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Linearized asymptotic stability for nabla Riemann-Liouville fractional difference equation

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In this paper, we present a theorem about stability of nonlinear fractional difference equation with Riemann-Liouvile difference operator. The result is a version of classical theorem on linear approximation and to derive them, we prove the variation of constants formula for nabla Riemann-Liouville fractional difference equations. We also present some results concerning the existence and uniqueness of the equation under consideration.

Key words: stability of nonlinear fractional difference equation, Riemann-Liouville difference operator, nabla Riemann-Liouville fractional difference equations

1. Introduction

The study of the stability of nonlinear systems through their linear approximation has a long history, formalized and structured by the well-known work of A.M. Lyapunov [1]. A comprehensive description of the state of the art of this field for systems described by differential equations can be found in [2], and for discrete systems in [3]. The classical results of this theory make it possible to infer the stability of a nonlinear system on the basis of the stability of its linear approximation and some assumptions about the nonlinear part.

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Recently, many researchers in the field of dynamic systems use models with derivatives or differences of fractional order, see e.g. [4–7] and the references therein. Among other reasons, this follows from many successful applications of fractional calculus to modeling of various problems via fractional differential and difference equations. These applications arise in various areas such as control theory, signal processing and the theory of viscoelasticity, see e.g. [8–14].

This paper deals with discrete time systems where different types of fractional differences are considered in the literature (forward/backward, Caputo, Riemann-Liouville, Grünwald-Letnikov), see [15–17]. In this paper we consider a discrete-time fractional nonlinear system with backward Riemann-Liouville differences which is also called nabla Riemann-Liouville system.

Stability as an important property of control systems [18] receives wide attention, and various stability issues of fractional order systems have been studied (see e.g. [19, 20]). The classic Lyapunov method becomes an attractive strategy, which is very convenient and practical because it does not need explicit solution of the differential equations. In continuous fractional-order nonlinear systems, many results for stability analysis have been obtained. The Mittag-Leffler stability definition is proposed to describe the dynamics of the system, and the fractional direct Lyapunov method is introduced creatively in [21, 22], which inspired many scientists and derived a series of pioneering work. Based on methodology of the frequency-distributed model, a Lyapunov approach has been presented to analyze the stability of fractional-order systems [23].

In terms of discrete fractional-order nonlinear dynamic systems, related work on stability analysis is scattered along the literature and faces many challenges. We cannot directly apply methods known from the theory of fractional differential equations. Discrete fractional direct Lyapunov method is consider in [24]. Besides, the authors extend an inequality from the continuous case [25] and give a sufficient condition for stability of Caputo delta fractional difference equations. The stability criterion under the definition of Riemann-Liouville difference is given in [26], yet, it should be mentioned that Refs. [24, 26] use noncausal forward difference. The definition of discrete Mittag-Leffler stability given in [27] also adopts the forward difference and limits the initial instant to 0, which makes the conclusion less general. To compensate for the difficulty in obtaining fractional difference of Lyapunov functions, in [28] some useful inequalities under different definitions, which bring great convenience to the use of discrete fractional direct Lyapunov method are proposed. Nevertheless, most current studies of stability analysis concentrate on convergence in the steady state, while the convergence rule is rarely investigated. Still an open problem is to how the dynamics of the system should be characterized. Thus far, for discrete fractional-order systems, systematic and complete framework for stability analysis has not been conducted.

The Schauder fixed point theorem in the asymptotic stability of nonlinear fractional difference equations is used [29]. The stability regions for linear fractional difference systems using Laplace transform is investigated in [30]. The authors of [31] established comparison theorems to extend the corresponding asymptotic result in [32]. The linearization to decide the stability of fractional difference systems is presented in [33].

In [34], a new definition of discrete Mittag-Leffler stability is proposed, producing a novel stability description of discrete fractional-order systems. Besides, using Lyapunov direct method, some useful criteria for analyzing stability of nabla discrete fractional-order systems are derived. The presented methods are applicable to both Caputo and Riemann-Liouville definitions, and a useful inequality is given to further improve the practicality of the discrete fractional direct Lyapunov method.

A new way to examine the asymptotic stability of nabla discrete fractional order systems is proposed in [35]. In this paper, several useful inequalities on fractional difference of Lyapunov functions have been investigated. Note that, all the inequalities are applicable for Riemann-Liouville, Caputo and Grünwald-Letnikov definitions. Applying these results, the classical Lyapunov theory can be used to analyze the stability of discrete fractional nonlinear systems without and with delays. The stability analysis for the discrete-time case has been also investigated in [36–38]. The explicit stability conditions for a linear fractional difference system with the Caputo-type operator are presented in [39, 40]. Additionally, in [39], the discussion concerning stability behavior of systems with the Riemann-Liouville-type difference operator is given. The main goal of the paper [41] is to formulate the stability conditions for the nonlinear systems with the difference fractional operators. The linearization of the considered nonlinear systems is used to formulate the conditions that guarantee the local asymptotical stability of nonlinear difference systems. Moreover, the fact that fractional derivatives can be approximate by fractional h-differences of corresponding types is used to show the relations between the stability of nonlinear fractional order differential systems and the stability of linear discrete-time systems with the fractional h-difference operators. In that paper the delta type fractional order systems are investigated. The main result of [41] contains a relatively strong assumption on decay of the utilized discrete Mittag-Leffler function (hence does not provide a direct fractional analogue to the classical linearization theorems known from the theory of first-order differential or difference systems). Therefore, the main goal of paper [42] is to prove the stability part of the linearization theorem for fractional difference equations, i.e., to provide a discrete analogue to [43]. The authors consider a system with the backward Caputo fractional difference operator due to its better numerical stability properties compared to the forward discretization.

