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A coin balanced on its edge is one way to imagine a state of quantum superposition

Typical and Atypical Evolutions of Quantum Systems

All data obtained from various micro-world physics experiments align with the predictions of quantum theory. But are there still areas in quantum mechanics yet to be explored?

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A central aim in physics is to describe a system and predict its evolution over time. For classical systems, we begin by determining the system's initial conditions, identifying the forces at play, formulating the equation of motion, and then solving it. If the system is non-chaotic – meaning its dynamics do not change significantly with slight adjustments to initial conditions – we can approximate the system's evolution over time.

Quantum physics, however, describes systems at a microscopic scale, at the level of single atoms or even smaller particles. In this realm, we cannot precisely determine the position or momentum of individual particles; instead, the theory allows us to analyze the probability of a particle being detected in a specific region of space. The core concept here is the *quantum state*, understood as a mathematical tool for calculating the likelihood of obtaining a particular measurement result. A quantum measurement alters the system's state, and the outcomes for two particles prepared in the same state will not necessarily be identical.

The quantum coin-toss

If a fair coin is tossed, whilst it is still spinning in the air, classical probability theory assigns it a state of ($^1/_2$,$^1/_2$), as both possible outcomes are equally likely. A certain event, like "heads" (H), can be represented by the vector $|0\rangle$= (1,0), while the opposite event, "tails" (T), is represented by the vector $|1\rangle$ = |(0,1). Quantum mechanics allows for the existence of a superposition state – written in this notation as $|+\rangle = (|0\rangle + |1\rangle)$ – which provides equal probabilities for both outcomes yet can be reversibly transformed into one of two certain events.

When tossing two distinguishable coins, we get one of four possible outcomes: HH, HT, TH, TT. Quantum theory, however, predicts a much larger set of possible states. The most interesting of these is the Bell state, written as $(|00\rangle + |11\rangle)$. If two coins were in such a quantum state, knowing the result of tossing one would allow us to precisely predict the result of the other, as it would be identical. Classical coins do not exhibit such properties, but such unusual result correlations do characterize so-called *quantum entangled states*. Quantum entanglement, involving strong non-classical correlations in measurement outcomes, is used in protocols for quantum teleportation, quantum key distribution, and various quantum computing algorithms. The effect of quantum entanglement does not violate relativity theory, as it does not allow information to be transferred at faster-than-light speeds.

> Quantum information processing can be faster and more efficient than classical information processing

In a 2022 study by Rather et al. (a team including the present author), a new quantum state was constructed involving a system of four subsystems, each with six levels. If four ordinary dice were put into this state, a measurement on any chosen pair of dice would determine the measurement results of the remaining two. This state enables the implementation of a new quantum teleportation protocol, generates a novel quantum error-correcting code, and can be used for testing quantum computers. It also allows for solving the quantum version of a combinatorial problem known as Euler's *problem of the thirty-six officers*.

If we take 9 cards – say, 3 aces, 3 kings, and 3 queens of three different suits – it's relatively easy to arrange them in a square so that each row and each column of the square contains only one card of each suit and

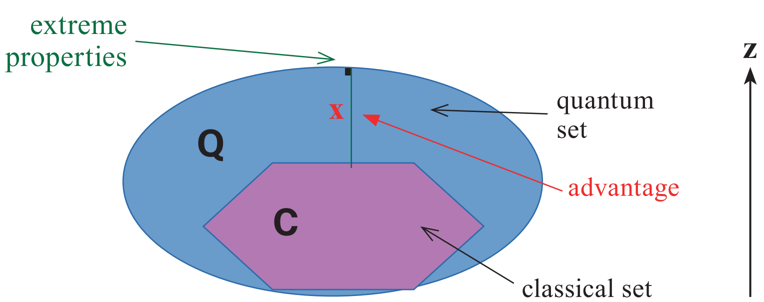
K♦	Q♠	A♣
Q♣	A♦	K♠
A♠	K♣	Q♦

A 3×3 Euler's square: in each row and column, the cards differ in terms of suit and rank

only one card of each rank. In 1779, Leonhard Euler observed that such arrangements are also possible with 16 cards (4 different suits and 4 different ranks) and 25 cards (5 suits and 5 ranks), but it seemed impossible to him to create such a square with 36 cards (6 suits and 6 ranks). The problem allegedly arose out of a practical challenge: how to optimally arrange a group of 36 officers during a military parade in St. Petersburg, with six representatives of different ranks coming from each of the six branches of the Russian military. Euler conjectured, but did not prove, that such a 6×6 combinatorial configuration was impossible; a rigorous proof was published by Gaston Tarry in 1900.

In our study (coming 121 years later), we used the quantum state of a four-die system to represent a solution to the quantum version of Euler's problem, where each of the 36 cells in the square allows for quantum entangled states, represented by combinations of several cards.

