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Drazin inverse matrix method for fractional descriptor continuous-time linear systems

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Abstract. The Drazin inverse of matrices is applied to find the solutions of the state equations of the fractional descriptor continuous-time systems with regular pencils. An equality defining the set of admissible initial conditions for given inputs is derived. The proposed method is illustrated by a numerical example.

Key words: Drazin inverse, descriptor, fractional, continuous-time, linear system, solution.

1. Introduction

Descriptor (singular) linear systems have been considered in many papers and books [1–15]. The eigenvalues and invariants assignment by state and output feedbacks have been investigated in [7, 10, 16, 17] and the minimum energy control of descriptor linear systems in [18, 19]. The computation of Kronecker's canonical form of singular pencil has been analyzed in [14]. The positive linear systems with different fractional orders have been addressed in [20]. Selected problems in theory of fractional linear systems has been given in monograph [21].

Descriptor standard positive linear systems by the use of the Drazin inverse has been addressed in [1–3, 9, 13, 22, 23]. The shuffle algorithm has been applied to checking the positivity of descriptor linear systems in [9]. The stability of positive descriptor systems has been investigated in [15]. Reduction and decomposition of descriptor fractional discrete-time linear systems have been considered in [11]. A new class of the descriptor fractional linear discrete-time systems has been introduced in [12].

In this paper the Drazin inverse of matrices is applied to find the solutions of the state equations of the fractional descriptor continuous-time linear systems with regular pencils.

The paper is organized as follows. In Sec. 2 the state equation of the fractional descriptor continuous-time linear systems and some basic definitions of the Drazin inverse are recalled. The solution to the state equation is given in Sec. 3. The proposed method is illustrated by numerical examples in Sec. 4. Concluding remarks are given in Sec. 5.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n \times m$ real matrices and $\Re^n = \Re^{n\times 1}$, Z_+ – the set of $n \times n$ nonnegative matrices, I_n – the $n \times n$ identity matrix, ker A – the kernel of the matrix.

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2. Preliminaries

Consider the fractional descriptor continuous-time linear system

$$E_0 D_t^\alpha x(t) = Ax(t) + Bu(t), \quad 0 < \alpha < 1, \tag{1}$$

where α is fractional order, $x(t) \in \Re^n$ is the state vector $u(t) \in \Re^m$ is the input vector, $E, A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$ and

$${}_0D_t^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\tau)^\alpha} \frac{dx(\tau)}{d\tau} d\tau \tag{2}$$

is the Caputo definition of $\alpha \in \Re$ order derivative of $x(t)$ and

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt \tag{3}$$

is the Euler gamma function.

It is assumed that det $E = 0$ but the pencil (E, A) of (1) is regular, i.e.

$$\det[Es - A] \neq 0 \quad \text{for some} \quad s \in \mathrm{C} \quad \text{(the field of complex numbers).} \tag{4}$$

Assuming that for some chosen $c \in \mathrm{C}$, $\det[Ec - A] \neq 0$ and premultiplying (1) by $[Ec - A]^{-1}$ we obtain

$$\overline{E}\, {}_0D_t^\alpha x(t) = \overline{A}x(t) + \overline{B}u(t), \tag{5a}$$

where

$$\overline{E} = [Ec-A]^{-1}E, \quad \overline{A} = [Ec-A]^{-1}A, \quad \overline{B} = [Ec-A]^{-1}B. \tag{5b}$$

Note that the Eqs. (1) and (4) have the same solution $x(t)$.

Definition 1. [13] The smallest nonnegative integer q is called the index of the matrix $\overline{E} \in \Re^{n\times n}$ if

$$\text{rank } \overline{E}^q = \text{rank } \overline{E}^{q+1}. \tag{6}$$

Definition 2. [13] A matrix $\overline{E}^D$ is called the Drazin inverse of $\overline{E} \in \Re^{n\times n}$ if it satisfies the conditions

$$\overline{E}\overline{E}^D = \overline{E}^D\overline{E}, \tag{7a}$$

$$\overline{E}^D\overline{E}\overline{E}^D = \overline{E}^D, \tag{7b}$$

$$\overline{E}^D\overline{E}^{q+1} = \overline{E}^q, \tag{7c}$$

where q is the index of $\overline{E}$ defined by (6).