The existence and uniqueness of solutions is the basis for studying the stability problem, but it is sometimes neglected. Differential and integral inequalities are important tools for exploring existence, uniqueness and stability of solutions of differential and difference equations (see, [44, 45] and the references therein). In [46], the local uniqueness of solutions of the fractional integro-differential equations was established with the help of Schauder's fixed-point theorem. In [47], local uniqueness and global uniqueness of solution of the initial value problem was given by use of the Gronwall inequality and the Bihari inequality. The works on uniqueness of nonlinear fractional difference systems can be divided mainly into two groups. The contractive mapping approach is used in first group, for instance [48–51]. The fractional Gronwall inequalities are used in second group, for example [52, 53]. However, most nonlinear functions in the existing literature on fractional difference systems are Lipschitz continuous.

With respect to paper [42], we deal with Riemann-Liouville equation instead of Caputo equation. Main result of this paper is the theorem about stability on the linear approximation. To obtain this result we establish a version of variation of constants formula for nonlinear nabla Riemann-Liouville fractional difference equations which is interesting by itself. In fact, this work can be considered as a discrete version of [54]. It should be emphasized, however, that certain difficulties arise for discrete systems compared to continuous systems. First, for discrete systems, expressing the solution through a discrete Mittag-Leffler function is possible only in a certain sub-area of the area in which the solution exists (see point 2 of Theorem 1 below). Second, the asymptotic properties of the discrete Mittag-Leffler function are much less studied than its continuous counterpart (see, [55, Eq. (4.1)]). The relations between Mittag-Leffler function and the solution of linear approximation equation as well as asymptotic growth rate are one of the main properties used in [54]. Despite these difficulties, we managed to show that if the eigenvalues of the matrix of the linear approximation equation are in the stability set of the linear equation and the nonlinear part satisfies a certain minor condition in the sense of the Lipschitz condition, then the null solution of the nonlinear equation is asymptotically locally stable.

The following notations will be used throughout this paper: by $\mathbb{R}$ we denote the set of real numbers, by $\mathbb{Z}$ the set of integers, by $\mathbb{N} := \mathbb{Z}_{\geqslant 0}$ the set of natural numbers $\{0, 1, 2, \dots\}$ including 0, and by $\mathbb{Z}_{\leqslant 0} := \{0, -1, -2, \dots\}$ the set of non-positive integers. For $a \in \mathbb{R}$ we denote by $\mathbb{N}_a := a + \mathbb{N}$ the set $\{a, a+1, \dots\}$. Moreover, we will use the following symbols:

$$\|x\| := \max_{1\leqslant i\leqslant d} |x_i| \quad \text{for} \quad (x_1, \dots, x_d) \in \mathbb{R}^d,$$

$$B_{\mathbb{R}^d(0,r)} := \left\{x \in \mathbb{R}^d : \ \|x\| \leqslant r\right\},$$

$$S_\alpha := \left\{ z \in \mathbb{C} \colon \; |\mathrm{Arg}(z)| > \frac{\alpha\pi}{2} \text{ or } |z| > \left(2\cos\left(\frac{\mathrm{Arg}(z)}{\alpha}\right)\right)^\alpha \right\}.$$

For a square matrix $A \in \mathbb{R}^{d\times d}$ we denote by $\sigma(A)$ the spectrum of A i.e. the set of all eigenvalues of A. By $I_d \in \mathbb{R}^{d\times d}$ we will denote the identity matrix. The set of all sequences $x : \mathbb{N}_1 \to \mathbb{R}^d$ will be denoted by $l(\mathbb{N}_1)$, and such that $\sum_{k=1}^{\infty} \|x(k)\|^p$ converges will be denoted by $l^p(\mathbb{N}_1)$. By $l^\infty(\mathbb{N}_1)$ we will denote the Banach space of all bounded sequences $x : \mathbb{N}_1 \to \mathbb{R}^d$ with the norm $\|x\|_\infty = \sup_{n\in\mathbb{N}_1} \|x(n)\|$.

For a matrix $C = \left[c_{ij}\right]$ the symbol $|C|$ denotes the matrix given by $|C| = (|c_{ij}|)$.

We recall some notions concerning fractional summation and fractional differences. By $\Gamma \colon \mathbb{R} \setminus \mathbb{Z}_{\leqslant 0} \to \mathbb{R}$ we denote the Euler- Gamma function defined by

$$\Gamma(\alpha) := \lim_{n\to\infty} \frac{n^\alpha n!}{\alpha(\alpha+1)\cdots(\alpha+n)}$$

for $\alpha \in \mathbb{R}\setminus\mathbb{Z}_{\leqslant 0}$, which is well-defined, since the limit exists, see e.g. [56, p. 156].

