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# On applications of quasi-Abelian Cayley graphs to Denial-of-Service protection

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**Abstract.** This paper addresses the problem of designing secure control for networked multi-agent systems (MASs) under Denial-of-Service (DoS) attacks. We propose a constructive design method based on the interaction topology. The MAS with a non-attack communication topology, modeled by quasi-Abelian Cayley graphs subject to DoS attacks, can be represented as a switched system. Using switching theory, we provide easily applicable sufficient conditions for the networked MAS to remain asymptotically stable despite DoS attacks. Our results are applicable to both continuous-time and discrete-time systems, as well as to discrete-time systems with variable steps or systems that combine discrete and continuous times.

**Key words:** multi-agent system; Denial-of-Service attack; asymptotic stability; switched systems; quasi-Abelian Cayley graphs

## 1. INTRODUCTION

The advent of powerful communication technologies has led to rapid development in cyber-physical systems, integrating physical plants with communication networks and computational devices. However, reliance on communication networks underscores the need to design control algorithms resilient against malfunctioning communication links. Periodic interruptions of communication links may serve as models of malicious attacks, where adversaries eliminate some communication links. This phenomenon is known as a Denial-of-Service (DoS) attack (see, e.g., [1] and references therein). Thus, there is a need for control algorithms capable of achieving synchronization even in the presence of temporarily unavailable communication links.

In recent years, networked security control subject to DoS attacks has garnered considerable attention (see [2–10] and references therein). Random DoS attacks have been considered in [4, 9, 11], while multi-agent systems subject to asynchronous Denial-of-Service attacks have been studied in [2, 5]. A dwell time approach [6, 7] and an event-triggered mechanism [3, 8, 10] have become effective strategies for researching secure control under DoS attacks. In [12–15], a switching approach was proposed, wherein the authors dealt with the presence of DoS attacks by considering two modes: stable and unstable. The control mechanisms designed in those papers can effectively mitigate the impact of frequency- and duration-limited DoS attacks.

In this paper, we propose a switching approach to address the problem of designing secure control for networked multi-agent systems (MASs) under DoS attacks. Unlike previous works, we present constructive design conditions based on the interaction topology of MASs. Specifically, we assume that the topology of the original network of agents is described by a quasi-Abelian Cayley graph that is undirected and connected. This assumption enables us to consider various models of DoS attacks, including stochastic and deterministic, synchronous (where all communication links are simultaneously disrupted) and asynchronous (where only part of the communication links are disrupted each time). Moreover, since the attacker's behavior is unpredictable, we do not impose any limitations on the frequency and duration of DoS attacks.

A characteristic feature of communication networks is data transmission in packets, where flow is not continuous. Thus, time intervals between consecutive moments of sending/receiving data arise. Furthermore, these time intervals are not necessarily constant. If such a network is used for control, the control algorithm must ensure satisfactory behavior even when receiving data at non-uniform time intervals. This design requirement motivated our consideration of MASs with continuous-time and discrete-time subsystems. Therefore, our results apply to continuous-time and discrete-time systems, as well as to discrete-time systems with variable steps or systems that are combinations of discrete

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and continuous times.

The main contributions of the paper are summarized as follows. Firstly, in Section 3, we provide sufficient conditions for switched systems with continuous-time and discrete-time subsystems to be asymptotically stable. This result generalizes Theorem 1 proved in [16] and is consistent with results presented in [17, 18], however, we do not use the time scale machinery and the positively regressivity assumption. Secondly, in Section 4, a switched system model is built to characterize the behavior of MAS under DoS attacks. In Theorems 18 and 20, conditions for the asymptotic stability of MASs under DoS attacks are given using switched systems. We emphasize that the traditional switching topologies differ from the changes caused by DoS attacks. Namely, switching caused by DoS attacks occurs among the original graph, which describes non-attack communication topology, and subgraphs describing different attack modes. Next, in Section 5, MASs with non-attack communication topology modeled by quasi-Abelian Cayley graphs are discussed. In this case, simple conditions ensuring the asymptotic stability of MASs despite DoS attacks are provided. Further, through examples, we explain a design method for MASs that achieve asymptotic stability in a malicious DoS attack environment. The basic idea is that the original non-attacked networked MAS should mimic the net of edges in a respective quasi-Abelian Cayley graph $G$. If any communication channel in any quasi-Abelian Cayley subgraph $\tilde{G}$ of graph $G$ is attacked, the automatic security system should turn off the whole block of physical channels modeled by subgraph $\tilde{G}$. In such a case, as long as the attacks stay within the block of communication links modeled by one of the subgraphs of $G$, the networked MAS will continue to work in an asymptotically stable manner. Since for any natural number $N$, there exists a quasi-Abelian Cayley graph with $N$ nodes, the method described above could be applied to any multi-agent system with $N$-agents.

## 2. PRELIMINARIES

In this paper, the information exchange among agents is modeled by undirected graphs and the systems are considered on arbitrary time domains. Therefore for a convenience of a reader we recall some notions and facts from graph theory and time scale calculus. Let $G = (V,E)$ be a weighted communication graph of $n$ agents, with the set of nodes (vertices) $V = \{v_1, v_2, ..., v_n\}$ and the set of edges $E \subseteq V \times V$. Each edge, denoted by $(i, j)$, means that there is information flow from agent $j$ to agent $i$. Matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ such that, $a_{ij} = 1$ if $(i, j) \in E$ and $a_{ij} = 0$ otherwise, we call the adjacency matrix. Matrix $L = [l_{ij}] \in \mathbb{R}^{n\times n}$ with $l_{ii} = \sum_{i\neq j} a_{ij}$ and $l_{ij} = -a_{ij}$, $i \neq j$, is called the graph Laplacian matrix induced by the topology $G$. By construction, $L$ has at least one zero eigenvalue with a corresponding eigenvector $1_n = [1, \dots, 1]^T$. If for a certain graph $G$ we have that $(j,i) \in E$ for every $(i, j) \in E$, then the graph is called undirected. Clearly, for an undirected graph matrices $A$ and $L$ are symmetric. Let $spec(L) := \{\lambda_j : j = 1,...,n\}$ be the set of all eigenvalues of $L$, which are ordered $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ for an undirected graph $G$. An undirected graph is connected if there exists a path between any two distinct nodes. For a connected graph we have $\lambda_2 > 0$, $\lambda_2 \in spec(L)$.

In order to analyze a special (practical) case of our main result we need the following definitions and facts.