The Drazin inverse $\overline{E}^D$ of a square matrix $\overline{E}$ always exists and is unique [4, 13]. If $\det \overline{E} \neq 0$ then $\overline{E}^D = \overline{E}^{-1}$. Some methods for computation of the Drazin inverse are given in [14, 22].

Lemma 1. [4, 13] The matrices $\overline{E}$ and $\overline{A}$ defined by (5b) satisfy the following equalities

1. $\overline{A}\overline{E} = \overline{E}\overline{A}$ and $\overline{A}^D\overline{E} = \overline{E}\overline{A}^D$, $\overline{E}^D\overline{A} = \overline{A}\overline{E}^D$, $\overline{A}^D\overline{E}^D = \overline{E}^D\overline{A}^D$, (8a)
2. $\ker \overline{A} \cap \ker \overline{E} = \{0\}$, (8b)
3. $\overline{E} = T\begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix}T^{-1}$, $\overline{E}^D = T\begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix}T^{-1}$, (8c)
   $\det T \neq 0$, $J \in \Re^{n_1\times n_1}$, is nonsingular, $N \in \Re^{n_2\times n_2}$ is nilpotent, $n_1 + n_2 = n$,
4. $(I_n - \overline{E}\overline{E}^D)\overline{A}\overline{A}^D = I_n - \overline{E}\overline{E}^D$
   and $(I_n - \overline{E}\overline{E}^D)(\overline{E}\overline{A}^D)^q = 0$. (8d)

3. Solution to the state equation by the use of Drazin inverse

In this section the solution to the state Eq. (1) will be presented by the use of the Drazin inverses of the matrices $\overline{E}$ and $\overline{A}$.

Theorem 1. The solution to the Eq. (1) is given by

$$x(t) = \Phi_0(t)\overline{E}\overline{E}^Dv + \overline{E}^D\int_0^t \Phi(t-\tau)\overline{B}u(\tau)d\tau + (\overline{E}\overline{E}^D - I_n)\sum_{k=0}^{q-1}(\overline{E}\overline{A}^D)^k\overline{A}^D\overline{B}u^{(k\alpha)}(t), \tag{9a}$$

where

$$\Phi_0(t) = \sum_{k=0}^{\infty}\frac{(\overline{E}^D\overline{A})^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \quad \Phi(t) = \sum_{k=0}^{\infty}\frac{(\overline{E}^D\overline{A})^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}, \tag{9b}$$

$$u^{(k\alpha)}(t) = {}_0D_t^{k\alpha}u(t) \tag{9c}$$

and the vector $v \in \Re^n$is arbitrary.

Proof. Proof is accomplished by showing that (9) satisfies the Eq. (5a). Substituting (9a) in the left side of the Eq. (5a), using (9b), Definition 2 and Lemma 1 we obtain

$$\begin{aligned}
&\overline{E}_0D_t^\alpha x(t) \\
&= \overline{E}_0D_t^\alpha\Bigg[\Phi_0(t)\overline{E}\overline{E}^Dv + \overline{E}^D\int_0^t\Phi(t-\tau)\overline{B}u(\tau)d\tau \\
&\quad + (\overline{E}\overline{E}^D - I_n)\sum_{k=0}^{q-1}(\overline{E}\overline{A}^D)^k\overline{A}^D\overline{B}u^{(k\alpha)}(t)\Bigg] \\
&= \overline{E}_0D_t^\alpha\left[\begin{array}{l}
\overline{E}\overline{E}^Dv + \sum_{k=1}^{\infty}\frac{(\overline{E}^D\overline{A})^k t^{k\alpha}}{\Gamma(k\alpha+1)}v \\
+\overline{E}^D\int_0^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\overline{B}u(\tau)d\tau \\
+\overline{E}^D\int_0^t\sum_{k=0}^{\infty}\frac{(\overline{E}^D\overline{A})^{k+1}(t-\tau)^{(k+2)\alpha-1}}{\Gamma[(k+2)\alpha]}\overline{B}u(\tau)d\tau \\
+(\overline{E}\overline{E}^D - I_n)\sum_{k=0}^{q-1}(\overline{E}\overline{A}^D)^k\overline{A}^D\overline{B}u^{(k\alpha)}(t)
\end{array}\right] \\
&= \sum_{k=0}^{\infty}\frac{\overline{E}(\overline{E}^D\overline{A})^{k+1}t^{k\alpha}}{\Gamma(k\alpha+1)}v + \overline{E}^D\overline{B}u(t) \\
&\quad + (\overline{E}^D)^2\overline{A}\int_0^t\Phi(t-\tau)\overline{B}u(\tau)d\tau \\
&\quad + (\overline{E}\overline{E}^D - I_n)\sum_{k=0}^{q-1}(\overline{E}\overline{A}^D)^k\overline{A}^D\overline{B}u^{(k\alpha)}(t) \\
&= A\Bigg[\Phi_0(t)\overline{E}\overline{E}^Dv + \overline{E}^D\int_0^t\Phi(t-\tau)\overline{B}u(\tau)d\tau \\
&\quad + (\overline{E}\overline{E}^D - I_n)\sum_{k=0}^{q-1}(\overline{E}\overline{A}^D)^k\overline{A}^D\overline{B}u^{(k\alpha)}(t)\Bigg] + Bu(t)
\end{aligned} \tag{10}$$