For $s \in \mathbb{R}$ with $s+1, s+1+\alpha \notin \mathbb{Z}_{\leqslant 0}$ the raising factorial power $(s)^{\overline{(\alpha)}}$ is defined by

$$(s)^{\overline{(\alpha)}} = \frac{\Gamma(s+\alpha)}{\Gamma(s)}$$

for $s \in (\mathbb{R} \setminus \mathbb{Z}_{\leqslant -1}) \cap (\mathbb{R} \setminus (-\alpha + \mathbb{Z}_{\leqslant -1}))$. For $r \in \mathbb{R}$ and $m \in \mathbb{Z}$ the binomial coefficient $\binom{r}{m}$ is defined as follows (see [57, Section 5.1, formula (5.1)])

$$\binom{r}{m} = \begin{cases} \dfrac{r(r-1)\cdots(r-m+1)}{m!} & \text{if } \; m \in \mathbb{Z}_{\geqslant 1}, \\ 1 & \text{if } \; m = 0, \\ 0 & \text{if } \; m \in \mathbb{Z}_{\leqslant -1}. \end{cases}$$

For $\nu \in \mathbb{R}_{\geqslant 0}$ and a function $x : \mathbb{N}_1 \to \mathbb{R}$, the ν-th nabla fractional sum $\nabla^{-\nu} x \colon \mathbb{N}_1 \to \mathbb{R}^d$ of order ν of x is defined as

$$(\nabla^{-\nu} x)(n) = \sum_{k=1}^{n} (-1)^{n-k} \binom{-\nu}{n-k} x(k) \tag{1}$$

for $n \in \mathbb{N}_1$ (see [16]). Let $\alpha \in (0,1)$ and $x \colon \mathbb{N}_1 \to \mathbb{R}^d$. The nabla Riemann-Liouville difference ${}_{\mathrm{R-L}}\nabla^\alpha x \colon \mathbb{N}_2 \to \mathbb{R}$ of x of order α is defined as

$${}_{\mathrm{R-L}}\nabla^\alpha = \nabla \circ \nabla^{-(1-\alpha)},$$

i.e.

$$({}_{\mathrm{R-L}}\nabla^\alpha x)(n) = (\nabla\nabla^{-(1-\alpha)} x)(n) \tag{2}$$

for $n \in \mathbb{N}_2$, where ∇ is a backward difference operator, i.e.

$$\nabla x(n) = x(n) - x(n-1),$$

see [16]. It can be shown [58] that

$$({}_{\mathrm{R-L}}\nabla^{\alpha} x)(n) = \sum_{k=1}^{n} (-1)^{n-k} \binom{\alpha}{n-k} x(k), \quad n \in \mathbb{N}_2. \tag{3}$$

The following definition can be found e.g. in [59] (see also [55]).

Definition 1. *For a matrix $A \in \mathbb{C}^{d\times d}$ such that $\sigma(A) \subset \{z \in \mathbb{C} : |z| < 1\}$ and $\alpha,\ \beta > 0$, the nabla discrete-time Mittag-Leffler type function is defined by*

$$E_{\overline{(\alpha,\beta)}}(A,n) = \sum_{k=0}^{\infty} A^k \frac{n^{\overline{k\alpha+\beta-1}}}{\Gamma(\alpha k+\beta)} = \sum_{k=0}^{\infty} A^k \binom{\alpha k+\beta-2+n}{\alpha k+\beta-1}, \quad n \in \mathbb{N}_1\,.$$

The condition $\sigma(A) \subset \{z \in \mathbb{C} : |z| < 1\}$ ensures the convergence of the series in definition of $E_{\overline{(\alpha,\beta)}}(A,n)$ (for more detail, see [55, Eq. (4.1)]).

2. Preliminaries

In this paper, we consider linear inhomogeneous fractional difference systems of the form

$$({}_{\mathrm{R-L}}\nabla^{\alpha} x)(n) = Ax(n) + f(x(n)), \quad n \in \mathbb{N}_2 \tag{4}$$

where $x\colon \mathbb{N}_1 \to \mathbb{R}^d$, ${}_{\mathrm{R-L}}\nabla^{\alpha}$ is Riemann-Liouville difference operator of a real order $\alpha \in (0,1)$, $f : \mathbb{R}^d \to \mathbb{R}^d$ is a continuous function and $A \in \mathbb{R}^{d\times d}$.

We denote by $\varphi(n,x_1)$ the solution of (4) with initial condition $\varphi(1,x_1) = x_1$. Observe that the problem of existence of the solution is not a trivial problem since $x(n)$ is on the left and right hand side of (4). In case of $f \equiv 0$, the solution of (4) with initial condition $x_1 \in \mathbb{R}^d$ will be denoted by $\overline{\varphi}(\cdot,x_1) : \mathbb{N}_1 \to \mathbb{R}^d$. We will use the following standard definition of stability of trivial solution of (4).

Definition 2. *Suppose that $f(0) = 0$. The trivial solution of (4) is called:*

1) *stable if for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for every $\|x_0\| < \delta$ we have*

$$\|\varphi(n,x_0)\| \leqslant \epsilon \text{ for } n \geqslant 0;$$

2) *unstable if it is not stable;*
3) *attractive if there exists $\widehat{\delta} > 0$ such that $\lim_{n\to\infty} \varphi(n,x_0) = 0$ whenever $\|x_0\| < \widehat{\delta}$;*
4) *asymptotically stable if it is both stable and attractive.*

Using the representation (3) we may rewrite (4) as a Volterra convolution equation as it is stated in the following lemma.

