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# Fractional order transient free-convection flow in a channel: application of the optimal homotopy asymptotic method

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## Abstract

In this study, a new physical model has been created to look into the behaviour of transient incompressible unsteady flow between two infinite parallel plates exposed to high temperatures. The model takes into consideration thermal radiation flux, chemical reaction, and mass diffusion at the boundaries. To handle non-integer behaviour, the model incorporates the Caputo notion of time fractional derivative. To solve this complex physical fractional order fluid model, a novel optimal homotopy asymptotic method and semi-analytical methodology is extended and utilized successfully. This method provides a third-order highly approximate solution, offering valuable insights into the behaviour within the system. The study comprehensively examines the effects of varied flow characteristics and fractional order on the dynamics of the system. The results are visually presented through graphs, offering a clear understanding of the system's response under different conditions. The effectiveness and ease of use of the optimal homotopy asymptotic method make it a valuable tool for solving boundary value fractional order problems encountered in scientific fields. The developed physical model and its fractional extension contribute significantly to the understanding of unsteady flow phenomena with thermal and chemical effects, advancing knowledge in this area of research.

**Keywords:** Parallel plates; Flow; Fractional model; Caputo derivative; Optimal homotopy asymptotic method

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## 1. Introduction

In the natural world, various processes entail the simultaneous transport of mass and heat. These occurrences arise not only due to temperature gradients but also from concentration differences or a combination of both factors. Over the past few decades, scientists have extensively studied buoyancy-driven flows influenced by mass and thermal diffusion working together. These investigations have found significant applications in diverse technological and engineering fields, including nuclear fuel

## Nomenclature

${}^{c}D^{\alpha}_{0,t}$ – Caputo derivative
$C_1$, $C_2$, $C_3$ – constants
$c_i$ – convergence control parameters
$c_p$ – specific heat at constant pressure, J/(kg K)
$D$ – coefficient of molecular diffusivity
$g$ – acceleration due to gravity, m/s$^2$
$H$ – nonlinear operator – Caputo fractional derivative operator (denotes space of functions)
$k$ – thermal conductivity, W/(m K)
$k^*$ – mean value of heat absorption parameter
$Kr$ – chemical reaction parameter
$L$ – Caputo fractional derivative operator
$n$ – order of approximation
$N_r$ – particles per unit volume, 1/m$^3$
Pr – Prandtl number
$q_{ry}$ – radiative heat flux, W/m$^2$
$R$ – Caputo fractional derivative operator (denotes space of functions that are Lebesgue integrable)
$s$ – parameter (that represents the convergence rate)
Sc – Schmidt number
$t$ – time, s
$u$ – velocity, m/s

### Greek symbols

$\alpha$ – fractional order
$\beta$ – volume expansion coefficient for the heat transfer
$\beta^*$ – volume expansion coefficient for the mass transfer
$\Gamma$ – gamma function
$\theta$ – fluid temperature, K
$\theta_d$ – far field temperature, K
$\theta_w$ – temperature of the wall, K
$\nu$ – kinematic viscosity, m$^2$/s
$\rho$ – density, kg/m$^3$
$\sigma^*$ – Stefan-Boltzmann constant, W/(m$^2$ K$^4$)
$\psi$ – concentration
$\psi_d$ – far field concentration
$\psi_w$ – wall-based nanoparticle concentration, mol/m$^3$

### Subscripts and Superscripts

$eff$ – effective order of approximation
$t$ – time variable or independent variable
$yy$ – second derivative

### Abbreviations and Acronyms

HPM – homotopy perturbation method
HVAC – heating, ventilation and air conditioning
OHAM – optimal homotopy asymptotic method