since

$$\begin{aligned}
&{}_0D_t^\alpha\overline{E}\overline{E}^Dv = 0, \quad \overline{E}(\overline{E}^D\overline{A})^{k+1} = \overline{A}^{k+1}(\overline{E}^D)^k, \\
&\Phi(t) = \sum_{k=0}^{\infty}\frac{(\overline{E}^D\overline{A})^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \\
&\quad + \sum_{k=0}^{\infty}\frac{(\overline{E}^D\overline{A})^{k+1}t^{(k+2)\alpha-1}}{\Gamma[(k+2)\alpha]}
\end{aligned} \tag{11}$$

and (8d) holds.

Therefore, the solution (9a) satisfies Eq. (5a).

From (9a) for t =0 we have

$$x(0) = x_0 = \overline{E}\overline{E}^Dv + (\overline{E}\overline{E}^D - I_n)\sum_{k=0}^{q-1}(\overline{E}\overline{A}^D)^k\overline{A}^D\overline{B}u^{(k\alpha)}(0). \tag{12}$$

Therefore, for given admissible $u(t)$ the consistent initial conditions should satisfy the equality (12). In particular case for $u(t) = 0$ we have $x_0 = \overline{E}\overline{E}^Dv$ and $x_0 \in Im(\overline{E}\overline{E}^D)$ where Im denotes the image of $\overline{E}\overline{E}^D$.

Theorem 2. Let

$$P = \overline{E}\overline{E}^D, \tag{13a}$$

$$Q = \overline{E}^D\overline{A}. \tag{13b}$$

Then:

1) $P^k = P$ for $k = 2,3,\ldots,$ (14)

2) $PQ = QP = Q,$ (15)

3) $P\Phi_0(t) = \Phi_0(t),$ (16)

4) $P\Phi(t) = \Phi(t).$ (17)

Proof. Using (13a) we obtain

$$P^2 = \overline{E}\overline{E}^D\overline{E}\overline{E}^D = \overline{E}\overline{E}^D = P \tag{18}$$

since by (7b) $\overline{E}^D\overline{E}\overline{E}^D = \overline{E}^D$ and by induction

$$P^k = P^{k-1}P = \overline{E}\overline{E}^D\overline{E}\overline{E}^D = P^2 = P \quad \text{for} \quad k = 2,3,\ldots. \tag{19}$$

Using (13a) and (13b) we obtain

$$PQ = \overline{E}\overline{E}^D\overline{E}^D\overline{A} = \overline{E}^D\overline{E}\overline{E}^D\overline{A} = \overline{E}^D\overline{A} = Q \tag{20}$$

and

$$QP = \overline{E}^D\overline{A}\overline{E}\overline{E}^D = \overline{E}^D\overline{E}\overline{A}\overline{E}^D = \overline{E}^D\overline{E}\overline{E}^D\overline{A} = \overline{E}^D\overline{A} = Q \tag{21}$$