Lemma 1. *The sequence $x : \mathbb{N}_1 \to \mathbb{R}^d$ is the solution of (4) with initial condition x_1 if and only if $x(1) = x_1$ and*

$$x(n) - Ax(n) - f(x(n)) = \sum_{k=1}^{n-1} (-1)^{n-1-k} \binom{\alpha}{n-k} x(k), \; n \in \mathbb{N}_2 . \tag{5}$$

Using this representation of solution of (4) we will prove the following result providing a necessary and sufficient condition for the existence and uniqueness of global solution of (4) for all initial conditions $x_1 \in \mathbb{R}^d$. Later, in Theorem 3, we will show another condition guaranteeing the existence of a global solution to equation (4) but only in some neighborhood of 0.

Lemma 2. *For each $x_1 \in \mathbb{R}^d$ there exists a unique global solution of (4) if and only if the function $g : \mathbb{R}^d \to \mathbb{R}^d$ given by*

$$g(x) = x - Ax - f(x)$$

is a bijection.

Proof. Suppose that g is a bijection and denote by $g^{-1} : \mathbb{R}^d \to \mathbb{R}^d$ the inverse function. Then for each $x_1 \in \mathbb{R}^d$, we may define a sequence $x(n)$, $n \in \mathbb{N}_1$ by $x(1) = x_1$ and

$$x(n) = g^{-1} \left(\sum_{k=1}^{n-1} (-1)^{n-1-k} \binom{\alpha}{n-k} x(k) \right), \quad n \in \mathbb{N}_2 .$$

From (5) it follows, that $x(n)$, $n \in \mathbb{N}_1$ is the unique and global solution of (4) corresponding to initial condition x_1. Suppose now that for each $x_1 \in \mathbb{R}^d$ there exists a unique global solution of (4) and g is not a surjection i.e. $\mathrm{Img} \neq \mathbb{R}^d$. Consider any $x_1 \in \mathbb{R}^d$ such that $\alpha x_1 \notin \mathrm{Img}$, then by (5) for $n = 2$ we have

$$g(x(2)) = \alpha x_1 .$$

This is a contradiction with $\alpha x_1 \notin \mathrm{Img}$, therefore $\mathrm{Img} = \mathbb{R}^d$. Suppose now that g is not a injection i.e. there exists a $\alpha x_1 \in \mathbb{R}^d$ and $\overline{x}_2, \widetilde{x}_2 \in \mathbb{R}^d$, $\overline{x}_2 \neq \widetilde{x}_2$ such that

$$g(\overline{x}_2) = g(\widetilde{x}_2) = \alpha x_1 .$$

The last inequality shows, in the context of (5) for $n = 2$, that the solution with initial conditions $x(1) = \alpha x_1$ is not uniquely defined for $n = 2$. This contradiction completes the proof. □

The next theorem collects some known facts about linear equation

$$({}_{\text{R-L}}\nabla^{\alpha}x)(n) = Ax(n), \quad 0 < \alpha < 1, \;\; n \in \mathbb{N}_2\,. \tag{6}$$

Theorem 1. *Consider (4) with $f \equiv 0$. The following facts hold*

1. *For each initial condition $x_1 \in \mathbb{R}^d$ there exists a unique and global solution $\overline{\varphi}(\cdot, x_1) : \mathbb{N}_1 \to \mathbb{R}^d$ if and only if $I - A$ is invertible;*
2. *If $\sigma(A) \subset \{z \in \mathbb{C} : |z| < 1\}$, then*
$$\overline{\varphi}(n, x_1) = (I_d - A)\, E_{\overline{(\alpha,\alpha)}}(A, n)x_1$$
for all $x_1 \in \mathbb{R}^d$
3. *If $\sigma(A) \subset S_\alpha$, then (6) is asymptotically stable and $\overline{\varphi}(\cdot, x_1) \in l^1(\mathbb{N}_1)$ for all $x_1 \in \mathbb{R}^d$;*
4. *If $\sigma\left(|I_d - A|^{-1}\right) \subset \{z \in \mathbb{C} : |z| < 1\}$, then for each $x_1 \in \mathbb{R}^d$ there exists a $C > 0$ such that*
$$\|\overline{\varphi}(n, x_1)\| \leqslant \frac{C}{n^{1+\alpha}}$$
for all $n \in \mathbb{N}_1$.

Proof. Point 1 follows form Lemma 2. Points 2, 3 and 4 are proved in [59], Theorem 18 and Theorem 6. Notice, however that in [59] the solution of (6) is defined on $\mathbb{N}_0$ with the condition that $x_1 = (I_d - A)^{-1}x_0$. □

We end this section with a theorem containing the so-called variation of constants formula. A particular case of this formula has been proved in [60, Theorem 1].

Theorem 2 (Variation of constants formula)**.** *Suppose that $I_d - A$ is invertible and function $g : \mathbb{R}^d \to \mathbb{R}^d$ given by $g(x) = x - Ax - f(x)$ is a bijection with the inverse function $g^{-1} : \mathbb{R}^d \to \mathbb{R}^d$. Then the unique global solutions $\varphi(\cdot, x_1)$, $\overline{\varphi}(\cdot, x_1')$, $x_1, x_1' \in \mathbb{R}^d$ of the initial value problem for (4) and (6), respectively, are related as follows*

$$\varphi(n, x_1) = \overline{\varphi}(n, x_1) + (I_d - A)^{-1} \sum_{k=1}^{n-1} \overline{\varphi}\left(k, f\left(\varphi(n-k+1, x_1)\right)\right) \tag{7}$$

for all $n \in \mathbb{N}_2$. In particular φ can be recursively expressed by $\overline{\varphi}$ as follows

$$\varphi(n, x_1) = g^{-1}\left((I_d - A)\,\overline{\varphi}(n, x_1) + \sum_{k=2}^{n-1} \overline{\varphi}\left(k, f\left(\varphi(n-k+1, x_1)\right)\right)\right), \tag{8}$$

for all $n \in \mathbb{N}_2$.

