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MEASUREMENTS OF THE AVERAGE FLOW VELOCITY WITH THE PITOT-PRANDTL TUBE

POMIARY WARTOŚCI ŚREDNIEJ PRĘDKOŚCI RURKĄ PITOTA-PRANDTLA

The input signal to the Pitot-Pradtl tube is the pressure difference between the tube inlet and side openings, which is proportional to the squared velocity. In the course of further processing the quantity is first averaged and then the square root is sought, hence the indicated average velocity value will be overstated. Conduits connecting the tube with the pressure difference transducers and the measuring chamber make up/act as a low-pass filter. Beyond the upper transmission limit, the filter implements the averaging procedure, thus generating certain errors. The paper explores the correlation theory in stochastic processes to introduce relevant formulas to compute the average velocities, taking into account the properties of the Pitot-Prandtl tube and dynamic behaviours of a pressure gauge and conduits, described with a linear differential equation of the first order.

Key words: fluid mechanics, metrology, stochastic processes

W artykule została opisana metoda wyznaczania wartości średniej prędkości powietrza z pomiarów dokonywanych rurką Pitota-Prandtla w strumieniach z silną turbulencją. Zawyżanie wartości średniej wynika z jej kwadratowej charakterystyki statycznej oraz z niewłaściwej kolejności uśredniania i linearyzacji (pierwiastkowania). Jeśli rurkę i stosowany miernik różnicy ciśnień można uważać za układ statyczny, to poprawka na wartość średnią prędkości dana jest wzorem (3). W przypadku gdy nie można pominąć właściwości dynamicznych układu wynikających np. z połączenia rurki Pitota-Prandtla z miernikiem różnicy ciśnień długimi wężami, układ pomiarowy opisuje równanie (18), w którego współczynniki opisują wyrażenia (17).

Aby wyprowadzić poprawny wzór dla obliczania wartości średniej prędkości, oblicza się wartość średnią lewej i prawej strony równania (18) — co pokazują równania (19) i (20), oraz autokorelację lewej strony równania (18) — co pokazuje wzór (31). Przyjmując w funkcji autokorelacji czas przesunięcia (opóźnienia) równy zero otrzymuje się równanie (32) wiążące wariancje obu stron równania (18). Po dalszych przekształceniach jego prawej strony otrzymuje się równanie (41), które wraz z równaniem (20) pozwala wyznaczyć wartość średnią prędkości powietrza wzorem (42). W tablicy 1 zestawiono wartości średnie prędkości i ich wariancje wyznaczone z jednoczesnych pomiarów dwoma

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termoanemometrami w tym samym przekroju chodnika, co pozwala oszacować wielkość poprawek według wzoru (3).

Słowa kluczowe: mechanika pyłów, metrologia, procesy stochastyczne

Nomenclature

D_p — variance of pressure difference between the chambers in the pressure gauge [Pa],
D_w — variance of measured velocity [m/s^2],
E — operator of the mean value [–],
$G(\omega)$ — characteristic function of probability density distribution [–],
$K_p(t_2 - t_1)$, $K_p(\tau)$ — pressure difference covariance function [Pa^2],
L — length of conduits connecting the Pitot-Prandtl tube with the pressure transducer [m],
Q — stream of mass of flowing fluid [kg/s],
R — individual gas constant [J/kgK],
S — cross-section area in conduits connecting the Pitot-Prandtl tube to the pressure transducer [m^2],
T_k, T_o, — absolute temperatures in pressure transducer chambers [K],
V_k, V_o — volume of pressure transducer chambers [m^3],
d — cross-section diameter in conduits connecting the Pitot-Prandtl tube to the pressure transducer [m],
$e(w)$ — function defining the pressure loss per unit length due to fluid velocity and viscosity [m/s^2],
g — acceleration of gravity [m/s^2],
i — complex unit [–],
m — mass of gas in the pressure transducer chamber [kg],
m_n — n-th order moment of a stochastic process [–],
p — pressure difference between the chambers in the pressure transducer [Pa],
p_k, p_o — absolute pressures in the pressure transducer chambers [Pa],
p_p — absolute pressure in the conduit connecting the Pitot-Prandtl tube to the pressure transducer [Pa],
p_s — static pressure of fluid in which velocity measurements are taken [Pa],
$p(t_1)$, $p(t_2)$ — pressure difference between the chambers of the pressure transducers at the instants t_1, t_2 [Pa],
$\overline{p(t_1)}$, $\overline{p(t_2)}$ — average values of the pressure difference between the chambers of the pressure transducers at the instants t_1, t_2 [Pa],
$p'(t_1)$, $p'(t_2)$ — time derivatives of the pressure difference between the chambers of the pressure transducers at the instants t_1, t_2 [Pa],
$\overline{p'(t_1)}$, $\overline{p'(t_2)}$ — average values of time derivatives of the pressure difference between the chambers of the pressure transducers at the instants t_1, t_2 [Pa],
p_1, p_2 — absolute pressure at the conduit inlet and outlet [Pa],

