BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 64, No. 1, 2016
DOI: 10.1515/bpasts-2016-0020

Green's function approach to frequency analysis of thin circular plates

K.K. ŻUR*

Faculty of Management, Bialystok University of Technology, 2 Ojca Stefana Tarasiuka St., 16-001 Kleosin, Poland

Abstract. The free vibration analysis of homogeneous and isotropic circular thin plates by using the Green's functions is considered. The formulae for construction of the influence function for all nodal diameters are presented in a closed form. The limited independent solutions of differential Euler equations were expanded in the Neumann power series using the method of successive approximation. This approach allows to obtain the analytical frequency equations as power series rapidly convergent to exact eigenvalues for different number of nodal diameters. The first ten dimensionless frequencies for eight different natural modes of circular plates are calculated. A part of obtained results have not been presented yet in open literature for thin circular plates. The results of investigation are in good agreement with selected results obtained by other methods presented in literature.

Key words: free vibration, circular plates, Green's function.

1. Introduction

In recent years, lightweight plate structures have been used in many engineering applications. Components of circular plates are commonly used in aerospace industries and aviation as well as in marine and civil engineering applications. Circular plates are the most critical structural elements in high speed rotating engineering systems such as circular saws, rotors, turbine flywheels, etc. The natural frequencies of the plates have been studied extensively for more than a century, if only because the frequency of an external load matches the natural frequency of the plate, destruction may occur.

Researchers have used various methods of analysis of dynamic behavior of plates with different boundary conditions. The work of Leissa [1] is an excellent source of information about methods used for free vibration analysis of plates. The free vibration analysis has been carried out by using a variety of weighting function methods [1] such as the Ritz method, the Galerkin method or the finite element method. In many works [1, 2] natural frequencies of circular plates are expressed in terms of the Bessel functions. Chakraverty et al. [3, 4] have studied the free vibration analysis of plates of various geometries by using two-dimensional boundary characteristic orthogonal polynomials in the Rayleigh-Ritz method. Wu and Liu [5, 6] proposed the generalized differential quadrature rule (GDQR) to a free vibration analysis of circular thin plates. Jaroszewicz and Zoryj [7] have studied free vibration of circular thin plates of constant and linearly variable thickness using the method of partial discretization (MPD). Ebrahimi and Rastgo [8] investigated the natural vibration behavior of circular functionally graded plates with clamped edges based on classical plate theory. Yalcin et al. [9] have studied free vibration of circular plates by using differential transformation method (DTM). Zhou et al. [10] applied the Hamiltonian approach to a solution of the free vibration problem of circular and annular thin plates. Kukla and Szewczyk [11] applied the Green's functions (influence function) and Bessel functions to the solution eigenvalue problem of annular thin plates with discrete elements such as oscillators. Similarly, Sorokin and Peake [12] have studied the free vibration analysis of sandwich plates with concentrated springs and mass by using Green's functions. An area of application of Green's functions in a free vibration analysis of isotropic beams and plates with constant and variable distribution of parameters is presented in the monograph of Kukla [13]. The application of Green's functions in free axisymmetric vibration of circular thin plates with clamped edges is presented in the book of Jaroszewicz and Zoryj [14].

In the present study, Green's functions are used to obtain ten lower natural frequencies for eight different natural modes of circular plates with different boundary conditions. A formula of construction of influence functions for different modes of uniform circular thin plates is obtained in closed form. The characteristic equations for different boundary conditions and different number of nodal lines of thin circular plates are defined. The numerical results of investigation are compared with results presented in literature.

2. Statement of the problem

Consider an isotropic, homogeneous circular thin plate of constant thickness h in cylindrical coordinate (r, θ, z) with the z-axis along the longitudinal direction. Geometry and coordinate system of considering plate as shown in Fig. 1. The partial differential equation for free vibration of thin plates has following form

$$\nabla^4 W + \frac{\rho h}{D}\frac{\partial^2 W}{\partial t^2} = 0, \tag{1}$$

---

*e-mail: k.zur@pb.edu.pl

where ρ is the mass density, $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity, E is Young's modulus, ν is the Poisson ratio, $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$ is the Laplacian and $W(r,\theta,t)$ is the small deflection compared with the thickness h of plate.