storage, underground coal gasification, groundwater hydrology, chemical engineering and processor cooling. Consequently, the study of free convective flow in vertical channels has garnered substantial attention because of its profound relevance in applied sciences and engineering [1]. Among the notable research efforts, Harris et al. [2,3] have examined the effects of transient free convective flow along a straight plate embedded in a porous medium that has been exposed to abrupt changes in heat flow and surface temperature. As the fluid flows towards the wall, heat generated through viscous temperature rises as a result of dissipation, resulting in reduced viscosity and significant flow stratification. These changes notably influence the heat transfer rate (HTR) [4,5], rendering this phenomenon practically important in various engineering applications. The study of fractional calculus has gained considerable interest in recent times, becoming a popular subject due to its widespread applications across scientific and engineering disciplines [6]. In this context, Sarwar et al. [7] investigated the behaviour of a fluid of the non-Newtonian fractional Brinkman type using the optimal homotopy asymptotic method (OHAM). OHAM has demonstrated to be a valuable tool for solving complex problems involving fractional calculus effectively implemented to various physical models, demonstrating its efficacy in providing accurate and efficient solutions [8]. Furthermore, references [9–12] delve into the time fractional operator's importance in heat transfer analysis, highlighting its usefulness in comprehending and modelling heat transfer processes involving fractional derivatives. In conclusion, the exploration of flows involving the interaction of mass and heat transport has enriched our understanding of intricate natural phenomena and has opened up numerous applications in engineering and applied sciences. By employing advanced mathematical techniques such as OHAM and studying fractional calculus, researchers have gained valuable insights and offered innovative solutions to challenging problems across various scientific and engineering domains.

Heat transfer analysis in a micro-channel was thoroughly investigated by Ojemeri and Hamza [13], with a focus on the complex interactions of Arrhenius-controlled processes, free convection, hydromagnetic flow and the effects of heat generation and absorption.

Through a microchannel that includes a non-Darcy porous medium, Bhatti et al. (his research team) [14] investigated the dynamics of natural convection in a non-Newtonian electromagnetohydrodynamics (EMHD) dissipative flow. Their research focused on using the homotopy perturbation method (HPM) to do the analysis.

A theoretical investigation on heat transfer in a micro-channel under Arrhenius control was carried out by Hamza et al. (team of experts) [15]. Their main focus was on how an artificial magnetic field affected natural convection.

In a composite channel that was partially filled with a porous material, Ajibade et al. [16] (research collaborators) conducted a thorough analysis of the effects of both viscous and Darcy dissipations on fully developed natural convection flow. They used HPM for analysis as part of their research technique.

In order to account for convective boundary conditions, Ray et al. [17] studied the non-similar solution of fluid flow and heat transfer for Eyring-Powell fluids. In their analysis, they used the homotopy analysis method (HAM).

Many studies [18–20] focus on the optimization of secondary energy sources in hydrostatic drive systems, which is relevant to machinery operating in tight spaces. In these specific operating circumstances, this study emphasizes the benefits of energy efficiency and custom technical solutions. The second collaborative study investigates the effect of blade design on fan efficiency using cutting-edge prototyping methods. These insights have an effect on a number of applications, such as ventilation, HVAC and cooling systems, and can increase the performance of fans. Increasing the maximum speed of underground suspended monorails is an essential part of safe crew mobility in the challenging environment of Polish underground mining. In mining, operational efficiency and safety are crucial, and this research looks into realistic approaches to achieve both. The fourth study evaluates a special energy recovery system created especially for hybrid cars in order to satisfy the pressing need for enhanced energy efficiency and sustainability within the automotive sector. By giving experts and researchers insightful information, these countless study discoveries significantly advance their respective fields.

In this study, we examine temperature and concentration gradient-driven unsteady free convection flow between two parallel plates. To take into consideration non-integer order derivatives in the equations, fractional order Caputo derivatives are introduced. Solving these difficult fractional differential equations and comprehending their ramifications for real-world applications are the main goals of the research. An important part of this inquiry involves validating and interpreting boundary conditions as well as taking into account their restrictions and potential future orientations.

This research introduces a fresh perspective by applying fractional calculus to transient free-convection flow in channels. It stands out due to the innovative use of OHAM, offering an efficient approach to complex fluid dynamics problems. The study's interdisciplinary potential, optimization focus and practical relevance make it a unique and impactful contribution to the field.

## 2. Materials and methods

Consider a free convective fluid that is incompressible, unstable, and flows between two parallel plates with a temperature gradient and mass diffusion, placed in the *xy*-Cartesian coordinate system. As seen in the image, one of the plates is fixed along the x-axis, while the *y*-axis is normal to the plate. Temperature and concentration variations have resulted in a free convection flow seen in Fig. 1.

