}_{\mu}(0)\\
&= A\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\mathscr{Q}^{A,B}_{i-1,j}\frac{[\mu(t)]^{i(\alpha-\beta)+j\alpha+n-\alpha-1}}{\Gamma(i(\alpha-\beta)+j\alpha+n-\alpha)}\phi^{[n-1]}_{\mu}(0)\\
&\quad+B\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\mathscr{Q}^{A,B}_{i,j-1}\frac{[\mu(t)]^{i(\alpha-\beta)+j\alpha+n-\alpha-1}}{\Gamma(i(\alpha-\beta)+j\alpha+n-\alpha)}\phi^{[n-1]}_{\mu}(0)\\
&= A\mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,n-\beta}(t)\phi^{[n-1]}_{\mu}(0)+B\mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,n}(t)\phi^{[n-1]}_{\mu}(0)
\end{aligned}
$$

We also easily compute

$$
\begin{aligned}
{}^{C}\mathfrak{D}^{\beta}_{0^+}\left(\sum_{k=1}^{2}z_k\right)(t) &= \sum_{i=0}^{n-2}\left(\mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,\alpha-\beta+i+1}(t)B\right)\phi^{[i]}_{\mu}(0)\\
&\quad+\mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,n-\beta}(t)\phi^{[n-1]}_{\mu}(0),
\end{aligned}
$$

and

$$
\begin{aligned}
\left(\sum_{k=1}^{2}z_k\right)(t) &= \sum_{i=0}^{n-2}\left(\frac{[\mu(t)]^{i}}{\Gamma(i+1)}+\mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,\alpha+i+1}(t)B\right)\phi^{[i]}_{\mu}(0)\\
&\quad+\mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,n}(t)\phi^{[n-1]}_{\mu}(0).
\end{aligned}
$$

It can be observed that

$$
{}^{C}\mathfrak{D}^{\alpha}_{0^+}\left(\sum_{k=1}^{2}z_k\right)(t)-A\,{}^{C}\mathfrak{D}^{\beta}_{0^+}\left(\sum_{k=1}^{2}z_k\right)(t)-B\left(\sum_{k=1}^{2}z_k\right)(t)=0.
$$

It is time to show that

$$
z_3(t)=\int_0^t \mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,\alpha}\left(\mu^{-1}(\mu(t)-\mu(s))\right)g(s)\,\mathrm{d}\mu(s)
$$

is a solution to Cauchy type system (7) with the initial condition $z^{(k)}(0)=\phi^{(k)}(0)=0$, $0\le k\le n-1$. To show this, we use the variation of constants technique. The structure of the solution is as in the following form

$$
z_3(t)=\int_0^t \mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,\alpha}\left(\mu^{-1}(\mu(t)-\mu(s))\right)\vartheta(s)\,\mathrm{d}\mu(s),
$$

where $\vartheta(s)$ is unknown for $s\in[0,t]$. Considering the well-known relation in Theorem 1 and the initial condition $z^{(k)}(0)=\phi^{(k)}(0)=0$, $0\le k\le n-1$ with the identity (2), we get

$$
{}^{C}_{0^+}\mathfrak{D}^{\alpha}_{\mu}z_3(t)={}^{RL}_{0^+}\mathfrak{D}^{\alpha}_{\mu}z_3(t),
$$