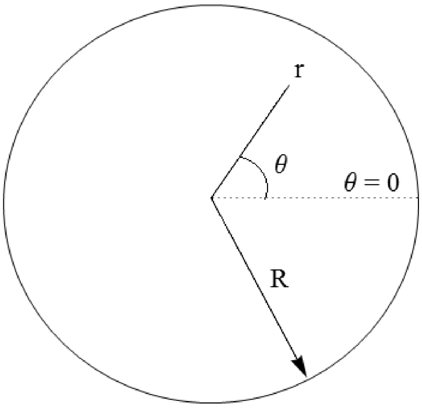

Fig. 1. Geometry and coordinate system of the circular plate

The deflection of a circular plate may be expressed as follows

$$W(r,\theta,t) = w(r)\cos(n\theta)e^{i\omega t}, \tag{2}$$

where n is the integer number of diagonal nodal lines, $w(r)$ is the radial mode function, ω is natural frequency, and $i^2 = -1$. Substituting Eq. (2) into Eq. (1) using the dimensionless coordinate $\xi = r/R$ the governing differential equation of the circular plate becomes:

$$L(w) - \lambda^2 w = 0, \tag{3}$$

where

$$L(w) \equiv \frac{d^4w}{d\xi^4} + \frac{2}{\xi}\frac{d^3w}{d\xi^3} - \frac{(1+2n^2)}{\xi^2}\frac{d^2w}{d\xi^2} + \frac{(1+2n^2)}{\xi^3}\frac{dw}{d\xi} + \frac{(n^4-4n^2)}{\xi^4}w \tag{4}$$

is differential operator and

$$\lambda = \omega R^2\sqrt{\rho h/D} \tag{5}$$

is dimensionless frequency of vibration.

The boundary conditions at the outer edge ($\xi = 1$) of the circular plate may be one of the following: clamped, simply supported, free, sliding supports and elastic supports. These conditions may be written in terms of the radial mode function $w(\xi)$ in the following form:

Clamped:

$$w(\xi)|_{\xi=1} = 0, \tag{6a}$$

$$\left.\frac{dw}{d\xi}\right|_{\xi=1} = 0. \tag{6b}$$

Simple supports:

$$w(\xi)|_{\xi=1} = 0, \tag{7a}$$

$$M(w)|_{\xi=1} = \left[\frac{d^2w}{d\xi^2} + \frac{\nu}{\xi}\frac{dw}{d\xi} - \frac{\nu n^2}{\xi^2}w\right]_{\xi=1} = 0. \tag{7b}$$

Free:

$$M(w)|_{\xi=1} = 0, \tag{8a}$$

$$V(w)|_{\xi=1} = \left[\frac{d^3w}{d\xi^3} + \frac{1}{\xi}\frac{d^2w}{d\xi^2} - \left(\frac{1+2n^2-\nu n^2}{\xi^2}\right)\frac{dw}{d\xi} + \left(\frac{3n^2-\nu n^2}{\xi^3}\right)w\right]_{\xi=1} = 0. \tag{8b}$$

Sliding supports:

$$\left.\frac{dw}{d\xi}\right|_{\xi=1} = 0, \tag{9a}$$

$$V(w)|_{\xi=1} = 0. \tag{9b}$$

Elastic supports:

$$\Phi(w)|_{\xi=1} = \left[\left(\frac{d^2w}{d\xi^2} + \nu\frac{dw}{d\xi} - \frac{\nu n^2}{\xi^2}w\right) + \phi\frac{dw}{d\xi}\right]_{\xi=1} = 0, \tag{10a}$$

$$\Psi(w)|_{\xi=1} = \left[\left(\frac{d^3w}{d\xi^3} + \frac{d^2w}{d\xi^2} + (\nu n^2 - 2n^2 - 1) \cdot\frac{dw}{d\xi} + (3n^2 - \nu n^2)w\right) - \psi w\right]_{\xi=1} = 0. \tag{10b}$$