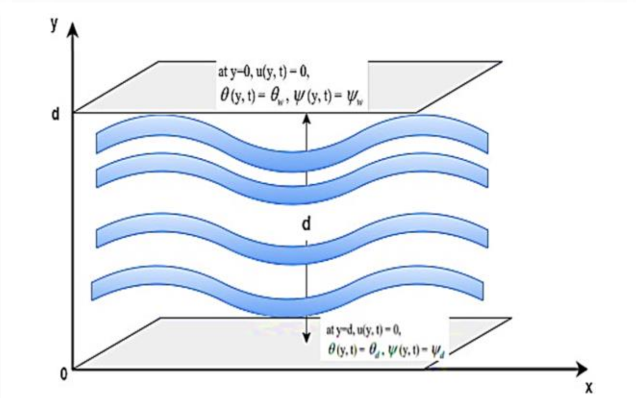


Fig. 1. Geometry of problem.

We make assumptions that at $t \leq 0$, the plates as well as the fluid are at ambient concentration of $\psi_d$ and temperature $\theta_d$. At time $t > 0$, fluid concentration and temperature at $y = 0$ are altered to $\psi_w$ and $\theta_w$, respectively.

The governing equations for unsteady flow are derived from Boussinesq's approximations as:

$$u_t - \nu u_{yy} - g\beta(\theta - \theta_d) - g\beta^*(\psi - \psi_d) = 0; \quad (1)$$

$$\rho c_p \theta_t - k\theta_{yy} + q_{ry} = 0, \;\; y, t > 0, \quad (2)$$

where $q_{ry}$ is radiative heat flux, and the above equation can be rewritten using the Roseland approximation:

$$\theta_t - \frac{k}{\rho c_p}\left(1 + \frac{16\sigma^* \psi_d^3}{3kk^*}\right)\psi_{yy} = 0, \quad (3)$$

where $\sigma^*$, $\rho$, $k$, $k^*$ and $c_p$ are Stefan-Boltzmann constant, density, thermal conduction, mean value of heat absorption parameter and specific heat with invariant pressure, respectively;

$$\psi_t - D\psi_{yy} + Kr(\psi - \psi_d) = 0, \quad (4)$$

where $Kr$ is the chemical reaction parameter. The appropriate boundary conditions are:

$$t > 0: \quad u(t,0) = 0, \quad \theta(t,0) = \theta_w, \quad \psi(t,0) = \psi_w, \quad (5)$$

$$t > 0: \quad u(t,d) = 0, \quad \theta(t,d) = \theta_d, \quad \psi(t,d) = \psi_d. \quad (6)$$

Equations (1), (3) and (4), after using the appropriate dimensionless variables and parameters, become

$$u_t - u_{yy} - \theta - N_r\psi = 0, \quad (7)$$

$$\theta_t - \frac{1}{\mathrm{Pr}_{eff}}\theta_{yy} = 0, \quad (8)$$

$$\psi_t + \frac{1}{\mathrm{Sc}}\psi_{yy} - \frac{Kr}{\mathrm{Sc}}\psi = 0, \quad (9)$$

with dimensionless boundary conditions:

$$t > 0: \quad u(t,0) = 0, \quad \theta(t,0) = 1, \quad \psi(t,0) = 1, \quad (10)$$

$$t > 0: \quad u(t,1) = 0, \quad \theta(t,1) = 0, \quad \psi(t,1) = 0. \quad (11)$$

In Eqs. (8)–(10), we replace the time derivative terms with the time fractional order Caputo derivative; then we have:

$${}^{C}D_{0,t}^{\alpha}u(t,y) - u_{yy}(t,y) - \theta(t,y) - N_r\psi(t,y) = 0, \quad (12)$$

$${}^{C}D_{0,t}^{\alpha}\theta(t,y) - \frac{1}{\mathrm{Pr}_{eff}}\theta_{yy}(t,y) = 0, \quad (13)$$

$${}^{C}D_{0,t}^{\alpha}\psi(t,y) + \frac{1}{\mathrm{Sc}}\psi_{yy}(t,y) - \frac{Kr}{\mathrm{Sc}}\psi(t,y) = 0, \quad (14)$$

where ${}^{C}D_{0,t}^{\alpha}$ is the Caputo derivative defined as:

$${}^{C}D_{0,t}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)f^{(n)}(\tau)d\tau, \;\; 0 < \alpha < 1. \quad (15)$$