**Definition 1.** *[19] Let $H$ be a finite group and let $S \subseteq H$ be a subset. The corresponding Cayley graph $Cay(H,S)$ has vertex set equal to $H$. Two vertices $(g,h) \in H$ are joined by a directed edge from $g$ to $h$ if and only if there exists $s \in S$ such that $g = sh$. Each edge is labeled to denote that it corresponds to $s \in S$. If $G$ is a graph such that there exists a group $H$ and generating set $S \subseteq H$ with $G \cong Cay(H,S)$, then $G$ is said to be Cayley.*

**Remark 2.** *[19] The Cayley graph $Cay(H,S)$ of a group $H$ is undirected if and only if $S = S^{-1}$. Moreover, if $S$ generates $H$, then the labeled Cayley graph $Cay(H,S)$ uniquely determines $H$.*

**Example 3.** [19] If $H = Z/nZ$ and $S = \{1,-1\}$ then $Cay(H,S)$ is the cycle on $n$ vertices.

In what follows we assume that $S$ does not contain the identity, so that $Cay(H,S)$ does not contain any loops. Under this assumption $Cay(H,S)$ is a connected and undirected regular graph of degree $|S|$ on $|H|$ vertices (without loops). Inside a single group one can often find different sets of generators with the same numbers of elements.

A Cayley graph $Cay(H,S)$ is called quasi-Abelian if $S$ is the union of some conjugacy of $H$.

**Theorem 4.** *[20] All quasi-Abelian Cayley graphs on finite group $H$ have a common basis of eigenfunctions and hence their Laplacian matrices commute.*

## 3. SWITCHED SYSTEMS WITH CONTINUOUS-TIME AND DISCRETE-TIME SUBSYSTEMS

Let us consider the switched system that is composed of continuous-time subsystem

$$\dot{x}(t) = A_c x(t) \tag{1}$$

and discrete-time subsystem

$$\Delta x(t) = A_d x(t), \tag{2}$$

where $x(t) \in \mathbb{R}^N$ is the state, $A_c, A_d \in \mathbb{R}^{N\times N}$. We do not assume that the sampling period of system (2) is constant, it can be a function of time. Moreover, we do not fix any switching law, that is, arbitrary switching is possible for the switched system composed of subsystems (1) and (2).
In the sequel, $\mathbb{T}$ denotes the time domain of the considered switched systems. Clearly, $\mathbb{T} \subseteq \mathbb{R}$. We assume that $0 \in \mathbb{T}$ is the initial time, and $\mathbb{T}$ is unbounded above and closed.

**Definition 5.** *The switched system composed of subsystems (1) and (2) is asymptotically stable on $\mathbb{T}$ if for any initial condition $x(0) = x_0$ the solution $\lim_{t\to\infty} \|x(t)\| = 0$.*

To study the stability of switched system composed of (1) and (2) we adopt notions known from time scales theory [21].

**Definition 6.** *Let $t \in \mathbb{T}$. The forward jump operator $\sigma(t) : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : t < s\}$.*

**Definition 7.** *The graininess function $\mu(t) : \mathbb{T} \to [0,\infty]$ is defined by $\mu(t) := \sigma(t) - t$, for all $t \in \mathbb{T}$.*

Let us observe that for continuous-time system (1), $\sigma(t) = t$ and $\mu(t) = 0$, while for discrete-time system (2), $\sigma(t) > t$ and $\mu(t) > 0$, respectively.

With the introduced notions we may write subsystem (2) in the equivalent form

$$x(\sigma(t)) = (I + \mu(t)A_d)x(t), \tag{3}$$

where $I$ is an identity matrix of dimension $N \times N$.

**Example 8.** Let $a$ and $b$ be positive numbers, and consider the switched system consisting of

$$\dot{x}(t) = A_c x(t), \quad t \in \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a[ \tag{4}$$

and

$$\Delta x(t) = A_d x(t) \quad t \in \bigcup_{k=0}^{\infty} \{k(a+b)+a\}. \tag{5}$$

Then $\mathbb{T} = \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a]$,

$$\mu(t) = \begin{cases} 0 & \text{if} \quad t \in \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a[ \\ b & \text{if} \quad t \in \bigcup_{k=0}^{\infty} \{k(a+b)+a\}, \end{cases}$$

and $x(t+b) = (I + bA_d)x(t)$ for $t \in \bigcup_{k=0}^{\infty} \{k(a+b)+a\}$.

We make the following five assumptions on subsystems (1) and (3).
Assumption 1: $A_c A_d = A_d A_c$.

Let us observe that under Assumption 1 it holds

$$e^{A_c t}(1 + \mu(t)A_d) = (1 + \mu(t)A_d)e^{A_c t} \tag{6}$$

for any scalar $t \in \mathbb{T}$.

Assumption 2: $A_c = A_c^T$ and $A_d = A_d^T$.
Observe that under Assumption 2 matrices $A_c$ and $A_d$ are diagonalizable, and all the eigenvalues of those matrices are real.
Assumption 3: The graininess function of system (3) fulfils $0 < \mu_{min} \leq \mu(t) \leq \mu_{max}$, where $\mu_{min} := \min_{t\in\mathbb{T}}\{\mu(t)\}$ and $\mu_{max} := \max_{t\in\mathbb{T}}\{\mu(t)\}$.

Let us denote $spec(A_c) = \{\lambda_j^c : j = 1,\ldots,N\}$ and $spec(A_d) = \{\lambda_j^d : j = 1,\ldots,N\}$.
Assumption 4: $A_c$ is Hurwitz stable, that is $\lambda_j^c < 0$, $j = 1,\ldots,N$.

In order to provide stability conditions for discrete-time subsystem (3) we have to analyze the values: $|1 + \mu(t)\lambda_j^d|$, $j = 1,\ldots,N$, for $t \in \mathbb{T}$ with $\mu(t) > 0$. To this end we consider the function $m(c) := |1 + c\lambda^d|$, $c \in [\mu_{min}, \mu_{max}]$. Observe that:

i) If $\lambda^d \geq 0$, then $\max_{\mu_{min}\leq c\leq\mu_{max}}\{|1+c\lambda^d|\} = |1+\mu_{max}\lambda^d|$.
ii) If $\lambda^d < 0$ and $\mu_{min} \geq -\frac{1}{\lambda^d}$, then $\max_{\mu_{min}\leq c\leq\mu_{max}}\{|1+c\lambda^d|\} = |1+\mu_{max}\lambda^d|$.
iii) If $\lambda^d < 0$ and $\mu_{min} < -\frac{1}{\lambda^d}$, then $\max_{\mu_{min}\leq c\leq\mu_{max}}\{|1 + c\lambda^d|\} = \max\{|1 + \mu_{min}\lambda^d|, |1+\mu_{max}\lambda^d|\}$.

Assumption 5: $A_d$ is stable, that is $\max_{\lambda_j^d\in spec(A_d)} \max_{\mu_{min}\leq c\leq\mu_{max}}\{|1+c\lambda_j^d|\} = \delta < 1$.