Proof. We will show (7) by mathematical induction. Using (5) for $n = 2$ we have

$$\varphi(2, x_1) - A\varphi(2, x_1) - f(\varphi(2, x_1)) = \alpha\varphi(1, x_1)$$

and therefore

$$\varphi(2, x_1) = (I_d - A)^{-1}\alpha x_1 + (I_d - A)^{-1} f(\varphi(2, x_1)).$$

Since by (5) for $n = 2$ and $f \equiv 0$ we know that $\overline{\varphi}(2, x_1) = (I_d - A)^{-1}\alpha x_1$, then

$$\varphi(2, x_1) = \overline{\varphi}(2, x_1) + (I_d - A)^{-1} f(\varphi(2, x_1))$$

and therefore (7) is true for $n = 2$. Suppose now that (7) is true for all $k = 2, \ldots, n$ and certain $n \in \mathbb{N}_2$. We will show that it is true also for $n + 1$. According to (5) we have

$$\varphi(n+1, x_1) - A\varphi(n+1, x_1) - f(\varphi(n+1, x_1)) = \sum_{k=1}^{n}(-1)^{n-k}\binom{\alpha}{n+1-k}\varphi(k, x_1)$$

and therefore

$$\begin{aligned}\varphi(n+1, x_1) = (I_d - A)^{-1}&\sum_{k=1}^{n}(-1)^{n-k}\binom{\alpha}{n+1-k}\varphi(k, x_1)\\ &+ (I_d - A)^{-1} f\big(\varphi(n+1, x_1)\big).\end{aligned} \tag{9}$$

We have

$$\varphi(1, x_1) = x_1$$

and by the induction hypothesis

$$\varphi(k, x_1) = \overline{\varphi}(k, x_1) + (I_d - A)^{-1}\sum_{j=1}^{k-1}\overline{\varphi}\big(j, f\big(\varphi(k - j + 1, x_1)\big)\big)$$

for $k = 2, \ldots, n$. Using the last two equalities in (9) we get

$$\begin{aligned}\varphi(n+1, x_1) = (I_d - A)^{-1}&\sum_{k=2}^{n}(-1)^{n-k}\binom{\alpha}{n+1-k}\Bigg(\overline{\varphi}(k, x_1)\\ &+ (I_d - A)^{-1}\sum_{j=1}^{k-1}\overline{\varphi}(j, f(\varphi(k - j + 1, x_1)))\Bigg)\\ &+ (I_d - A)^{-1}(-1)^{n-1}\binom{\alpha}{n}x_1 + (I_d - A)^{-1} f(\varphi(n+1, x_1))\end{aligned}$$

$$
\begin{aligned}
&= (I_d - A)^{-1} \sum_{k=2}^{n} (-1)^{n-k} \binom{\alpha}{n+1-k} \overline{\varphi}(k, x_1) \\
&\quad + (I_d - A)^{-1} \sum_{k=2}^{n} (-1)^{n-k} \binom{\alpha}{n+1-k} (I_d - A)^{-1} \sum_{j=1}^{k-1} \overline{\varphi}\left(j, f\left(\varphi(k-j+1, x_1)\right)\right) \\
&\quad + (I_d - A)^{-1} (-1)^{n-1} \binom{\alpha}{n} \overline{\varphi}(1, x_1) + (I_d - A)^{-1} f\left(\varphi(n+1, x_1)\right) \\
&= (I_d - A)^{-1} \sum_{k=1}^{n} (-1)^{n-k} \binom{\alpha}{n+1-k} \overline{\varphi}(k, x_1) \\
&\quad + (I_d - A)^{-1} \sum_{k=2}^{n} \sum_{j=1}^{k-1} (-1)^{n-k} \binom{\alpha}{n+1-k} (I_d - A)^{-1} \overline{\varphi}\left(j, f\left(\varphi(k-j+1, x_1)\right)\right) \\
&\quad + (I_d - A)^{-1} f\left(\varphi(n+1, x_1)\right).
\end{aligned}
$$

From (5) with $f \equiv 0$ we know that

$$
\overline{\varphi}(n+1, x_1) = (I_d - A)^{-1} \sum_{k=1}^{n} (-1)^{n-k} \binom{\alpha}{n+1-k} \overline{\varphi}(k, x_1).
$$

Therefore to complete the induction proof it is enough to show that

$$
\begin{aligned}
(I_d - A)^{-1} \sum_{k=2}^{n} \sum_{j=1}^{k-1} (-1)^{n-k} \binom{\alpha}{n+1-k} \overline{\varphi}\left(j, f\left(\varphi(k-j+1, x_1)\right)\right) \\
+ f\left(\varphi(n+1, x_1)\right) = \sum_{k=1}^{n} \overline{\varphi}\left(k, f\left(\varphi(n-k+2, x_1)\right)\right),
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
(I_d - A)^{-1} \sum_{k=2}^{n} \sum_{j=1}^{k-1} (-1)^{n-k} \binom{\alpha}{n+1-k} \overline{\varphi}\left(j, f\left(\varphi(k-j+1, x_1)\right)\right) \\
= \sum_{k=2}^{n} \overline{\varphi}\left(k, f\left(\varphi(n-k+2, x_1)\right)\right).
\end{aligned}
$$