$M(w)$ and $V(w)$ are the normalized radial bending moment and the normalized effective shear force, respectively. $\phi = K_\phi R/D$ and $\psi = K_\psi R^3/D$ are normalized parameters of elastic supports. K_ϕ and K_ψ are rotational and translational spring constants (Fig. 2), respectively.

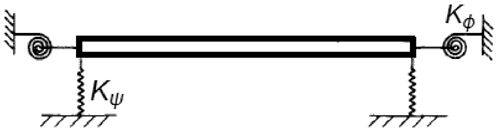

Fig. 2. The cross-section of uniform circular plate with elastic supports

3. Mathematical background

The formula of construction of the Green's function (general solution) for homogeneous differential equations

$$L(y) \equiv \sum_{k=0}^{n} p_k(x)\frac{d^ky}{dx^k} = 0, \quad a < x < b \tag{11}$$

has following form

$$K(x,\alpha) = \sum_{k=1}^{n} C'_k y_k(x), \tag{12}$$

where coefficients $p_k(x)$ are continuous functions, $p_n(x) \neq 0$ for $x \in [ab]$ and α is a position where all discrete elements could be mounted [11, 14].

Constants C'_k have form [13]

$$C'_k = \frac{(-1)^{n+k}}{p_n(\alpha)\, W_n(\alpha)}$$

$$\cdot \begin{vmatrix} y_1(\alpha) & \cdots & y_{k-1}(\alpha) & y_{k+1}(\alpha) & \cdots & y_n(\alpha) \\ y'_1(\alpha) & \cdots & y'_{k-1}(\alpha) & y'_{k+1}(\alpha) & \cdots & y'_n(\alpha) \\ \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-2)}(\alpha) & \cdots & y_{k-1}^{(n-2)}(\alpha) & y_{k+1}^{(n-2)}(\alpha) & \ldots & y_n^{(n-2)}(\alpha) \end{vmatrix},$$

$$k = 1, 2, \ldots, n, \tag{13}$$

where W_n is Wronskian

$$W_n(x) = \left| y_k^{(i-1)}(x) \right|_{1 \le i,\, k \le n} \tag{14}$$

and $y_k(x)$ are linear independent solutions of Eq. (1), then Wronskian satisfies condition $W_n(\alpha) \neq 0$.

The other formula of construction of the Green's function for homogeneous differential equations has form [14]

$$K(x, \alpha) = \frac{|A|}{W(\alpha) p_n(\alpha)}, \tag{15}$$

where

$$|A| = \begin{vmatrix} y_1(\alpha) & y_2(\alpha) & \ldots & y_n(\alpha) \\ y'_1(\alpha) & y'_2(\alpha) & \cdots & y'_n(\alpha) \\ \vdots & \vdots & \ldots & \vdots \\ y_1^{(n-2)}(\alpha) & y_2^{(n-2)}(\alpha) & \cdots & y_n^{(n-2)}(\alpha) \\ y_1(x) & y_2(x) & \cdots & y_n(x) \end{vmatrix}, \tag{16}$$

$$W(\alpha) = \begin{vmatrix} y_1(\alpha) & y_2(\alpha) & \ldots & y_n(\alpha) \\ y'_1(\alpha) & y'_2(\alpha) & \cdots & y'_n(\alpha) \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)}(\alpha) & y_2^{(n-1)}(\alpha) & \ldots & y_n^{(n-1)}(\alpha) \end{vmatrix}. \tag{17}$$