**Lemma 9.** *[22] Let F be a commuting family of diagonalizable linear operators on the finite-dimensional vector space $V$. There exists an ordered basis for $V$ such that every operator in F is represented in that basis by a diagonal matrix.*

**Lemma 10.** *[23] If two matrices commute with each other, every non degenerate eigenvector of one is also an eigenvector of the other.*

**Lemma 11.** *[24] If $F \subset \mathbb{R}^{n\times n}$ is a commuting family of matrices, then there is a vector $v \in \mathbb{C}^n$ that is an eigenvector of every $A \in F$.*

The following Theorem 12 and Theorem 13 provide sufficient conditions for switched systems with continuous-time (1) and discrete-time subsystems (3) to be asymptotically stable.

**Theorem 12.** *Under Assumptions 1–5, the switched system composed of subsystems* (1) *and* (3) *is asymptotically stable on* $\mathbb{T}$.

*Proof.* First let us observe that for any time $\mathbb{T} \ni t > 0$ we can divide the interval $[0,t]$ in such a way that $t = t_c + \sum_{i=1}^{m} \mu(t_i)$, where $t_c$ is the total duration time for continuous-time subsystems on $[0,t]$ and $\sum_{i=1}^{m} \mu(t_i)$ is the total duration time for discrete-time subsystems on the interval $[0,t]$. Furthermore, according to Assumptions 1 and 2, and by Lemmas 9, 10, and 11, one can express the solution to the considered switched system in the form

$$x(t) = x_0 \sum_{j=1}^{N} c_j v_j e^{\lambda_j^c t_c} \prod_{i=1}^{m} (1 + \mu(t_i)\lambda_j^d), \tag{7}$$

where $x_0 = x(0)$, $c_j$ are constants, $v_j$ are shared eigenvectors corresponding to $\lambda_j^c$ and $\lambda_j^d$. Thus

$$\|x(t)\| \le \|x_0\| \sum_{j=1}^{N} \|c_j\| \|v_j\| e^{\lambda_j^c t_c} \prod_{i=1}^{m} |1 + \mu(t_i)\lambda_j^d|. \tag{8}$$

Now, let us observe that for the fixed $j$ and according to Assumptions 4 and 5, we have

$$e^{\lambda_j^c t_c} \prod_{i=1}^{m} |1 + \mu(t_i)\lambda_j^d| \le e^{\lambda_j^c t_c} \delta^m = \\ = e^{\lambda_j^c t_c} e^{\sum_{i=1}^{m} \mu(t_i) \frac{\ln \delta}{\mu_{min}}} \le e^{\lambda_j t},$$

where $\lambda_j = \max\{\lambda_j^c, \frac{\ln \delta}{\mu_{min}}\} < 0$. Proceeding analogously for all $j \in \{1,\ldots,N\}$, we get

$$\|x(t)\| \le \|x_0\| \sum_{j=1}^{N} \|c_j\| \|v_j\| e^{\lambda_j t},$$

where $\lambda_j < 0$ for all $j = 1,\ldots,N$. Therefore, $\lim_{t\to\infty} \|x(t)\| = 0$. This completes the proof. □

**Theorem 13.** *Let* $\mu(t_i) = a_i$. *If* $\sum_{i=1}^{\infty} a_i < \infty$, *then under Assumptions 1–4, the switched system composed of subsystems* (1) *and* (3) *is asymptotically stable on* $\mathbb{T}$.

*Proof.* Similarly to the proof of Theorem 12 the solution to the considered switched system is given by (7) and inequality (8) holds. Now, let us observe that for the fixed $j$ we have

$$e^{\lambda_j^c t_c} \prod_{i=1}^{m} |1 + \mu(t_i)\lambda_j^d| \le \\ e^{\lambda_j^c t_c} \left( \max_{\mu_{min} \le c \le \mu_{max}} \{|1 + c\lambda_j^d|\} \right)^m \\ = e^{\lambda_j^c t} e^{\sum_{i=1}^{m} \mu(t_i)(-\lambda_j^c + \frac{\ln \max_{\mu_{min} \le c \le \mu_{max}} \{|1+c\lambda_j^d|\}}{\mu_{min}})}.$$

Proceeding analogously for all $j \in \{1,\ldots,N\}$, we get

$$\|x(t)\| \le \|x_0\| \sum_{j=1}^{N} \|c_j\| \|v_j\| e^{\lambda_j^c t} \cdot \\ \cdot e^{\sum_{i=1}^{m} \mu(t_i)(-\lambda_j^c + \frac{\ln \max_{\mu_{min} \le c \le \mu_{max}} \{|1+c\lambda_j^d|\}}{\mu_{min}})}.$$

Since $\lambda_j^c < 0$, $j = 1,\ldots,N$, and $\sum_{i=1}^{\infty} \mu(t_i) < \infty$, it follows that $\lim_{t\to\infty} \|x(t)\| = 0$. This completes the proof. □

**Example 14.** We consider the switched system composed of

$$\dot{x}(t) = A_c x(t), \quad t \in \bigcup_{k=1}^{\infty} [2^k + \frac{1}{2^k}, 2^{k+1}[ \tag{9}$$

and

$$\Delta x(t) = A_d x(t) \quad t \in \{0\} \cup \bigcup_{k=1}^{\infty} \{2^{k+1}\}. \tag{10}$$

Then $\mathbb{T} = \{0\} \cup \bigcup_{k=1}^{\infty} [2^k + \frac{1}{2^k}, 2^{k+1}]$ and

$$\mu(t) = \begin{cases} 0 & \text{if} \quad t \in \bigcup_{k=1}^{\infty} [2^k + \frac{1}{2^k}, 2^{k+1}[ \\ \frac{1}{2^{k+1}} & \text{if} \quad t \in \bigcup_{k=1}^{\infty} \{2^{k+1}\} \\ 2\frac{1}{2} & \text{if} \quad t = 0. \end{cases}$$

Then one gets $\sum_{i=1}^{\infty} \mu(t_i) = 2\frac{1}{2} + \sum_{i=1}^{\infty} \frac{1}{2^{i+1}} < \infty$.

Note that, Theorems 12 and 13 evidently show that the stability of the discrete-time subsystem essentially depends on the size of the discrete steps. In consequence, when the discrete step size is variable in time, the stability conditions strongly depend on $\mu_{min}$ and $\mu_{max}$.

**Remark 15.** *Theorems 12 and 13 remain true for more than two subsystems satisfying assumptions of those theorems.*

## 4. DOS ATTACKS

Let $\mathbb{T} \subseteq \mathbb{R}$ be a time-domain such that $0 \in \mathbb{T}$, and $\mathbb{T}$ is unbounded above and closed. Consider a set of $N$ agents interacting over a communication network. The interaction topology of the network is represented by undirected graph $\mathscr{G} = (V, E)$ with the set

of nodes $V = \{1,2,\dots,N\}$ and edges $E \subseteq V \times V$. We are interested in a very general situation where DoS attacks could occur on some or all transmission channels at any time.
Let us define:

- $\Psi_{(i,j)}[a,b]_{\mathbb{T}}$, $i<j$, as the union of moments of DoS attacks on edge (channel) $(i,j) \in E$ over $[a,b]_{\mathbb{T}}$;
- $\Upsilon(t) := \{(i,j) \in E \mid t \in \Psi_{(i,j)}[0,\infty[_{\mathbb{T}}\}$ as the set of edges (channels) which are attacked at time $t$.