since (8a) holds. From (9b) and (13a) we have

$$\begin{aligned} P\Phi_0(t) &= \sum_{k=0}^{\infty}\frac{P(\overline{E}^D\overline{A})^k t^{k\alpha}}{\Gamma(k\alpha+1)} \\ &= \sum_{k=0}^{\infty}\frac{P\overline{E}^D\overline{A}(\overline{E}^D\overline{A})^{k-1}t^{k\alpha}}{\Gamma(k\alpha+1)} \\ &= \sum_{k=0}^{\infty}\frac{(\overline{E}^D\overline{A}^D)^k t^{k\alpha}}{\Gamma(k\alpha+1)} = \Phi_0(t) \end{aligned} \tag{22}$$

since by (20) $P\overline{E}^D\overline{A} = \overline{E}^D\overline{A}$. Proof of (17) is similar.

4. Example

Consider Eq. (1) with the matrices

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad 0 < \alpha < 1. \tag{23}$$

The pencil of (23) is regular since

$$\det[Es - A] = \begin{vmatrix} s+1 & 0 \\ 0 & 2 \end{vmatrix} = 2(s+1) \neq 0. \tag{24}$$

We chose $c = 1$ and the matrices (5b) take the forms

$$\begin{aligned} \overline{E} &= [Ec - A]^{-1}E \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}, \\ \overline{A} &= [Ec - A]^{-1}A \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1}\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix}, \\ \overline{B} &= [Ec - A]^{-1}B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}. \end{aligned} \tag{25}$$

Using (7b) and (23) we obtain

$$\overline{E}^D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \overline{A}^D = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}. \tag{26}$$

It is easy to check that the matrices (26) satisfying the conditions (7) and (8).

Using (26) and (9b) we obtain

$$\Phi_0(t) = \sum_{k=0}^{\infty}\frac{(\overline{E}^D\overline{A})^k t^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\begin{bmatrix} -1^k & 0 \\ 0 & 0 \end{bmatrix} \tag{27a}$$

and

$$\begin{aligned} \Phi(t) &= \sum_{k=0}^{\infty}\frac{(\overline{E}^D\overline{A})^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \\ &= \sum_{k=0}^{\infty}\frac{t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}\begin{bmatrix} -1^k & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{27b}$$

since

$$\overline{E}^D\overline{A} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{28}$$

From (9a) and (27) we have the desired solution in the form

$$\begin{aligned} x(t) &= \Phi_0(t)\overline{E}\overline{E}^D v + \overline{E}^D\int_0^t \Phi(t-\tau)\overline{B}u(\tau)d\tau \\ &+ (\overline{E}\overline{E}^D - I_n)\sum_{k=0}^{q-1}(\overline{E}\overline{A}^D)^k\overline{A}^D\overline{B}u^{(k\alpha)}(t) \\ &= \sum_{k=0}^{\infty}\left[\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\begin{bmatrix} -1^k & 0 \\ 0 & 0 \end{bmatrix} v + \frac{1}{\Gamma[(k+1)\alpha]}\right. \\ &\left.\begin{bmatrix} (-0.5)^k \\ 0 \end{bmatrix}\int_0^t (t-\tau)^{(k+1)\alpha-1}u(\tau)d\tau\right] + \begin{bmatrix} 2 \\ 1 \end{bmatrix}u(t) \end{aligned} \tag{29}$$

since

$$\overline{E}\overline{E}^D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \overline{E}\overline{E}^D - I_2 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad q = 1$$

$$\text{and} \quad \overline{A}^D\overline{B} = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \tag{30}$$

for arbitrary $v \in \Re^2$.

5. Concluding remarks

The Drazin inverse of matrices has been applied to find the solutions of the state equations of the descriptor fractional continuous-time systems with regular pencils. The equality (12) defining the set of admissible initial conditions for given inputs has been derived. The proposed method has been illustrated by a numerical example. Some properties of the matrices P, Q, $\Phi_0(t)$ and $\Phi(t)$ have been established (Theorem 2).

Comparing the presented method with the method based on the Weierstrass decomposition of the regular pencil [12] we may conclude that the proposed method is computationally attractive since the Drazin inverse of matrices can be computed by the use of well-known numerical methods [13, 14]. The presented method can be extended to the positive descriptor fractional continuous-time linear systems. An open problem is an extension of the considerations for standard and positive continuous-discrete descriptor fractional linear systems [13, 24–26].

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