let us go on the calculation

$$
\begin{aligned}
&= {}^{RL}_{0^+}\mathfrak{D}^{\alpha}_{\mu}\left(\int_0^t\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathscr{Q}^{A,B}_{i,j}\frac{(\mu(t)-\mu(s))^{i(\alpha-\beta)+j\alpha+\alpha-1}}{\Gamma(i(\alpha-\beta)+j\alpha+\alpha)}\vartheta(s)\,\mathrm{d}\mu(s)\right)\\
&= A\,{}^{RL}_{0^+}\mathfrak{D}^{\alpha}_{\mu}\left(\int_0^t\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\mathscr{Q}^{A,B}_{i-1,j}\frac{(\mu(t)-\mu(s))^{i(\alpha-\beta)+j\alpha+\alpha-1}}{\Gamma(i(\alpha-\beta)+j\alpha+\alpha)}\vartheta(s)\,\mathrm{d}\mu(s)\right)\\
&\quad+B\,{}^{RL}_{0^+}\mathfrak{D}^{\alpha}_{\mu}\left(\int_0^t\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\mathscr{Q}^{A,B}_{i,j-1}\frac{(\mu(t)-\mu(s))^{i(\alpha-\beta)+j\alpha+\alpha-1}}{\Gamma(i(\alpha-\beta)+j\alpha+\alpha)}\vartheta(s)\,\mathrm{d}\mu(s)\right)\\
&\quad+{}^{RL}_{0^+}\mathfrak{D}^{\alpha}_{\mu}\left({}^{RL}_{0^+}\mathfrak{I}^{\alpha}_{\mu}\vartheta\right)(t)\\
&= A\,{}^{RL}_{0^+}\mathfrak{D}^{\alpha}_{\mu}\left(\int_0^t\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathscr{Q}^{A,B}_{i,j}\frac{(\mu(t)-\mu(s))^{i(\alpha-\beta)+j\alpha+2\alpha-\beta-1}}{\Gamma(i(\alpha-\beta)+j\alpha+2\alpha-\beta)}\vartheta(s)\,\mathrm{d}\mu(s)\right)\\
&\quad+B\,{}^{RL}_{0^+}\mathfrak{D}^{\alpha}_{\mu}\left(\int_0^t\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathscr{Q}^{A,B}_{i,j}\frac{(\mu(t)-\mu(s))^{i(\alpha-\beta)+j\alpha+2\alpha-1}}{\Gamma(i(\alpha-\beta)+j\alpha+2\alpha)}\vartheta(s)\,\mathrm{d}\mu(s)\right)\\
&\quad+{}^{RL}\mathfrak{D}^{\alpha}_{0^+}\left({}^{RL}\mathfrak{I}^{\alpha}_{0^+}\vartheta\right)(t).
\end{aligned}
$$

By applying the μ-Riemann Liouville type fractional derivative and Lemma 11, we reach to

$$
\begin{aligned}
&= A\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathscr{Q}^{A,B}_{i,j}\left(\frac{d}{d\mu(t)}\right)^{n}\\
&\quad\int_0^t\frac{(\mu(t)-\mu(s))^{i(\alpha-\beta)+j\alpha+\alpha+n-\beta-1}}{\Gamma(i(\alpha-\beta)+j\alpha+\alpha+n-\beta)}\vartheta(s)\,\mathrm{d}\mu(s)\\
&\quad+B\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathscr{Q}^{A,B}_{i,j}\left(\frac{d}{d\mu(t)}\right)^{n}\\
&\quad\int_0^t\frac{(\mu(t)-\mu(s))^{i(\alpha-\beta)+j\alpha+\alpha+n-1}}{\Gamma(i(\alpha-\beta)+j\alpha+\alpha+n)}\vartheta(s)\,\mathrm{d}\mu(s)+\vartheta(t).
\end{aligned}
$$

By using Leibniz rule for higher order derivatives under a integral sign, we acquire

$$
\begin{aligned}
&\left({}^{C}\mathfrak{D}^{\alpha}_{0^+}z_3\right)(t)\\
&\quad= A\int_0^t\mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,\alpha-\beta}\left(\mu^{-1}(\mu(t)-\mu(s))\right)\vartheta(s)\,\mathrm{d}\mu(s)+\vartheta(t)\\
&\quad\quad+B\int_0^t\mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,\alpha}\left(\mu^{-1}(\mu(t)-\mu(s))\right)\vartheta(s)\,\mathrm{d}\mu(s).
\end{aligned}
$$

Similarly, we have

$$
A\left({}^{C}\mathfrak{D}^{\beta}_{0^+}z_3\right)(t)=A\int_0^t\mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,\alpha-\beta}\left(\mu^{-1}(\mu(t)-\mu(s))\right)\vartheta(s)\,\mathrm{d}\mu(s),
$$

and

$$
Bz_3(t)=B\int_0^t\mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,\alpha}\left(\mu^{-1}(\mu(t)-\mu(s))\right)\vartheta(s)\,\mathrm{d}\mu(s).
$$

So, linear combinations of above last three results yield that

$$\left({}_{0^+}^{C}\mathfrak{D}_{\mu}^{\alpha} z_3\right)(t) - A\left({}_{0^+}^{C}\mathfrak{D}_{\mu}^{\beta} z_3\right)(t) - Bz_3(t) = \vartheta(t).$$

Therefore, $\vartheta(t) = g(t)$ which is what we want to show. □

Now, we can share an exact analytical closed solution formula of Sobolev-type multi-term μ-evolution equations with general fractional orders in the coming theorem.