Expressing $\overline{\varphi}\left(\cdot, f\left(\varphi(n-k+2, x_1)\right)\right)$ according to (5) with $f \equiv 0$ we get

$$\sum_{k=2}^{n} \overline{\varphi}\left(k, f\left(\varphi(n-k+2, x_1)\right)\right)$$
$$= (I_d - A)^{-1} \sum_{k=2}^{n} \sum_{j=1}^{k-1} (-1)^{k-1-j} \binom{\alpha}{k-j} \overline{\varphi}\left(j, f\left(\varphi(n-k+2, x_1)\right)\right).$$

Let us fix $n \in \mathbb{N}_2$ and suppose that for any $k \in \{2, \ldots, n\}$ and $j \in \{1, \ldots, k-1\}$ we define a real number $a(k, j)$. It is clear that

$$\sum_{k=2}^{n} \sum_{j=1}^{k-1} a(k, j) = \sum_{k=2}^{n} \sum_{j=1}^{k-1} a(n+1-k+j, j).$$

Applying the last identity with

$$a(k, j) = (-1)^{k-1-j} \binom{\alpha}{k-j} \overline{\varphi}\left(j, f\left(\varphi(n-k+2, x_1)\right)\right)$$

we get

$$\sum_{k=2}^{n} \sum_{j=1}^{k-1} (-1)^{k-1-j} \binom{\alpha}{k-j} \overline{\varphi}\left(j, f\left(\varphi(n-k+2, x_1)\right)\right)$$
$$= \sum_{k=2}^{n} \sum_{j=1}^{k-1} (-1)^{n-k} \binom{\alpha}{n+1-k} \overline{\varphi}\left(j, f\left(\varphi(k-j+1, x_1)\right)\right).$$

This completes the proof of (7). And, (8) follows immediately from (7). $\square$

3. Main result

From now, we assume that $f : \mathbb{R}^d \to \mathbb{R}^d$ is a locally Lipschitz continuous function satisfying that

$$f(0) = 0, \quad \lim_{r \to 0} l_f(r) = 0, \tag{10}$$

where $l_f(r)$ is denoted to be the Lipschitz constant

$$l_f(r) := \sup_{x,y \in B_{\mathbb{R}^d}(0;r); x \neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|} \tag{11}$$

of f on the ball $B_{\mathbb{R}^d(0,r)}$.

The main result of this paper is contained in the following theorem.

Theorem 3. *Suppose that $f : \mathbb{R}^d \to \mathbb{R}^d$ satisfies conditions (10) and that $\sigma(A) \subset S_\alpha$. There exists $\varepsilon > 0$ such that for all $x_1 \in \mathbb{R}^d$, $\|x_1\| < \varepsilon$ there exists the unique global solutions $\varphi(\cdot, x_1)$, of the initial value problem for (4) and the trivial solution of (4) is asymptotically stable.*

Before presenting the proof of this theorem we will introduce for all $x \in \mathbb{R}^d$ an operator $T_x : l(\mathbb{N}_1 \to l(\mathbb{N}_1)$. Assume that $I_d - A$ is invertible. Let us fix $\varphi \in l(\mathbb{N}_1)$ and define

$$(T_x\varphi)(n) = \overline{\varphi}(n,x) + (I_d - A)^{-1} \sum_{k=1}^{n-1} \overline{\varphi}\left(k, f\left(\varphi(n-k+1)\right)\right), \quad n \in \mathbb{N}_1,$$

where $\overline{\varphi}(\cdot, x) : \mathbb{N}_1 \to \mathbb{R}^d$ is the unique global solution of initial value problem

$$\left({}_{\text{R-L}}\nabla^\alpha x\right)(n) = Ax(n), \quad n \in \mathbb{N}_2, \tag{12}$$

$x(1) = x$. In the definition of T_x we use the convention that $\Sigma_{k=1}^0 := 0$. The operator T_x is called Lyapunov-Perron operator, and its role is stated in the following theorem, which follows from definition of T_x and the variation of constants formula (7).

Proposition 1. *Let $x_1 \in \mathbb{R}^d$ be arbitrary and $\varphi \in l(\mathbb{N}_1, \mathbb{R}^d)$ is such that $\varphi(1) = x_1$. Then, the following statements are equivalent:*

1. *φ is a solution of the initial value problem for (4) with $\varphi(1) = x_1$.*
2. *φ is a fixed point of the operator T_{x_1}.*

Next we show some estimates on T_x in particular we will show that $T_x : l^\infty(\mathbb{N}_1) \to l^\infty(\mathbb{N}_1)$.