Here and subsequently $[a,b]_{\mathbb{T}} := [a,b] \cap \mathbb{T}$.

**Remark 16.** *Since edges $(i,j)$ and $(j,i)$ are seen as one channel, only edge $(i,j)$ with $i<j$ is considered and $\Psi_{(i,j)} = \Psi_{(j,i)}$.*

Let $\Theta$ be a set of all subsets of the set of all connections between every two different nodes in graph $\mathscr{G}$. In other words, setting $\mathscr{E} = \{(i,j) : 1 \le i,j \le N \wedge i<j\}$ we can say that $\Theta$ is the set of all subsets of the set $\mathscr{E}$ and $|\Theta| = 2^{|\mathscr{E}|}$. Obviously, $\Upsilon(t)$, $t \in [0,\infty[_{\mathbb{T}}$ are elements of $\Theta$. Note that, at a given time $t \in [0,\infty[_{\mathbb{T}}$ one and only one from $2^{|\mathscr{E}|} = 2^{\frac{|E|}{2}}$ possible attacks modes may happen. Hence, to every $t \in [0,\infty[_{\mathbb{T}}$ corresponds a unique element of the set $\Theta$. Now, let $A \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ be the adjacency matrix of $\mathscr{G}$ and $L$ its Laplacian. To describe a DoS attack at time $t$ we introduce a matrix $L_{\Upsilon(t)}$ that is defined as $L$ with $a_{ij} = 0$ for $(j,i) \notin \Upsilon(t)$. Precisely, if channel $(j,i)$ is not attacked at time $t$, then in matrix $L_{\Upsilon(t)}$ we have $a_{ij} = 0$. By introducing bijection map $f : \Theta \to \{1,\dots,2^{|\mathscr{E}|}\} \subset \mathbb{N}$ numbering the elements of set $\Theta$), we define switching signal $\kappa : \mathbb{T} \to \{1,\dots,2^{|\mathscr{E}|}\}$ as $\kappa(t) := f(\Upsilon(t))$. By definition, $\kappa$ is piecewise continuous. Consequently, with the attack mode at $t \in [0,\infty[_{\mathbb{T}}$, we associate a matrix $A_{\kappa(t)}$ defined by

$$A_{\kappa(t)} := L - L_{\Upsilon(t)}. \tag{11}$$

### A. Stability under DoS attack

Suppose each node of a graph $\mathscr{G} = (V,E)$ represents a dynamic agent with dynamics described by continuous-time equations

$$\dot{x}_i(t) = p x_i(t) + q u_i(t)\,, \tag{12}$$

and discrete-time equation

$$x_i(\sigma(t)) = x_i(t) + \mu(t)(p x_i(t) + q u_i(t))\,, \tag{13}$$

where $x_i(t) \in \mathbb{R}$ and $u_i(t) \in \mathbb{R}$ denote the state and the control input at time $t$ of node $i$, $i = 1,\dots,N$, respectively. The two parameters $p,q \in \mathbb{R}$ will be specified later. We consider the switched dynamics for each agent and a corresponding switching law is governed by a particular time-domain $\mathbb{T}$. Namely, on continuous part of $\mathbb{T}$ we deal with equation (12) while on discrete part of $\mathbb{T}$ with equation (13). The state-feedback distributed control for multi-agent system composed of (12) and (13) is proposed as follows

$$u_i(t) = \sum_{j=1,(j,i)\notin\Upsilon(t)}^{N} a_{ij}(x_j(t) - x_i(t))\,, \quad i = 1,\dots,N\,, \tag{14}$$

where $A = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the adjacency matrix of graph $\mathscr{G}$ describing network of multi-agent system composed of (12) and (13), $u_i : \mathbb{T} \to \mathbb{R}$. Accordingly, the network dynamics becomes

$$\dot{x}(t) = (pI_N - q(L - L_{\Upsilon(t)}))x(t)\,, \tag{15}$$

for continuous-time and

$$x(\sigma(t)) = \big(I_N + \mu(t)(pI_N - q(L - L_{\Upsilon(t)}))\big)\,x(t)\,, \tag{16}$$

for discrete-time, where $x = [x_1,\dots,x_N]^T$. On account of formula (11), we obtain the switched system composed of continuous-time subsystems

$$\dot{x}(t) \in \{(pI_N - qA_\kappa)x(t)\}_{\kappa\in I}\,, \tag{17}$$

and discrete-time subsystems

$$x(\sigma(t)) \in \{(I_N + \mu(t)(pI_N - qA_\kappa))x(t)\}_{\kappa\in I}\,, \tag{18}$$

that represent multi-agent system system (12)–(13) under DoS attack.

Below we state conditions on parameters $p$ and $q$ that guarantee asymptotic stability of switched multi-agent system (12)–(13) under DoS attacks. To this end, let us denote $spec(A_\kappa) = \{\lambda_j^\kappa : j = 1,\dots,N\}$, $\kappa \in I$, and

$$\lambda_{max} = \max_{j\in\{1,\dots,N\},\kappa\in I} \lambda_j^\kappa\,. \tag{19}$$

**Remark 17.** *Observe that, by formula (11), matrices $A_\kappa$ are principal submatrices of Laplacian matrix $L$, it imlpies that $\lambda_{max}$ is the largest eigenvalue of $L$.*

**Theorem 18.** *Let Assumptions 1–3 hold for all matrices from the family $\{A_\kappa,\ \kappa \in I\}$, and one of the following conditions is fulfilled:*

**i)** *if $q > 0$, then $\frac{1}{q}\left(\frac{2}{\mu_{max}} + p\right) > \lambda_{max}$,*
**ii)** *if $q < 0$, then $\frac{p}{q} > \lambda_{max}$.*

*Then multi-agent system composed of subsystems (15) and (16) under DoS attacks is asymptotically stable provided $p \in \left(-\frac{2}{\mu_{max}}, 0\right)$.*

*Proof.* First, let us observe that the matrices $A_\kappa$ are positive semi-definite. We notice that $\lambda_{\max} > 0$; otherwise, all matrices $A_\kappa$ would have zero entries.