## 3. Implementation of OHAM

Equations (8) and (9) describe the optimal homotopy asymptotic approach (OHAM). To the best of our knowledge, Sarwar et al. [10], Sarwar and Rashidi [11] and Sarwar and Iqbal [12] firstly formulated this approach applicable to fractional order partial differential equations. The key steps of OHAM for fractional order partial differential equations are as follows (for details we refer to [10–12]):

(a) construct the governing fractional order differential equation as follows (see Eq. (16)):

$$ {}^{C}D_{0,t}^{\alpha} u(r,t) = L\big(u(r,t)\big) + N\big(u(r,t)\big) + f(r,t), 0 < \alpha < 1, \tag{16} $$

with boundary condition $B(u, u_t) = 0$, where ${}^{C}D_{0,t}^{\alpha}$ denotes the Caputo fractional derivative operator defined in Eq. (15);

(b) construct an optimal homotopy $H(\psi; c_i)$, defined as $\psi(r,t;p)\colon \Omega \times [0,1] \longrightarrow R$ which satisfies Eq. (17)

$$ H(\psi; c_i) = (1-p)\left( {}^{C}D_{0,t}^{\alpha}\psi(r,t) - f(r,t)\right) - H(p;c_i)\times $$
$$ \left( {}^{C}D_{0,t}^{\alpha}\psi(r,t) - (L(\psi(r,t)) + (N(\psi(r,t)) + f(r,t))\right) = 0, \tag{17} $$

where $r \in \Omega$ and $p \in [0, 1]$ is an embedding parameter, $c_i$ are convergence control parameters;

(c) expand $\psi(r, t; p, c)$ in Taylor's series about $p$, to get approximate solutions;

(d) equate the coefficients of like powers of $p$, if necessary, zeroth order, first order, second order, and higher order issues can be obtained;

(e) apply the fractional order integral operator on the obtained problems and with the boundary conditions and get the desired outcomes.

## 4. Numerical example

In this section, we will look for a solution to the problem (12–14) with boundary conditions (10–11). To begin the simulations, the following starting guesses are suggested: $u_0 = y - y^2$, $\theta_0 = 1 - y^2$, $\psi_0 = 1 - y^2$. The solution for the fractional concentration, temperature and velocity can be found in the form as below in Eqs. (18–20). These equations encapsulate the comprehensive understanding of the system's dynamics, offering insights into the interplay between various parameters and phenomena.

**Results of Fractional Concentration Field:**

$$ \psi(t,y) = \frac{t^{\alpha}}{\mathrm{Sc}^3}\left[ -\frac{2C_1(C_1^2+C_1+C_2)Kr\mathrm{Sc}t^{\alpha}\big((y^2-1)Kr+4\big)}{\Gamma(2\alpha+1)} + \frac{(C_2+C_1((C_1+1)^2+2C_2)+C_3)\mathrm{Sc}^2\,\big((y^2-1)Kr+2\big)}{\Gamma(\alpha+1)} + \frac{C_1^3Kr^{\mathbf{2}}t^{2\alpha}\big((y^2-1)Kr+6\big)}{\Gamma(3\alpha+1)} \right]. \tag{18} $$

**Results of Fractional Temperature Field:**

$$ \theta(t,y) = \frac{2(C_1(C_1(C_1+3)+3)-2C_2-C_3)t^{\alpha}}{\Gamma(\alpha+1)\mathrm{Pr}} - y^2 + 1. \tag{19} $$