Additionally, the functions $K(x, \alpha)$ must satisfy conditions presented in the following form [13]

$$\left.\frac{\partial^i K(x, \alpha)}{\partial x^i}\right|_{x=\alpha} = 0, \quad i = 0, 1, \ldots, n-2, \tag{18a}$$

$$\left.\frac{\partial^{n-1} K(x, \alpha)}{\partial x^{n-1}}\right|_{x=\alpha} = \frac{1}{p_n(\alpha)}. \tag{18b}$$

4. Green's function

The characteristic equation of homogeneous differential Euler equation

$$L(w) \equiv \frac{d^4 w}{d\xi^4} + \frac{2}{\xi}\frac{d^3 w}{d\xi^3} - \frac{(1+2n^2)}{\xi^2}\frac{d^2 w}{d\xi^2}$$
$$+ \frac{(1+2n^2)}{\xi^3}\frac{dw}{d\xi} + \frac{(n^4 - 4n^2)}{\xi^4} w = 0 \tag{19}$$

has following form

$$s^4 - 4s^3 + (4 - 2n^2)s^2 + 4n^2 s - n^2(4 - n^2) = 0. \tag{20}$$

The roots of Eq. (20) have form

$$s_1 = n, \quad s_2 = -n, \quad s_3 = 2 - n, \quad s_4 = 2 + n. \tag{21}$$

The linear independent solutions of Eq. (15) are

$$w_1(\xi) = \xi^n, \quad w_2(\xi) = \xi^{-n},$$
$$w_3(\xi) = \xi^{2-n}, \quad w_4(\xi) = \xi^{2+n}. \tag{22}$$

The Green's function (solution of homogeneous Euler equation $L(K(\xi, \alpha)) = 0$) for different nodal diameters may be received from Eq. (15) presented in the following form

$$K_n(\xi, \alpha) = \frac{|A_n|}{W(\alpha)_n p_n(\alpha)}, \tag{23}$$

where $p_n(\alpha) = 1$ is coefficient placed on before highest order of derivative of Euler differential Eq. (19) and

$$|A_n| = \begin{vmatrix} \alpha^n & \alpha^{-n} & \alpha^{2-n} & \alpha^{2+n} \\ \frac{d(\alpha^n)}{d\alpha} & \frac{d(\alpha^{-n})}{d\alpha} & \frac{d(\alpha^{2-n})}{d\alpha} & \frac{d(\alpha^{2+n})}{d\alpha} \\ \frac{d^2(\alpha^n)}{d\alpha^2} & \frac{d^2(\alpha^{-n})}{d\alpha^2} & \frac{d^2(\alpha^{2-n})}{d\alpha^2} & \frac{d^2(\alpha^{2+n})}{d\alpha^2} \\ \xi^n & \xi^{-n} & \xi^{2-n} & \xi^{2+n} \end{vmatrix}, \tag{24}$$

$$W(\alpha)_n = \begin{vmatrix} \alpha^n & \alpha^{-n} & \alpha^{2-n} & \alpha^{2+n} \\ \frac{d(\alpha^n)}{d\alpha} & \frac{d(\alpha^{-n})}{d\alpha} & \frac{d(\alpha^{2-n})}{d\alpha} & \frac{d(\alpha^{2+n})}{d\alpha} \\ \frac{d^2(\alpha^n)}{d\alpha^2} & \frac{d^2(\alpha^{-n})}{d\alpha^2} & \frac{d^2(\alpha^{2-n})}{d\alpha^2} & \frac{d^2(\alpha^{2+n})}{d\alpha^2} \\ \frac{d^3(\alpha^n)}{d\alpha^3} & \frac{d^3(\alpha^{-n})}{d\alpha^3} & \frac{d^3(\alpha^{2-n})}{d\alpha^3} & \frac{d^3(\alpha^{2+n})}{d\alpha^3} \end{vmatrix}, \tag{25}$$