A key parameter describing any system is its dimension d, equal to the number of distinguishable states. For example, the dimension d is 2 for a single coin, 4 for two coins, 6 for a die, and 36 for a two-die system. A quantum state is described by a density matrix ρ of dimension d (a positive semi-definite Hermitian matrix with unit trace). In the simplest case, $d = 2$, the quantum system is called a qubit, and the set $\Omega 2$ of all quantum states forms a three-dimensional Bloch sphere. For $d > 2$, the set Ω_d is a compact convex set embedded in real space of dimension $d^2 - 1$. A typical quantum state is one randomly chosen according to a uniform measure over the entire set Ω_d.



The set of classical states C is contained within the set Q of quantum states. This same framework represents classical and quantum operations, correlations, algorithms, and technologies. The *z*-axis denotes a selected objective function, while the distance *x* illustrates what is known as the *quantum advantage*

The quantum state of a four-particle system discussed here is unusual, as its degree of entanglement is much higher than the reference value characterizing typical random states. Such states with extreme properties and a high degree of quantum entanglement, located on the boundary of the set Ω, can be highly useful in quantum information processing.

Physics describes the future

The evolution of quantum systems over time can be represented by describing how the density matrix ρ, which reflects the probabilities of obtaining particular measurement outcomes, changes over time. For a quantum system isolated from its environment, its unitary evolution, described by the Heisenberg equation, can be visualized as a rigid rotation of the set Ω_d within a $(d^2 - 1)$-dimensional space. In a more general case, where the system interacts with its environment, the dynamics are not reversible, and the set of states Ωd undergoes contraction.

In a stroboscopic description of a system's dynamics, we compare consecutive frames as if in a film, measuring time in discrete units, $t = 1, 2, 3, \ldots$ In this case, the unit evolution of the density matrix is defined by a quantum operation Φ (a completely positive trace-preserving map) through the relation $\rho_{t+1} = \Phi(\rho_t)$. In an alternative approach, we increase the sampling frequency, and in the limiting case, we move to a continuous-time description of the system, $\rho(t) = e^{tL}\rho(0)$ where t represents time and L is the Lindblad generator, determining the trajectory within the set Ω_d of density matrices.

Worth noting here are the significant contributions of the Polish school in Toruń, led by Roman Stanisław Ingarden (1920–2011), a pioneer in theoretical physics, to the development of the quantum theory of open systems. Andrzej Jamiołkowski's 1972 work on positive maps, for instance, led to the construction of the widely-used Choi-Jamiołkowski isomorphism, which associates each quantum operation Φ with a quantum state σ in the extended set Ω_{d^2}. Furthermore, in 1976, Andrzej Kossakowski (1938–2021) co-authored a seminal work that provided a complete description of the non-unitary evolution of quantum systems in continuous time (independently of a parallel paper by Goran Lindblad)

The set of quantum operations Φ that transform the set of states Ω_d into itself forms a convex set of dimension $d^4 - d^2$. Just as with states, typical operations can be considered, generated using random matrix theory; operator theory allows for the anal-

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The quantum version of Euler's problem of 36 officers can be solved by placing an entangled state of 2 or 4 cards in each cell of a 6×6 square. Each cell then corresponds to a state of 4 dice, representing the row of the square, its column, the color of the card, and its rank, respectively

ysis of their spectra and the set of invariant states. The results obtained serve as reference points for investigating atypical operations with very specific properties. If we consider the state space Ω_d as the "stage" on which the action of a quantum algorithm unfolds, then quantum operations act as the elemental episodes, out of which an author can craft a "script."

Classical and quantum computation

Quantum information processing can be faster and more efficient than classical information processing, as the set of quantum states – which includes quantum superposition and entangled states – is significantly larger than the set of classical states. Similarly, the set of classical state transformations represents only a small portion of the much larger set of quantum operations. In other words, a "scriptwriter" working within the framework of quantum mechanics has a far broader "stage" and more options for developing their narrative than a fellow composer designing a classical algorithm.

However, implementing quantum algorithms in practice is challenging: while classical computations are deterministic and, when performed correctly, yield the desired result, quantum computations produce the expected outcome only with a certain probability, meaning the process must be repeated. The inevitable interaction between a quantum processor and its surroundings introduces errors and leads to a decay of the system's quantum properties, causing it to revert to a classical state. This effect, known as decoherence, happens relatively quickly. At present, experimental quantum processors are only capable of performing a few dozen operations on a few dozen qubits, limiting their capacity for complex calculations.

It remains uncertain if or when we will reach the critical threshold of quantum computational advantage – where a quantum computer can solve a specific, useful computational task faster than classical computers. But be that as it may, quantum information theory, which bridges theoretical and experimental physics, computer science, and various fields of mathematics, is a rapidly advancing scientific discipline with increasing practical relevance.

In 2024, the European Research Council awarded an ERC Advanced Grant to the author's project titled "Typical and Atypical Structures in Quantum Theory" (inspired by the Atypowa Foundation, which supports individuals on the autism spectrum). The research project will be conducted at the Institute of Theoretical Physics at Jagiellonian University. Its primary goal is to investigate the properties of typical quantum states and operations and to identify exceptional structures with extreme properties that are valuable for quantum information processing and the development of new quantum technologies.

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