**Results of Fractional Velocity Field:**

$$ \begin{aligned} u(t,y) = {} & y - y^2 + \frac{C_1\big(N_1(y^2-1)+y^2+1\big)t^{\alpha}}{\alpha\Gamma(\alpha)} + t^{\alpha}\Bigg(\frac{(C_2+C_1((C_1+1)^2+2C_2)+C_3)\big(N_1(y^2-1)+y^2+1\big)}{\Gamma(\alpha+1)} - \\ & \frac{\sqrt{\pi}2^{1-2\alpha}C_1t^{\alpha}\Big(2\mathrm{Sc}\big((C_1^2+C_1+C_2)(N_1+1)\mathrm{Pr}+C_1^2+C_1\big)+(C_1^2+C_1+C_2)N_1\mathrm{Pr}\big((y^2-1)Kr+2\big)\Big)}{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(\alpha+1)\mathrm{ScPr}}\,\frac{C_1^3N_1Krt^{2\alpha}\big(2(\mathrm{Sc}+2)+(y^2-1)Kr\big)}{\Gamma(3\alpha+1)\mathrm{Sc}^2}\Bigg) + \\ & \frac{t^{\alpha}}{\Gamma(\alpha+1)}\left((C_1^2+C_1+C_2)(N_1(y^2-1)+y^2+1) - \frac{\sqrt{\pi}4^{-\alpha}C_1^2t^{\alpha}\Big(N_1\mathrm{Pr}\big(2(\mathrm{Sc}+1)+(y^2-1)Kr\big)+2\mathrm{Sc}(\mathrm{Pr}+1)\Big)}{\Gamma\left(\alpha+\frac{1}{2}\right)\mathrm{ScPr}}\right). \end{aligned} \tag{20} $$

## 5. Analysis of OHAM

This research delves into the dynamics of transient, viscous, and incompressible flow in the absence of restrictions between two parallel upright plates. This research's main focus is identifying the length component. We consider the consequences of mass dispersion and thoroughly examine the uniform temperature distribution of the system.

We used an analytical strategy and OHAM to solve the resulting model, allowing us to identify the solutions for all boundary and beginning conditions. The research provided important new understandings of the temperature, concentration, and velocity expressions. These results are graphically shown in Chapter 6, allowing us to see and comprehend how these parameters varied over the flow domain. The system's complicated interactions between temperature, concentration, and velocity are made clearer by the graphical representation, which also highlights significant patterns and trends. We are able to fully comprehend the transient, viscous and free convective flow behaviour between the parallel plates by applying the OHAM technique to this particular issue.

This study offers a vital basis for future research and applications in the fields of engineering and science by adding to our understanding of fluid dynamics and convective heat and mass transfer.

## 6. Results

Figures 2–4 present the velocity, temperature, and concentration field behaviour. Figure 2 is diagrammed for varied effective Pr numbers against $y$ by restricting $t$ = 0.01 and $\alpha$ = 0.9 on the velocity. We can see that by increasing the Pr number, the velocity is decreased. In Fig. 3, as the Pr number increases, the

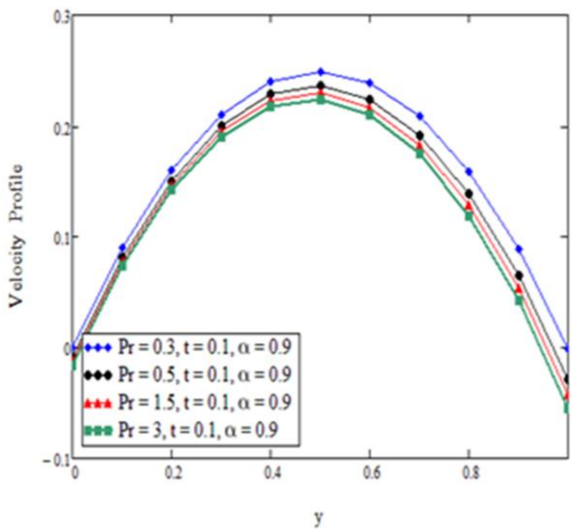


Fig. 2. Velocity profile for Sc = 0.22, $N$ = 0.5, $N_r$ = 1, $Kr$ = 1.5.

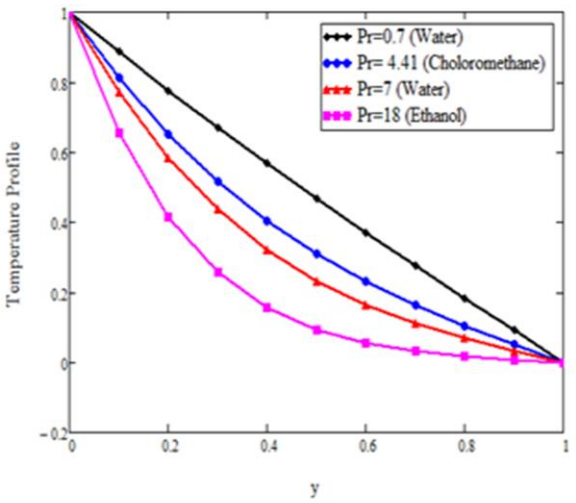


Fig. 3. Temperature profile for Sc = 0.22, $N$ = 0.5, $N_r$ = 1, $Kr$ = 1.5.