t — time [s],
w, $w(t)$ — measured velocity [m/s],
w — mean value of measured velocity [m/s],
w_p — average velocity of fluid flow (in the cross-section) in the conduits connecting the Pitot-Prandtl tube to the pressure transducer [m/s],
x — horizontal coordinate [m],
— vertical coordinate [m],
z_1, z_2 — inlet and outlet levels in the conduits connecting the Pitot-Prandtl tube to the pressure transducer [m],
Δp — pressure difference between the inlet and outlets of the conduit connecting the Pitot-Prandtl tube to the pressure transducer [Pa],
$\overline{\Delta p^*}$ — mean value of the pressure difference between the inlet and outlet of the Pitot--Prandtl tube [Pa],
Δp_k, Δp_o — infinitely small pressure increments in pressure transducer chambers [Pa],
Δt — infinitely small time increments
μ — coefficient of dynamic viscosity of the measured fluid [Ns/m^2],
ρ — density of the measured fluid [kg/m^3].

1. Introduction

Flows in ventilation networks in mine are always fluctuating, due to the travels of railway and hoist installations, opening of dams, and emission of gases. There are also some external factors, such as wind speed variations. Those fluctuations are rather slow due to the presence of huge amounts of air inside the ventilation system.

Turbulence is responsible for greater fluctuations of the velocity vector. Velocity measurements are mostly taken to evaluate the slow-changing stream of mass, hence the measured parameter is the slow-changing velocity vector component normal to the channel cross-section. The normal component is usually measured with the use of measuring instruments, where the direction characteristics resembles the cosine function pattern. It is of major importance as the velocity vector in the turbulent stream will change its direction and length. Major discrepancies between the direction characteristics and the cosine function pattern may lead to errors in airflow measurements. Vane anemometers and Pitot-Prandtl tubes are excellent for these applications as long as their dimensions are sufficiently small in relation to the vortex size.

In the presence of fast-changing fluctuations, dynamic properties of measuring instruments may become another source of errors. In mining ventilation practice vane anemometers and Pitot-Prandtl tubes are most commonly used. The tubes are used for measuring velocities in excess of 2 m/s. This study is mostly concerned with those devices.

Velocity indicated by the Pitot-Prandtl tube is usually derived from the formula:

$$\overline{w} = \sqrt{\frac{2\overline{\Delta p^*}}{\rho}} \tag{1}$$

In the case of huge fluctuations, such as those presented in Table 1, calculation of the average velocity from (1) involves a major error because the tube characteristic is quadratic. The uncertainty might be evaluated by averaging the left-hand and right-hand side of the equation describing the static properties of the tube:

$$E[\Delta p^*] = E\left[\frac{\rho w^2}{2}\right] = \frac{\rho}{2} E[(w - \overline{w} + \overline{w})^2] = \frac{\rho}{2}(\overline{w}^2 + D_w) \tag{2}$$

The operator $E[\Delta p^*]$ indicates an average-value operation while the slash denotes the averaged value.

After further transformations we get:

$$\sqrt{\frac{2\overline{\Delta p^*}}{\rho}} = \sqrt{\overline{w}^2 + D_w} \cong \overline{w} + \frac{D_w}{2\overline{w}} \tag{3}$$

This result would be obtained through averaging the pressure differences measured with infinitely fast devices. In order to determine the order/magnitude of uncertainty due to turbulence, the average values of velocity and their variances obtained from simultaneous measurements with use of two hot wire anemometers in the headings of identical cross-section are compiled in Table 1 (Cierniak i in. 1996). The widths of the frequency spectra of measured signals are also provided. The limit frequency is taken to be that at which the amplitude would reach about 10 % of the maximal value. The hot wire anemometer A3 was positioned nearer the heading wall than A2 (Chrzanowski 1961).

In real life situations the processes are much more complex than in an idealised, non-existing infinitely fast anemometer discussed above. It is worthwhile, however, to consider such hypothetical instruments as they allow for very fast, though approximate, evaluation of real processes.