After calculations, the Green's function (GF) for different number nodal diameters n has following form

$$K_n(\xi, \alpha) = \frac{\alpha^{n+1}\xi^{-n+2} - \xi^n \alpha^{-n+3}}{8n^2 - 8n}$$
$$+ \frac{\alpha^{-n+1}\xi^{n+2} - \xi^{-n}\alpha^{n+3}}{8n^2 + 8n}, \quad n \ge 2 \tag{26}$$

and satisfies conditions from Eq. (18)

$$K_n(\alpha, \alpha) = \frac{\partial K_n(\xi, \alpha)}{\partial \xi}|_{\xi=\alpha}$$
$$= \frac{\partial^2 K_n(\xi, \alpha)}{\partial \xi^2}|_{\xi=\alpha} = 0, \tag{27a}$$

$$\frac{\partial^3 K_n(\xi, \alpha)}{\partial \xi^3}|_{\xi=\alpha} = 1. \tag{27b}$$

The function $K_n(\xi, \alpha)$ is indeterminate for $n = 0$ and $n = 1$, then the Green's functions for this values have following form

$$\lim_{n \to 0} K_n(\xi, \alpha) = \frac{\alpha}{4}\left[\alpha^2 - \xi^2 + (\xi^2 + \alpha^2) ln \frac{\xi}{\alpha}\right], \tag{28}$$

$$\lim_{n \to 1} K_n(\xi, \alpha) = \frac{\xi^4 - \alpha^4 + 4\xi^2\alpha^2 ln \frac{\alpha}{\xi}}{16\xi}. \tag{29}$$

In Table 1 the examples of formulas of Green's function for seven nodal diameters are presented.

Table 1
The formulas of Green's functions for the different number of nodal diameters

n	$K_n(\xi,\alpha)$
0	$K_0(\xi,\alpha)=\frac{\alpha}{4}\left[\alpha^2-\xi^2+(\xi^2+\alpha^2)ln\frac{\xi}{\alpha}\right]$
1	$K_1(\xi,\alpha)=\frac{\xi^4-\alpha^4+4\xi^2\alpha^2ln\frac{\alpha}{\xi}}{16\xi}$
2	$K_2(\xi,\alpha)=\frac{\left(\xi^2-\alpha^2\right)^3}{48\xi^2\alpha}$
3	$K_3(\xi,\alpha)=\frac{\left(\xi^2-\alpha^2\right)^3(\xi^2+\alpha^2)}{96\xi^3\alpha^2}$
4	$K_4(\xi,\alpha)=\frac{1}{96}\left(\frac{\alpha^5}{\xi^2}-\frac{\xi^4}{\alpha}\right)+\frac{1}{160}\left(\frac{\xi^6}{\alpha^3}-\frac{\alpha^7}{\xi^4}\right)$
5	$K_5(\xi,\alpha)=\frac{1}{160}\left(\frac{\alpha^6}{\xi^3}-\frac{\xi^5}{\alpha^2}\right)+\frac{1}{240}\left(\frac{\xi^7}{\alpha^4}-\frac{\alpha^8}{\xi^5}\right)$
6	$K_6\left(\xi,\alpha\right)=\frac{1}{240}\left(\frac{\alpha^7}{\xi^4}-\frac{\xi^6}{\alpha^3}\right)+\frac{1}{336}\left(\frac{\xi^8}{\alpha^5}-\frac{\alpha^9}{\xi^6}\right)$
7	$K_7(\xi,\alpha)=\frac{1}{336}\left(\frac{\alpha^8}{\xi^5}-\frac{\xi^7}{\alpha^4}\right)+\frac{1}{448}\left(\frac{\xi^9}{\alpha^6}-\frac{\alpha^{10}}{\xi^7}\right)$

5. Solution

The ordinary differential equations with constant or variable coefficients can be transformed to the Fredholm or the Volterra integral equations using e.g. Fubini's method [17]. The solutions of this equations are solutions of the transformed ordinary differential equation. If the Green's function (kernel of integral equation) is well known, the linear independent solutions can be expanded in the Neumann power series rapidly convergent to eigenvalues (spectrum of integral kernel) based on the method of successive approximations [18].