Proposition 2. *Consider system (4) and suppose that $\sigma(A) \subset S_\alpha$. Then for each $\varphi \in l^\infty(\mathbb{N}_1)$ and $x \in \mathbb{R}^d$ we have $T_x\varphi \in l^\infty(\mathbb{N}_1)$, moreover there exists a constant $C(\alpha, A) > 0$ such that for each $x_1, x_2 \in \mathbb{R}^d$ and $\varphi_1, \varphi_2 \in l^\infty(\mathbb{N}_1)$ the following inequality holds*

$$\begin{aligned}\left\|T_{x_1}\varphi_1 - T_{x_2}\varphi_2\right\|_\infty \leqslant C(\alpha, A)\big(&\|x_1 - x_2\| \\ &+ l_f\left(\max\left\{\|\varphi_1\|_\infty, \|\varphi_2\|_\infty\right\}\right)\|\varphi_1 - \varphi_2\|_\infty\big).\end{aligned} \tag{13}$$

In particular

$$\|T_x\varphi_1 - T_x\varphi_2\|_\infty \leqslant C(\alpha, A) l_f\left(\max\left\{\|\varphi_1\|_\infty, \|\varphi_2\|_\infty\right\}\right)\|\varphi_1 - \varphi_2\|_\infty \tag{14}$$

for all $x \in \mathbb{R}^d$.

Proof. Let us fix $x_1, x_2 \in \mathbb{R}^d$ and $\varphi_1, \varphi_2 \in l^\infty(\mathbb{N}_1)$. We have

$$\begin{aligned}
\left\|\left(T_{x_1}\varphi_1\right)(n) - \left(T_{x_2}\varphi_2\right)(n)\right\| &\leqslant \left\|\overline{\varphi}\left(n, x_1\right)_{\mathrm{R-L}} - \overline{\varphi}\left(n, x_2\right)\right\| \\
&\quad + \left\|\left(I_d - A\right)^{-1}\right\| \left\|\sum_{k=1}^{n-1} \overline{\varphi}\big(k, f(\varphi_1(n-k+1))\big) - \sum_{k=1}^{n-1} \overline{\varphi}\big(k, f(\varphi_2(n-k+1))\big)\right\| \\
&= \left\|\overline{\varphi}\left(n, x_1 - x_2\right)\right\| \\
&\quad + \left\|\left(I_d - A\right)^{-1}\right\| \left\|\sum_{k=1}^{n-1} \overline{\varphi}\left(k, f\left(\varphi_1(n-k+1)\right) - f\left(\varphi_2(n-k+1)\right)\right)\right\| \\
&\leqslant \left\|\overline{\varphi}\left(n, x_1 - x_2\right)\right\| \\
&\quad + \left\|\left(I_d - A\right)^{-1}\right\| \sum_{k=1}^{n-1} \left\|\overline{\varphi}\left(k, f\left(\varphi_1(n-k+1)\right) - f\left(\varphi_2(n-k+1)\right)\right)\right\|. \quad (15)
\end{aligned}$$

Consider any basis $\{e_1, \ldots, e_d\}$ of $\mathbb{R}^d$. Then for any $\overline{x}_1 \in \mathbb{R}^d$, $\overline{x}_1 = \sum_{i=1}^{d} a_i e_i$, $\|\overline{x}_1\| = 1$, we have

$$\max_{i=1,\ldots,d} |a_i| \leqslant 1$$

and therefore

$$\left\|\overline{\varphi}\left(n, \overline{x}_1\right)\right\| = \left\|\sum_{i=1}^{d} a_i \overline{\varphi}\left(n, e_i\right)\right\| \leqslant \sum_{i=1}^{d} |a_i| \left\|\overline{\varphi}\left(n, e_i\right)\right\| \leqslant \sum_{i=1}^{d} \left\|\overline{\varphi}\left(n, e_i\right)\right\|.$$

Applying the last inequality to $\overline{x}_1 = x_1 / \|x_1\|$, where $x_1 \in \mathbb{R}^d$, $x_1 \neq 0$, we get

$$\left\|\overline{\varphi}\left(n, x_1\right)\right\| \leqslant \sum_{i=1}^{d} \left\|\overline{\varphi}\left(n, e_i\right)\right\| \|x_1\|. \quad (16)$$

Since $\sigma(A) \subset S_\alpha$, then by point 3 of Theorem 1 the sequences $\overline{\varphi}(\cdot, e_i)$, $i = 1, \ldots, d$ are in $l^1(\mathbb{N}_1, \mathbb{R}^d)$ and therefore from (16) we get

$$\left\|\overline{\varphi}\left(n, x_1\right)\right\| \leqslant C_1(\alpha, A) \|x_1\| \quad (17)$$

for any $n \in \mathbb{N}_1$ and any $x_1 \in \mathbb{R}^d$, where

$$C_1(\alpha, A) = \sum_{i=1}^{d} \sup_{n \in \mathbb{N}_1} \left\|\overline{\varphi}\left(n, e_i\right)\right\|.$$

From the point 3 of Theorem 1 we know that $\overline{\varphi}(\cdot, e_i) \in l^1(\mathbb{N}_1, \mathbb{R}^d)$. Using the same arguments as above we can show that there is a constant $C_2(\alpha, A) > 0$ such