TABLE 1

Main parameters of airflow velocity in the selected heading

TABLICA 1

Zestawienie parametrów charakteryzujących prędkość powietrza w wybranym chodniku

	Anemometer A2			Anemometer A3		
	Average velocity $\overline{w}$ [m/s]	Velocity variance D_w [m/s]^2	Spectrum width Δf [Hz]	Average velocity $\overline{w}$ [m/s]	Velocity variance D_w [m/s]^2	Spectrum width Δf [Hz]
1	0.22	0.0018	0–0.54	0.15	0.0037	0–1.0
2	1.23	0.076	0–0.30	1.01	0.037	0–0.5
3	1.84	0.20	0–1.50	1.38	0.058	0–2.0

Further analysis taking into account dynamic processes requires several assumptions as to the measuring circuit: both measuring circuits supplying the total and static pressure and both chambers of the pressure gauge are assumed to have the identical dynamical properties, flows in the conduits are laminar and changes in measurement chamber volume due to pressure changes are negligible. It is also assumed that pressure transducer is sufficiently fast so that its dynamical parameters might be neglected. Further studies require a mathematical model of the measuring circuit of a specific design. First of all it is necessary to describe the flow of fluids in the conduits.

In many cases flows in thin tubes are adequately described with a 1D equation of motion:

$$\frac{\partial w_p}{\partial t} + w_p \frac{\partial w_p}{\partial x} + \frac{1}{\rho}\frac{\partial p_p}{\partial x} + g\frac{dz}{dx} + f(w_p) = 0 \tag{4}$$

When pressure and temperature fluctuations are insignificant and tube cross-section is constant, the term $w_p = \frac{\partial w_p}{\partial x}$ can be omitted without the risk of a major error. Integration of both sides of thus simplified equation with respect to x yields:

$$L\frac{dw_p}{dt} + g(z_2 - z_1) + Lf(w_p) = -\frac{1}{\rho}(p_2 - p_1) \tag{5}$$

For simplicity, the stream of mass will be used in further calculations instead of velocity. The relationship between the stream of mass and velocity is given as:

$$Q = \rho S w_p \tag{6}$$

Accordingly, eq. (5) can be rewritten as:

$$A\frac{dQ}{dt} + BQ + H = \Delta p \tag{7}$$

For round conduits the coefficients in this equation are given by the formula:

$$A = \frac{4L}{\pi d^2}; \qquad B = \frac{128\mu L}{\pi d^4 \rho}; \qquad H = \rho g(z_1 - z_2); \qquad \Delta p = p_1 - p_2 \tag{8}$$

There is pressure difference Δp between the tube inlet and measuring chambers of the pressure gauge. Assuming the process to be isothermal, the starting point in description of pressure variations in the measuring chambers is the Clapeyron equation:

$$p_k V_k = mRT_k \tag{9}$$

Hence pressure increase is given by the formula:

$$\Delta p_k V_k = RT_k Q \Delta t \tag{10}$$

Moving towards the limit value, we get:

$$\frac{dp_k}{dt} = \frac{RT_k Q}{V_k} \tag{11}$$

The following relationship applies to the part of the measuring circuit connecting the tube inlet with the measuring chamber of the pressure gauge:

$$\frac{\rho w^2}{2} + p_s - p_k = \Delta p = A\frac{dQ}{dt} + BQ + H \tag{12}$$

Substituting (11) into (12) and ordering the terms, we obtain:

$$\frac{\rho w^2}{2} + p_s = \frac{AV_k}{RT_k}\frac{d^2 p_k}{dt^2} + \frac{BV_k}{RT_k}\frac{dp_k}{dt} + p_k + H \tag{13}$$

The second part of the measuring circuit, i.e. that connecting the side openings with the other chamber (the reference chamber) of the pressure gauge is described with the equation:

$$p_s = \frac{A_o V_o}{RT_o}\frac{d^2 p_o}{dt^2} + \frac{B_o V_o}{RT_o}\frac{dp_o}{dt} + p_o + H_o \tag{14}$$

When all the assumptions are fulfilled:

$$\frac{AV_k}{RT_k} = \frac{A_o V_o}{RT_o}; \qquad \frac{BV_k}{RT_k} = \frac{B_o V_o}{RT_o}; \qquad H = H_o \tag{15}$$

Subtracting eq. (13) and (14) by sides yields a relatively simple equation describing the links between the Pitot-Prandtl tube, conduits and the pressure gauge.

$$a\frac{d^2 p}{dt^2} + b\frac{dp}{dt} + ep = w^2 \tag{16}$$

where:

$$p = p_k - p_o; \qquad a = \frac{8LV_k}{\pi d^2 \rho RT}; \qquad b = \frac{256\mu V_k L}{\pi d^4 \rho RT}; \tag{17}$$

$$e = \frac{2}{\rho}; \qquad T = T_k - T_o; \qquad T = T_k = T_o$$

When the requirements set forth in (15) are not fulfilled, the readouts will be affected by static pressure and temperature in both circuits delivering pressure to the pressure gauge.