We begin with assumption i). From inequality $\frac{1}{q}\left(\frac{2}{\mu_{max}}+p\right)>\lambda_{max}$ we get

$$\frac{2}{\mu(t_i)}+p>q\cdot\lambda_j^\kappa \text{ for all } j=1,\ldots,N,\kappa\in I \text{ and } i\geq 1. \tag{20}$$

On the other hand, $q>\frac{p}{\lambda_{max}}$ for $p<0$, thus

$$\mu(t_i)(b\lambda_j^\kappa-p)>0 \text{ for all } j=1,\ldots,N,\kappa\in I \tag{21}$$

and $i\geq 1$. From (20) and (21) we conclude that

$$|1+\mu(t_i)(p-q\lambda_j^\kappa)|<1, \tag{22}$$

what together with the observation $spec\{I_N+\mu(t_i)(pI_N-qA_\kappa)\}=\{1+\mu(t_i)(p-q\lambda_j^\kappa):j=1,\ldots,N,\kappa\in I\}$ means that system (16) fulfils Assumption 5. Moreover, one can notice that in the considered case of assumptions, system (15) is always asymptotically stable. Indeed, inequality $q>\frac{p}{\lambda_{max}}$ implies that $p-q\lambda_j^\kappa<0$, for all $j=1,\ldots,N,\kappa\in I$. Since $spec\{pI_N-qA_\kappa\}=\{p-q\lambda_j^\kappa: j=1,\ldots,N,\kappa\in I\}$, we conclude that system (15) fulfils Assumption 4. By Theorem 12 we get the thesis.

Next, let us consider assumption ii): $q<0$, then $q>\frac{p}{\lambda_{max}}$. Since $p\in\left(-\frac{2}{\mu_{max}},0\right)$, it means that $p-q\lambda_j^\kappa<0$, for all $j=1,\ldots,N,\kappa\in I$. It implies that system (15) is Hurwitz stable (Assumption 4 holds). Moreover, from $q>\frac{p}{\lambda_{max}}$ one gets $\mu(t_i)(p-q\lambda_j^\kappa)<0$, what leads us to

$$1+\mu(t_i)(p-q\lambda_j^\kappa)<1. \tag{23}$$

Furthermore, observe that $q\lambda_j^\kappa<p+\frac{2}{\mu(t_i)}$, what implies that for all $j=1,\ldots,N,\kappa\in I$,

$$1+\mu(t_i)(p-q\lambda_j^\kappa)>-1. \tag{24}$$

By inequalities (23) and (24) we conclude that system (16) fulfils Assumption 5. By Theorem 12 we get that multi-agent system composed of subsystems (15) and (16) under DoS attacks is asymptotically stable, what finishes the proof. □

**Remark 19.** *Observe that if $q=0$ in system (12)–(13), then it is enough that $p\in\left(-\frac{2}{\mu_{max}},0\right)$ for the system composed of systems (15) and (16) to be asymptotically stable under DoS attacks.*

**Theorem 20.** *Let $\mu(t_i)=a_i$ and $\sum_{i=1}^\infty a_i<\infty$. Moreover, let Assumptions 1–3 be satisfied for all matrices from the family $\{A_\kappa,\ \kappa\in I\}$. If one of the following conditions holds:*

**i)** *$p<0$ and $q\geq 0$,*
**ii)** *$p<0$, $q<0$ and $\frac{p}{q}>\lambda_{max}$,*

*then multi-agent switched system composed of subsystems (15) and (16) is asymptotically stable under DoS attacks.*

*Proof.* We lead the proof only for condition ii) since condition i) trivially implies the thesis. First we show that system (15) is Hurwitz stable. As before, we observe that $spec\{pI_N-qA_\kappa\}=\{p-q\lambda_j^\kappa: j=1,\ldots,N,\kappa\in I\}$, so it is enough to show that $p-q\lambda_j^\kappa<0$ for all $j=1,\ldots,N,\kappa\in I$. $A_\kappa$ are positive semi-definite matrices, so $\lambda_j^\kappa\geq 0$ for every $\lambda_j^\kappa\in spec\{A_\kappa\}$. Moreover, condition $\frac{p}{q}>\lambda_{max}$ implies that $p<\lambda_j^\kappa q$ for all $j=1,\ldots,N,\kappa\in I$. Thus $p-q\lambda_j^\kappa<0$ only if $p,q<0$, for all $j=1,\ldots,N,\kappa\in I$. The latter means that matrices $pI_N-qA_\kappa$ are Hurwitz stable. Since Assumptions 1–4 are satisfied and $\sum_{i=1}^\infty a_i<\infty$ for $\mu(t_i)=a_i$, by Theorem 13, the thesis holds. □

## 5. APPLICATIONS OF CAYLEY GRAPHS TO DOS PROTECTION

In this section, we assume that the network of agents' topology is described by the Cayley graph $G$, with the corresponding adjacency matrix $A$ having entries of 0 and 1.

**Lemma 21.** *(cf. [25]) If $\{A_\kappa,\ \kappa\in I\}$ is a family of matrices given by formula (11), where $L$ is Laplacian of the Cayley graph, then the following inequalities hold true:*

**i)** $\lambda_{max}\geq 1+\max_x d_x$,
**ii)** $\lambda_{max}\geq\frac{\sum_x d_x}{N-1}$,

*where $\lambda_{max}$ is given by formula (19), $d_x$ is the degree of the vertex $x$.*

**Theorem 22.** *Let Assumption 3 be satisfied and one of the following conditions holds:*

**i)** *if $q>0$, then $\frac{1}{q}\left(\frac{2}{\mu_{max}}+p\right)>\lambda_{max}$,*
**ii)** *if $q<0$, then $\frac{p}{q}>\lambda_{max}$.*

*Then multi-agent system composed of subsystems (15) and (16) under DoS attacks is asymptotically stable provided $p\in\left(-\frac{2}{\mu_{max}},0\right)$.*

*Proof.* First, let us observe that Assumptions 1–2 are fulfilled since the matrices $A_\kappa,\kappa\in I$ correspond to Cayley graphs and subgraphs. Furthermore, assumptions i) and ii) imply, as shown in Theorem 18, that Assumptions 4 and 5 are satisfied, so the thesis holds. □

**Theorem 23.** *Let $\mu(t_i)=a_i$ and $\sum_{i=1}^\infty a_i<\infty$. Moreover, let Assumptions 3 be satisfied. If one of the following conditions holds:*

**i)** $p < 0$ *and* $q \geq 0$,
**ii)** $p < 0$, $q < 0$ *and* $\frac{p}{q} > \lambda_{max}$,

*then multi-agent switched system composed of subsystems* (15) *and* (16) *is asymptotically stable under DoS attacks.*

*Proof.* Again, Assumptions 1–2 are fulfilled for the matrices $A_\kappa, \kappa \in I$ corresponding to Cayley graphs and subgraphs. Thus, the thesis holds by Theorem 20. □

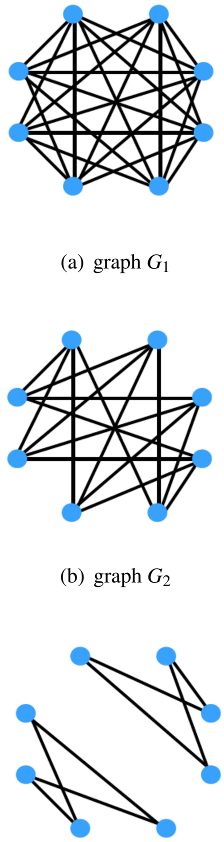
**Remark 24.** *Let us observe that in Theorems 22 and 23, one can use the inequalities provided in Lemma 21 instead of* $\lambda_{max}$.