Theorem 9. Assume that $\mu \in C^1([-h,T])$ such that μ is increasing and $\mu'(t) \neq 0$ for every $t \in [-h,T]$ and that A, $B \in B(Y,Y)$. An exact analytical closed solution formula $x \in AC^n([-h,T],X)$ of Cauchy type system (1) can be given as noted below:

$$\begin{aligned} x(t) &= \sum_{i=0}^{n-2} E^{-1}\left(\frac{\mu^i(t)}{\Gamma(i+1)} + \mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,\alpha+i+1}(t)B\right)\phi_\mu^{[i]}(0) \\ &\quad + E^{-1}\mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,n}(t)\phi_\mu^{[n-1]}(0) \\ &\quad + \int_0^t E^{-1}\mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,\alpha}\left(\mu^{-1}(\mu(t)-\mu(s))\right)g(s)\,\mathrm{d}\mu(s), \quad t > 0. \end{aligned}$$

Proof. It immediately follows from applying the substitution $z(t) = E^{-1}x(t)$ to Theorem 8 □

7. SPECIAL CASES

In this section, we present special cases depending on special choices of the existent parameters in (1).

Example 1. Let us consider the Sobolev-type multi-term μ-evolution equations (1) of orders $1 < \alpha \leq 2$, $0 < \beta \leq 1$. In this case, Theorem 9 could be rewritten as follows.

Proposition 1. Assume that $\mu \in C^1([0,T])$ such that μ is increasing and $\mu'(t) \neq 0$ for every $t \in [0,T]$ and that $A, B \in B(Y,Y)$. An explicit solution formula of the initial value system (1) with $n = 2$ has the form

$$\begin{aligned} z(t) &= \left(E^{-1} + E^{-1}\mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,\alpha+1}(t)B\right)\phi(0) \\ &\quad + E^{-1}\mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,n}(t)\frac{\phi'(0)}{\mu'(0)} \\ &\quad + \int_0^t E^{-1}\mathscr{X}^{A,B,\mu}_{\alpha-\beta,\alpha,\alpha}\left(\mu^{-1}(\mu(t)-\mu(s))\right)g(s)\,\mathrm{d}\mu(s), \quad t > 0, \end{aligned}$$

where $\phi(0) = \phi_\mu^{[0]}(0)$, $\phi'(0)/\mu'(0) = \phi_\mu^{[1]}(0)$.

Example 2. Let us consider the Sobolev-type multi-term evolution equations (1) with $\mu(t) = t$

$$\begin{cases} {}_{0^+}^{C}\mathfrak{D}_t^{\alpha}(Ex)(t) - A_0\left({}_{0^+}^{C}\mathfrak{D}_t^{\beta}x\right)(t) - B_0x(t) = g(t), \ t \in (0,T], \\ Ex(0) = \phi(0), \ \ (Ex(0))^{(k)} = \phi^{(k)}(0), \quad k = 1,2,\dots,n-1. \end{cases} \tag{10}$$

Proposition 2. Let $A, B \in B(Y,Y)$. An explicit solution formula of the initial value system (10) has the form

$$\begin{aligned} x(t) &= \sum_{i=0}^{n-2}\left(E^{-1}\frac{t^i}{\Gamma(i+1)} + E^{-1}\mathscr{X}^{A,B,t}_{\alpha-\beta,\alpha,\alpha+i+1}(t)B\right)\phi_0^{(i)} \\ &\quad + \int_0^t E^{-1}\mathscr{X}^{A,B,t}_{\alpha-\beta,\alpha,\alpha}(t-s)g(s)\,\mathrm{d}s + E^{-1}\mathscr{X}^{A,B,t}_{\alpha-\beta,\alpha,n}(t)\phi_0^{(n-1)}, \end{aligned}$$

for $t > 0$, and $\phi_0^{(k)} = \phi^{(k)}(0)$, $k = 0,1,\dots,n-1$.

8. CONCLUSIONS AND FUTURE WORK

We introduced the Sobolev-type multi-term μ-fractional evolution with generalized fractional orders with respect to another function and presented an analytical form of solution of (7).

One can expect the findings of the current paper to hold for a class of problems governed by different type of fractional evolution systems such as μ-Hilfer type, Hadamard type fractional differential equations. All possibilities as noted above can be questioned once again for this new system.

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