In the conditions of slow changes and small conduit cross-sections the inertia forces are negligible since they are quite insignificant in relation to the forces due to fluid viscosity, which is best seen when a is divided by b in formula (17). When the conduit diameter is $d = 4$, the ratio a/b would equal 0.03. Spectral analysis, being the part of earlier measurements, reveals that a significant portion of a spectrum is less than 2 Hz. This is indirectly confirmed by silence prevailing in most headings and shafts in the mine. As the frequencies are that low, the term including the second derivative of pressure can be easily neglected. The inertia being neglected, eq. (16) can be written in a simplified form:

$$b\frac{dp}{dt} + fp = w^2 \tag{18}$$

In order to find the average value of velocity two operations have to be performed on (18) and several assumptions are made. The considered stochastic process is assumed to be stationary and differentiable in terms of a mean square derivative. The distribution of the probability density function is taken to be normal. The first operation ascertains the mean value.

$$E[bp'+fp] = E[w^2] \tag{19}$$

The mean value of the derivative in a stationary process equals zero, hence the equation can be rewritten as:

$$fE[p] = f\overline{p} = E[(w - \overline{w} + \overline{w})^2] = E[(w - \overline{w})^2 + 2\overline{w}(w - \overline{w}) + \overline{w}^2] = \tag{20}$$

$$= E(w - \overline{w})^2 + 2\overline{w}E(w - \overline{w}) + \overline{w}^2 = \overline{w}^2 + D_w$$

The other operation involves computing the covariance of both sides of the equation (18); it is much more complex as seen from the formula:

$$E\{[bp'(t_1) + fp(t_1) - \overline{bp'(t_1) + fp(t_1)}][bp'(t_2) + fp(t_2) - \overline{bp'(t_2) + fp(t_2)}]\} = \tag{21}$$

$$= E\{[w^2(t_1) - \overline{w^2(t_1)}][w^2(t_2) - \overline{w^2(t_2)}]\}$$

Assuming the process to be stationary, we get:

$$\overline{p'(t_1)} = \overline{p'(t_2)} = \overline{p}' = 0 \tag{22}$$

$$\overline{p(t_1)} = \overline{p(t_2)} = \overline{p} \tag{23}$$

$$\overline{w^2(t_1)} = \overline{w^2(t_2)} = \overline{w}^2 + D_w \tag{24}$$

Accordingly, eq. (21) can be rewritten as:

$$E\{[bp'(t_1)-b\overline{p(t_1)}+fp(t_1)-f\,\overline{p}][bp'(t_2)-bp(t_2)+fp(t_2)-f\,\overline{p}]\}= \tag{25}$$

$$=E\{[w^2(t_1)-\overline{w}^2-D_w][w^2(t_2)-\overline{w}^2-D_w]\}$$

The terms in square brackets being duly multiplied, the left-hand side can be rewritten as:

$$E\begin{Bmatrix} b^2[p'(t_1)-\overline{p'}][p'(t_2)-\overline{p'}]+bf[p'(t_1)-\overline{p'}][p(t_2)-\overline{p}]+ \\ +bf[p'(t_2)-\overline{p'}][p(t_1)-\overline{p}]+f^2[p(t_1)-\overline{p}][p(t_2)-\overline{p}] \end{Bmatrix}=L \tag{26}$$

Changing the sequence of procedures: summation and differentiation with averaging, we get:

$$b^2\frac{\partial^2}{\partial t_1\partial t_2}E\{[p(t_1)-\overline{p}][p(t_2)-\overline{p}]\}+bf\frac{\partial}{\partial t_1}E\{[p(t_1)-\overline{p}][p(t_2)-\overline{p}]\}+ \tag{27}$$

$$+bf\frac{\partial}{\partial t_2}E\{[p(t_1)-\overline{p}][p(t_2)-\overline{p}]\}+f^2E\{[p(t_1)-\overline{p}][p(t_2)-\overline{p}]\}=L$$

By definition

$$E\{[p(t_1)-\overline{p}][p(t_2)-\overline{p}]\}=K_p(t_2-t_1)=K_p(\tau) \tag{28}$$

is the function of pressure covariance.