### A. Examples

In this section, in order to keep the network working, we consider the situation, when the deployment of agents (devices) is modelled by the quasi-Abelian Cayley graphs $G_\kappa = Cay_\kappa(H_\kappa, S_\kappa)$, $\kappa \in I$. It implies that vertex sets can be any finite simple non-abelian group. As an example of generators of such a group, let us consider $S_1, S_2, S_3$ as follows: $S_1 = \{e, y, y^2, y^3, x, xy, xy^2, xy^3\}$, $S_2 = \{xy, xy^2, xy^3\}$, $S_3 = \{y, y^3\}$. It means that the network consists of eight cooperating devices (agents). In Figure 1, we present three graphs $G_1, G_2, G_3$ that correspond to generators $S_1, S_2, S_3$, respectively. Graph $G_1$ illustrates the original communication topology (the situation without DoS attacks), while by $G_2$ different $2^{16}$ attack modes are modelled and the rest of $2^8$ possible different attack modes by graph $G_3$. It should be understood as follows: if any communication channels of the subgraphs $G_2$ or $G_3$ are attacked, the automatic security system should turn off all the physical channels (between cooperating devices) represented by all the edges of the corresponding subgraph $G_2$ or $G_3$.

Consequently, there are three matrices: $A_1$ is the Laplacian matrix of the original communication topology, $A_2$ that is associated with different $2^{16}$ attack scenarios, $A_3$ that is associated with the rest of $2^8$ possible attack scenarios. Therefore $\{A_\kappa, \kappa \in I\}$, $I = \{1, 2, 3\}$, and $\lambda_{max} = 6$.

$$A_1 = \begin{bmatrix} 6 & 0 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 6 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & 6 & 0 & -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & 6 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 6 & 0 & -1 & -1 \\ -1 & -1 & -1 & -1 & 0 & 6 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 6 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 & 6 \end{bmatrix},$$



(a) graph $G_1$

(b) graph $G_2$

(c) graph $G_3$

**Fig. 1.** Three considered situations for the network of eight collaborating devices.

$$A_2 = \begin{bmatrix} 4 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 4 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 4 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 4 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 4 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 4 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 4 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 & 4 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 2 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 2 \end{bmatrix},$$

In what follows, we show how to apply Theorem 22 and Theorem 23 in particular situations. Namely, we consider two time domains: $\mathbb{T}_1 =$

$\bigcup_{k=0}^{\infty}[k(0,000003), k(0,000003)+0,000001]$ (see Example 8) and $\mathbb{T}_2 = \{0\} \cup \bigcup_{k=1}^{\infty}[2^k + \frac{1}{2^k}, 2^{k+1}]$ (see Example 14). In the case of $\mathbb{T}_1$, we use Theorem 22 with $\mu_{max} = 0,000002$. Since for $\mathbb{T}_2$ we have $\sum_{i=1}^{\infty}\mu(t_i) = 2\frac{1}{2} + \sum_{i=1}^{\infty}\frac{1}{2^{i+1}} < \infty$, Theorem 23 is applicable.

**Example 25.** Let us analyze the situation when there is no attack in the first interval $[0, 0.000001]$. In the second interval $[0.000003, 0.000004]$, at least one of the physical channels that correspond to edges in the subgraph $G_3$ is attacked, while in the next interval $[0.000006, 0.000007]$, DoS attack appears on at least one of physical channels that correspond to edges of subgraph $G_2$. In the interval $[0.000009, 0.00001]$, once again, at least one of the links in the subgraph $G_3$ is attacked and so on, such that attacks are launched in any order. It follows that the switching signal is:

$$\kappa(t) = \begin{cases} 1, & t \in [0, 0.000001] \\ 2, & t \in [0.000003, 0.000004] \\ 3, & t \in [0.000006, 0.000007] \\ 2, & t \in [0.000009, 0.00001] \\ \vdots \end{cases},$$

with the graph presented in Figure 2.

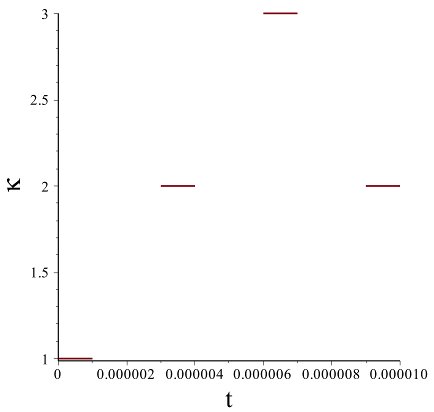
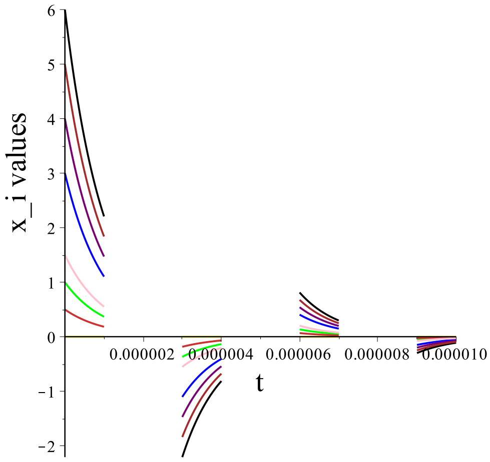
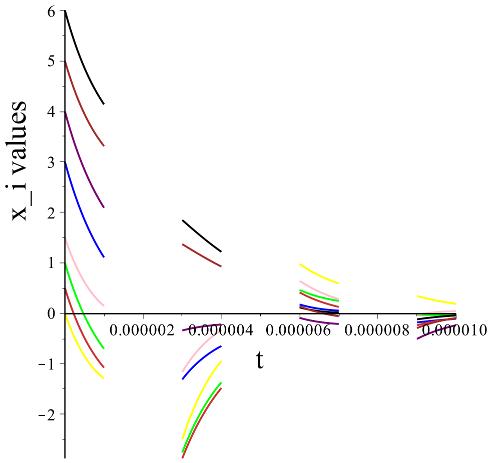



**Fig. 2.** Switching signal for the system with eight agents subject to DoS attacks.

The behavior of the network of agents is illustrated in Figure 3 and Figure 4. Figure 3 shows asymptotic stability, when: (a) $p = -999999, q = \frac{1}{7}$; (b) $p = -999999, q = -150000$. In Figure 4 one can see instability, when: (a) $p = -999999, q = -200000$; (b) $p = -999999, q = -300000$. In both cases, the initial conditions are $x_1(0) = 1, x_2(0) = 3, x_3(0) = 6, x_4(0) = 0, x_5(0) = \frac{1}{2}, x_6(0) = 4, x_7(0) = \frac{3}{2}, x_8(0) = 5$.