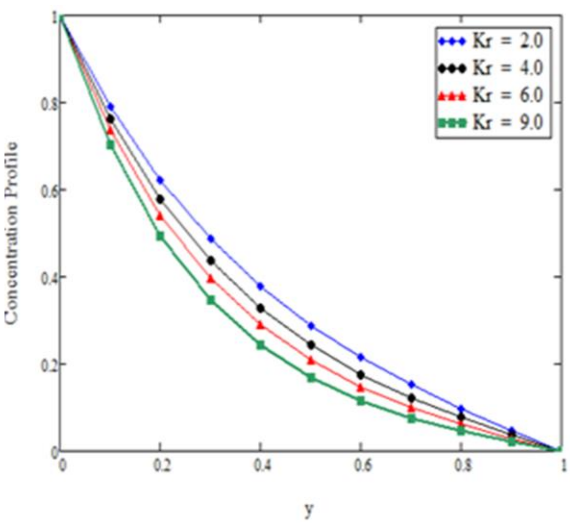


Fig. 4. Concentration profile for Sc = 0.22, $N$ = 0.5, $N_r$ = 1, Pr = 1.5.

Table 1. Values of convergence control parameters for velocity, temperature and concentration field at fractional order $\alpha$ = 0.9.

| Parameter | $u$ | $\theta$ | $\psi$ |
|---|---|---|---|
| $C_1$ | −1.0026 | −2.77074 | 0.3003 |
| $C_2$ | −0.00001 | −1.9757 | −0.3247 |
| $C_3$ | −5.65601 | −1.60073 | −0.0121 |

liquid temperature decreases. Raising Pr, as expected, shortens thermic conduction and lowers the thickness of the caloric border layer due to its high viscosity. Figure 4 depicts the impact of the chemical reaction parameter $Kr$. It is apparent that as $Kr$ increases, the flow concentration diminishes. Table 1 provides values of convergence control parameters.

## 7. Conclusions

The optimal homotopy asymptotic method (OHAM) is utilised in this paper to solve a new fractional model based on the Caputo derivative. In order to observe the heat and momentum boundary layer concentration thickness, the model includes a fractional-order parameter. The analysis demonstrates that the fluid temperature increases as time values increase and decrease. In contrast, the temperature drops when Pr (probably a related metric) increases in value. These results imply that higher temperatures in the system are caused by lower values and longer time periods. Additionally, the fractional parameter has a substantial impact on the concentration growth and increases noticeably with longer time values. The concentration rise becomes more obvious as the values drop. Along with the decreasing boundary layer thickness, the concentration also decreases with the increasing Schmidt number (Sc) and chemical reaction parameter ($Kr$) values. This implies that narrower boundary layers are produced as a result of suppressing the concentration growth at higher Sc and $Kr$ values.

Overall, these research results offer insightful information on how the fractional-order parameter and other significant characteristics, such as fluid temperature, concentration and velocity affect the system's behaviour. These findings greatly advance our knowledge of fractional models and their prospective applications to fluid dynamics and related fields of study.

A fractional model is created using the Caputo derivative, and it is solved by a semi-analytical method OHAM. The concentration thickness of heat and momentum boundary layers is observed by the fractional-order parameter in this model. The fluid temperature falls for larger Pr. The concentration increases with decreasing fractional parameter values and increasing time values. Furthermore, the concentration decreases by increasing Sc and $Kr$, as well as the boundary layer thickness. The velocity increases by increasing the parametric values of $N$ and time $t$, while it decreases by increasing Pr, $Kr$, and Sc.

Future research should investigate the influence of different fractional orders on transient free convection in channels and determine their practical significance. Additionally, the impact of nanofluids and non-Newtonian fluids on heat and mass transfer should be examined, while optimizing channel designs. Collaborating across disciplines can enhance the practical relevance of the research.

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