Accordingly, eq. (27) can be rewritten as:

$$b^2\frac{\partial^2}{\partial t_1\partial t_2}K_p(t_2-t_1)+bf\frac{\partial}{\partial t_1}K_p(t_2-t_1)+ \tag{29}$$

$$+bf\frac{\partial}{\partial t_2}K_p(t_2-t_1)+f^2K_p(t_2-t_1)=L$$

even the eq.:

$$\frac{\partial}{\partial t_1}K_p(t_2-t_1)=-\frac{\partial}{\partial t_2}K_p(t_2-t_1) \tag{30}$$

the eq. (29) can be simplified and written as:

$$-b^2\frac{\partial^2}{\partial \tau^2}K_p(\tau)+f^2K_p(\tau)=L \tag{31}$$

Assuming $\tau = t_2 - t_1 = 0$ and making use of (31), the eq (25) can be rewritten as:

$$-b^2 \frac{\partial^2}{\partial \tau^2} K_p(\tau)\Big|_{\tau=0} + f^2 D_p = E\{[w^2(t) - \overline{w}^2 - D_w]^2\} \tag{32}$$

as the covariance function for the zero argument becomes the variance function.

The terms in square brackets being computed, the right-hand side of (32) can be rewritten as:

$$P = E\{[w^2(t) - \overline{w}^2 - D_w]^2\} = E[w^4(t)] - 2(\overline{w}^2 + D_w)E(w^2(t)] + (\overline{w}^2 - D_w)^2 \tag{33}$$

The first term of the right-hand side in (33) is an ordinary fourth-order moment. The second term includes a second-order moment. These two moments are directly computable by the characteristic function of velocity distribution. For the normal distribution it has the form (Swisznikow 1965):

$$G(\omega) = \exp\left(i\overline{w}\omega - \frac{D_w \omega^2}{2} \right) \tag{34}$$

The moment of the m-th order is obtained from the formula (Swisznikow 1965):

$$m_m = \frac{1}{i^m} \frac{d^m}{d\omega^m} G(\omega)\Big|_{\omega=0} \tag{35}$$

The second and fourth derivative of the covariance function (given in (34)) are given as:

$$\frac{d^2}{d\omega^2} G(\omega) = \left[-D_w + (i\overline{w} - D_w \omega)^2\right] \exp\left(i\overline{w}\omega - \frac{D\omega^2}{2} \right) \tag{36}$$

$$\frac{d^4}{d\omega^4} G(\omega) = [3D_w^2 - 6D_w(i\overline{w} - D_w\omega)^2 + (i\overline{w} - D_w\omega)^4] \exp\left(i\overline{w} - \frac{D_w \omega^2}{2} \right) \tag{37}$$

Given eq. (35), it is readily apparent that:

$$m_2 = D_w + \overline{w}^2 \tag{38}$$

$$m_4 = 3D_w^2 + 6D_w \overline{w}^2 + \overline{w}^4 \tag{39}$$

while eq. (33) is rewritten as:

$$P = E\{[w^2(t) - \overline{w}^2 - D_w]^2\} = 3D_w^2 + 6D_w \overline{w} + \overline{w}^4 - (\overline{w}^2 + D_w)^2 = \tag{40}$$

$$= 2D_w^2 + 4\overline{w}^2 D_w$$

Substituting this formula into eq. (32) yields:

$$-b^2 \frac{\partial^2}{\partial \tau^2} K_p(0) + f^2 D_p = 2D_w^2 + 4\overline{w}^2 D_w \tag{41}$$

When the systems of equations (20) and (41) are solved, the average velocity can be derived from the formula:

$$\overline{w} = \sqrt[4]{(f\,\overline{p})^2 - \frac{1}{2}\left[-b^2 \frac{\partial^2}{\partial \tau^2} K_p(0) + f^2 D_p\right]} \tag{42}$$

Conclusions

The formula (42) yields the correct value of the average velocity, taking into account the properties of a stochastic process i.e. the function of velocity and properties of the Pitot-Prandtl tube, a pressure gauge and connecting conduits. When an electronic pressure gauge, an analogue/digital processor and a processor are connected to the Pitot-Prandtl tube, a new device can be built, implementing the formula (42) and hence implementing the correct procedure of average velocity measurements.

This study is a part of the research activities pursued at the Institute.

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