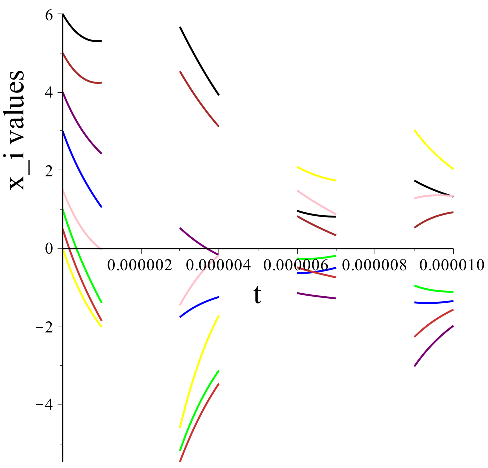

(a) $p = -999999, q = \frac{1}{7}$



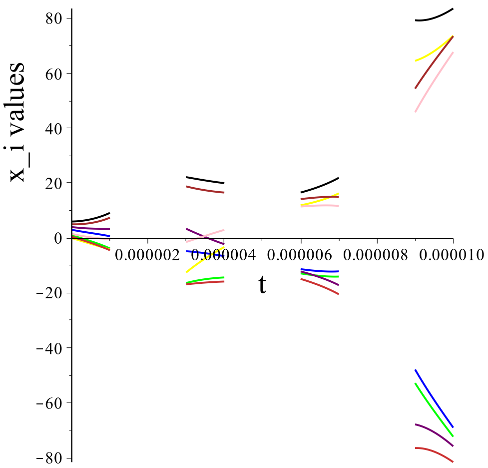

(b) $p = -999999, q = -150000$

**Fig. 3.** The evolution in time of the network of eight devices under DoS attacks is asymptotically stable.

**Example 26.** In this example, we analyze the situation when there is no attack at the initial time, and in the time interval $[\frac{5}{2}, 4]$. Then in the interval $[\frac{17}{4}, 8]$, at least one of the physical channels that correspond to edges in the subgraph $G_2$ is attacked, while in the interval $[\frac{65}{8}, 16]$, DoS attack appears on at least one of the physical channels that correspond to edges in the subgraph $G_3$ and so on, such that attacks are launched in any order. What follows, the switching signal is:




(a) $p = -999999, q = -200000$




(b) $p = -999999, q = -300000$

**Fig. 4.** The evolution in time of the network of eight devices under DoS attacks that is not asymptotically stable.

$$\kappa(t) = \begin{cases} 1, & t \in \{0\} \cup [\frac{5}{2}, 4] \\ 3, & t \in [\frac{17}{4}, 8] \\ 2, & t \in [\frac{65}{8}, 16] \\ \vdots & \end{cases},$$

with the graph presented in Figure 5.

The behavior of the network of agents is illustrated in Figure 6 and Figure 7. Figure 6 shows asymptotic stability, when: (a) $p = -10, q = -1$; (b) $p = -1, q = 1$. In Figure 7 one can see instability, when $p = -1, q = -1$. In both cases, the initial conditions are $x_1(0) = 1, x_2(0) = 3, x_3(0) = 6, x_4(0) = 0, x_5(0) = \frac{1}{2}, x_6(0) = 4, x_7(0) = \frac{3}{2}, x_8(0) = 5$.

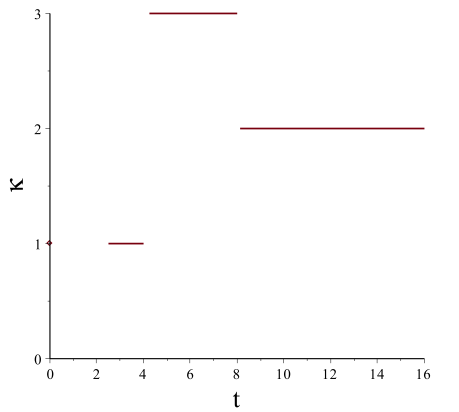
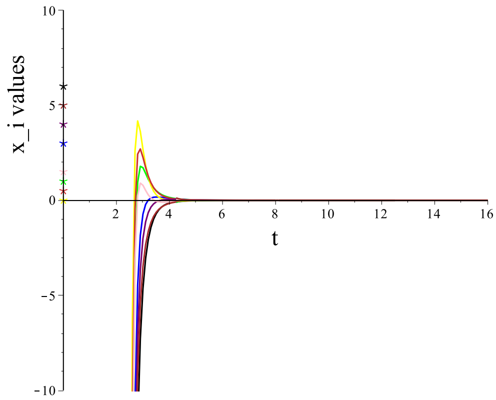
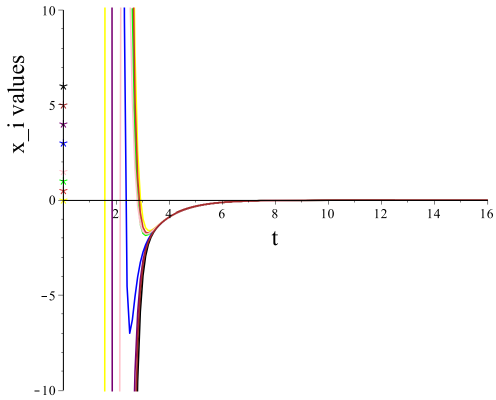



**Fig. 5.** Switching signal for the system with eight agents subject to DoS attacks.

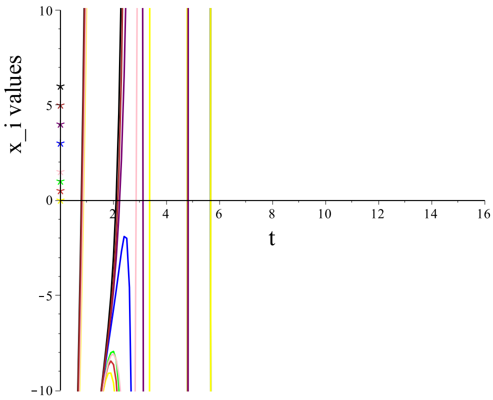



(a) $p = -10, q = -1$

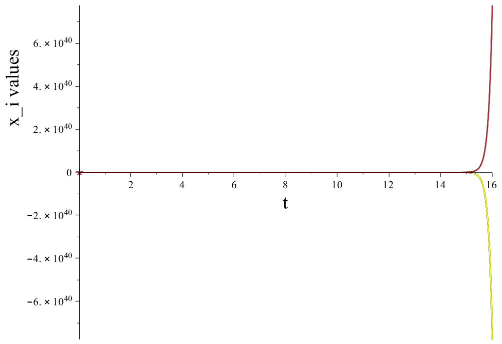



(b) $p = -1, q = 1$

**Fig. 6.** The evolution in time of the network of eight devices under DoS attacks is asymptotically stable.




(a) $p = -1, q = -1$, zoomed




(b) $p = -1, q = -1$

**Fig. 7.** The evolution in time of the network of eight devices under DoS attacks that is not asymptotically stable.

### B. How to apply the presented design method in practice?

The main idea of applying the presented method in practice is as follows: the original non-attacked communication topology of the MAS should mimic the network of edges in a quasi-Abelian Cayley graph $G$ with the appropriate number of nodes. In the case where at least one of the physical channels corresponding to edges in any quasi-Abelian Cayley subgraph of graph $G$ is attacked, the automatic security system should turn off the entire block of communication links modeled by this subgraph.