that
$$\sum_{k=1}^{n} \|\overline{\varphi}(k, x_1)\| \leqslant C_2(\alpha, A)\, \|x_1\| \tag{18}$$
for all $x_1 \in \mathbb{R}^d$ and $n \in \mathbb{N}_1$. Using (17) and (18) in (15) we obtain
$$\begin{aligned}\left\| \left(T_{x_1}\varphi_1\right)(n) - \left(T_{x_2}\varphi_2\right)(n)\right\| &\leqslant C_1(\alpha, A)\, \|x_1 - x_2\| \\ &+ \left\|(I_d - A)^{-1}\right\| C_2(\alpha, A) \sup_{n\in\mathbb{N}_1} \|f(\varphi_1(n)) - f(\varphi_2(n))\|.\end{aligned}$$
The definition (11) of $l_f(r)$ gives
$$\begin{aligned}\sup_{n\in\mathbb{N}_1} \left\|\left(T_{x_1}\varphi_1\right)(n) - \left(T_{x_2}\varphi_2\right)(n)\right\| &\leqslant C_1(\alpha, A)\, \|x_1 - x_2\| \\ &+ \left\|(I_d - A)^{-1}\right\| l_f\left(\max\{\|\varphi_1\|_\infty, \|\varphi_2\|_\infty\}\right) \|\varphi_1 - \varphi_2\|_\infty\, C_2(\alpha, A).\end{aligned}$$
From the last inequality with $x_2 = 0$ and $\varphi_2 = 0$ we obtain that $T_{x_1}\varphi_1 \in l^\infty(\mathbb{N}_1)$. The last inequality implies also (13) and (14) with
$$C(\alpha, A) = \max\{C_1(\alpha, A), C_2(\alpha, A)\}.$$

Now we are in position to prove Theorem 3.

Proof. [Proof of Theorem 3] At first observe that $\sigma(A) \subset S_\alpha$ implies that $I_d - A$ is invertible and therefore the initial value problem for equation has a unique global solution for all $x_1 \in \mathbb{R}^d$. Let us fix $r > 0$ such that $C(\alpha, A) l_f(r) < \frac{1}{2}$, define
$$\varepsilon = \frac{r}{2\, C(\alpha, A)}$$
and consider any $x \in \mathbb{R}^d$ with $\|x\| \leqslant \varepsilon$ and $\varphi \in l^\infty(\mathbb{N}_1)$ with $\|\varphi\|_\infty \leqslant r$. According to (13) we get
$$\|T_x\varphi\|_\infty \leqslant C(\alpha, A)\, \|x\| + C(\alpha, A) l_f(r)\, \|\varphi\|_\infty \leqslant C(\alpha, A) \frac{r}{2\, C(\alpha, A)} + \frac{r}{2} = r.$$
The last inequality proves that
$$T_x\left(B_\infty(r)\right) \subset B_\infty(r),$$
where
$$B_\infty(r) := \{\varphi \in l^\infty(\mathbb{N}_1) : \|\varphi\|_\infty \leqslant r\}.$$
We get that for all $x \in B_{\mathbb{R}^d(0,\varepsilon)}$ and $\varphi_1, \varphi_2 \in B_\infty(r)$ we have
$$\|T_x x_1 - T_x x_2\|_\infty \leqslant C(\alpha, A) l_f(r)\, \|\varphi_1 - \varphi_2\|_\infty = q\, \|\varphi_1 - \varphi_2\|_\infty,$$

where $q = C(\alpha, \lambda) l_f(r) = \frac{1}{2}$. By the Contraction Mapping Principle, there exists a unique fixed point $\varphi \in B_\infty(r)$ of T_x, which is by Theorem 1, the unique solution of (4) with initial condition $x(1) = x$. The uniqueness of initial value problem for (4) implies the stability of trivial solution. To complete the proof we have to show that the trivial solution is attractive.

For arbitrary $x \in B_{\mathbb{R}^d(0,\varepsilon)}$ let $\varphi \in B_\infty(r)$ be the unique solution of (4) satisfying $x(1) = x$. Put

$$a := \limsup_{n\to\infty} \|\varphi(n)\| \leqslant r$$

and fix a $\eta > 0$. Then there exists $n_0 \in \mathbb{N}_1$ such that

$$\|\varphi(n)\| \leqslant a + \eta$$

for all $n \in \mathbb{N}_{n_0}$. We will estimate

$$\limsup_{n\to\infty} \|\varphi(n)\| .$$

To do this, we use the variation of constants formula (7) which gives

$$x(n) = \overline{\varphi}(n, x) + (I_d - A)^{-1} \sum_{k=1}^{n-1} \overline{\varphi}\left(k, f\left(\varphi(n - k + 1)\right)\right)$$

or equivalently

$$x(n) = \overline{\varphi}(n, x) + (I_d - A)^{-1} \sum_{l=1}^{n-1} \overline{\varphi}\left(n - l, f\left(\varphi(l + 1)\right)\right).$$

Since (4) is asymptotically stable, then

$$\lim_{n\to\infty} \left\| (I_d - A)^{-1} \sum_{l=1}^{n_0-1} \overline{\varphi}\left(n - l, f\left(x(l + 1)\right)\right) \right\| = 0.$$

Therefore, from the fact that

$$\lim_{n\to\infty} \overline{\varphi}(n, x) = 0$$

and

$$\varphi(n) = T_x\varphi(n)$$

we have

$$\begin{aligned} \limsup_{n\to\infty} \|\varphi(n)\| &\leqslant \limsup_{n\to\infty} \left\|(I_d - A)^{-1}\right\| \left\| \sum_{l=n_0}^{n-1} \overline{\varphi}\left(n - l, f\left(\varphi(l + 1)\right)\right) \right\| \\ &\leqslant C(\alpha, c\lambda) l_f(r)(a + \eta). \end{aligned}$$

Thus

$$a \leqslant C(\alpha,\lambda)l_f(r)(a+\eta).$$

By letting $\eta \to 0^+$ and due to the assumption $C(\alpha,\lambda)l_f(r) < \dfrac{1}{2}$ we get that $a = 0$ and the proof is complete. □

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