By applying Theorems 22 or 23, as long as the attacks stay within the block of physical channels modeled by any quasi-Abelian Cayley subgraph of graph $G$, the entire networked MAS will continue to work in an asymptotically stable manner.

The crucial factors that guarantee the success of this design method are:

- Laplacian matrices of quasi-Abelian Cayley graphs on the finite group commute.
- For any natural number $N$ there exists a quasi-Abelian Cayley graph with $N$ nodes.

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## REFERENCES

[1] A. Cetinkaya, H. Ishii, and T. Hayakawa, "An overview on denial-of-service attacks in control systems: attack models and security analyses", *Entropy*, vol. 21, no. 2, 210, 2019.

[2] B. Chang, X. Mu, Z. Yang, and J. Fang, "Event-based secure consensus of muti-agent systems under asynchronous DoS attacks", *Appl. Math. Comput.* vol. 401, 126120, 2021.

[3] Z. Feng and G. Hu, "Secure Cooperative Event-Triggered Control of Linear Multiagent Systems Under DoS Attacks", *IEEE Trans. Control Syst. Technol.*, vol. 28, no. 3, pp. 741–752, 2020.

[4] Y. Liu, "Secure Control of Networked Switched Systems with Random DoS Attacks via Event-triggered Approach", *Int. J. Control Autom. Syst.*, vol. 18, 2572–2579, 2020.

[5] J. Liu and X. Wang, "Secure Consensus Control for Multi-agent Systems Subject to Consecutive Asynchronous DoS Attacks", *Int. J. Control Autom. Syst.*, vol. 21,pp. 61–70, 2023.

[6] H. Liu, "Event-triggering-based leader-following bounded consensus of multi-agent systems under DoS attacks", *Commun. Nonlinear Sci. Numer. Simul.*, vol. 89, 105342, 2020. 105342,

[7] P. Wang, G. Wen, and W. Yu, "Consensus tracking of linear multi-agent systems with undirected switching communication topologies under impulsive disturbances" in *15th International Conference on Control, Automation, Robotics and Vision (ICARCV)*, Singapore, 2018, pp. 815–819.

[8] Y. Xu, M. Fang, Z. -G. Wu, Y. -J. Pan, M. Chadli, and T. Huang, "Input-based event-triggering consensus of multiagent systems under denial-of-service attacks", *IEEE Trans. Syst. Man Cybern.*, vol. 50, no. 4, pp. 1455-1464, 2020.

[9] Y. Yang, H. Xu, and D. Yue, "Observer-based distributed secure consensus control of a class of linear multi-agent systems subject to random

attacks", *IEEE Trans. Circuits Syst. I, Regul. Pap.*, vol. 66, no. 8, pp. 3089–3099, 2019.

[10] L. Zha, J. Liu, and J. Cao, "Resilient event-triggered consensus control for nonlinear muti-agent systems with DoS attacks", *J. Franklin Inst.*, vol. 356, no. 13, pp. 7071–7090, 2019.

[11] H. Zhang, Y. Qi, J. Wu, L. Fu and L. He, "DoS Attack Energy Management Against Remote State Estimation", *IEEE Trans. Control Netw. Syst.*, vol. 5, no. 1, pp. 383–394, 2018.

[12] P. H. S. Coutinho, I. Bessa, P. S. P. Pessim, and R. M. Palhares, "A switching approach to event-triggered control systems under denial-of-service attacks", *Nonlinear Anal., Hybrid Syst.*, vol. 50, 101383, 2023.

[13] C. De Persis, and P. Tesi, "Input-to-state stabilizing control under denial-of-service", *IEEE Trans. Automat. Control*, vol. 60, no. 11, pp. 2930–2944, 2015.

[14] C. De Persis, and P. Tesi, "Networked control of nonlinear systems under denial-of-service", *Systems Control Lett.*, vol. 96, pp. 124–131, 2016.

[15] V. S. Dolk, P. Tesi, C. De Persis, and W.P.M.H. Heemels, "Event-triggered control systems under denial-of-service attacks", *IEEE Trans. Control Netw. Syst.*, vol. 4, no. 1, pp. 93–105, 2016.

[16] G. Zhai, Hai Lin, A. N. Michel and K. Yasuda, "Stability analysis for switched systems with continuous-time and discrete-time subsystems" in *Proceedings of the 2004 American Control Conference*, Boston, MA, USA, 2004, pp. 4555–4560, vol. 5.

[17] G. Eisenbarth, J.M. Davis, and I. Gravagne, "Stability of simultaneously triangularizable switched systems on hybrid domains", *Electron. J. Differ. Equ.*, vol. 63, pp. 1–19, 2014.

[18] F.Z. Taousser, M. Defoort and M. Djemai, "Stability analysis of a class of switched linear systems on non-uniform time domains", *Syst. Control Lett.*, vol. 74, pp. 24–31, 2014.

[19] A. V. Kelarev, "On undirected Cayley graphs",*Australas. J. Comb.*, vol. 25, pp. 73–78 2002.

[20] T. Biyikoglu, J. Leydold and P. F. Stadler, *Laplacian Eigenvectors of Graphs: Perron-Frobenius and Faber-Krahn Type Theorems*, Springer Berlin Heidelberg New York, 2007.

[21] M. Bohner and A. Peterson, *Dynamic equations on time scales*, Birkhäuser Boston, Boston, MA, 2001.

[22] K. Hoffman and R. Kunze, *Linear Algebra*, Prentice-hall, inc., Englewood Cliffs, New Jersey, 1961.

[23] A.W. Joshi, *Matrices and Tensor in Physics*, New Age International Publishers, 1995.

[24] R.A. Horn and Ch.R. Johnson, *Matrix analysis*, Cambridge University Press, Cambridge, 2013.

[25] W. H. Haemers, On some problems of Lovász concerning the Shannon capacity of graphs, *IEEE Trans. Inf. Th.* 25, 231–232, 1979.