The limited (for $\xi=0$) independent solutions of Eq. (19) are $w_1(\xi)=\xi^n$ and $w_2(\xi)=\xi^{n+2}$. This solutions were expanded in the Neumann power series *in the following form*

$$K_n(\xi,\lambda)_u=\xi^n+\sum_{i=1}^{\eta}K_i(\xi)_u\cdot\lambda^{2i},\qquad\lambda\in R^+\tag{30a}$$

$$K_n(\xi,\lambda)_v=\xi^{n+2}+\sum_{i=1}^{\eta}K_i(\xi)_v\cdot\lambda^{2i},\tag{30b}$$

where $K_i(\xi)_u$ and $K_i(\xi)_v$ are iterated kernels [18] presented in the following form

$$K_i(\xi)_u=\int_0^{\xi}K_n(\xi,\alpha)K_{i-1}(\alpha)_u d\alpha,\tag{31a}$$

$$K_i(\xi)_v=\int_0^{\xi}K_n(\xi,\alpha)K_{i-1}(\alpha)_v d\alpha\tag{31b}$$

and η is the degree of approximations.

$K_0(\alpha)_u$ and $K_0(\alpha)_v$ are the limited independent solutions depending on number of nodal lines presented in the following form

$$K_0(\alpha)_u=\alpha^n,\tag{32a}$$

$$K_0(\alpha)_v=\alpha^{n+2}.\tag{32b}$$

The characteristic equations $\Delta_n=0$ for different boundary conditions and different values of parameter n are obtained from well known the characteristic determinants given by:
Clamped:

$$\Delta_n(\lambda)\equiv\left|\begin{array}{cc}K_n(\xi,\lambda)_u & K_n(\xi,\lambda)_v\\ \frac{\partial K_n(\xi,\lambda)_u}{\partial\xi} & \frac{\partial K_n(\xi,\lambda)_v}{\partial\xi}\end{array}\right|_{\xi=1};\tag{33}$$

Simply supported:

$$\Delta_n(\lambda)\equiv\left|\begin{array}{cc}K_n(\xi,\lambda)_u & K_n(\xi,\lambda)_v\\ M\left[K_n(\xi,\lambda)_u\right] & M\left[K_n(\xi,\lambda)_v\right]\end{array}\right|_{\xi=1};\tag{34}$$

Free:

$$\Delta_n(\lambda)\equiv\left|\begin{array}{cc}M\left[K_n(\xi,\lambda)_u\right] & M\left[K_n(\xi,\lambda)_v\right]\\ V\left[K_n(\xi,\lambda)_u\right] & V\left[K_n(\xi,\lambda)_v\right]\end{array}\right|_{\xi=1};\tag{35}$$

Sliding supports:

$$\Delta_n(\lambda)\equiv\left|\begin{array}{cc}\frac{\partial K_n(\xi,\lambda)_u}{\partial\xi} & \frac{\partial K_n(\xi,\lambda)_v}{\partial\xi}\\ V\left[K_n(\xi,\lambda)_u\right] & V\left[K_n(\xi,\lambda)_v\right]\end{array}\right|_{\xi=1};\tag{36}$$

Elastic supports:

$$\Delta_n(\lambda)\equiv\left|\begin{array}{cc}\Phi[K_n(\xi,\lambda)_u] & \Phi[K_n(\xi,\lambda)_v]\\ \Psi[K_n(\xi,\lambda)_u] & \Psi[K_n(\xi,\lambda)_v]\end{array}\right|_{\xi=1}.\tag{37}$$

For all boundary conditions formula of Δ_n has following form

$$\Delta_n=a_0+\sum_{i=1}^{\eta}(-1)^i a_i\lambda^{2i},\tag{38}$$

where $a_0,\,a_1,\ldots,a_\eta$ are coefficients of characteristic equations depending on boundary conditions and number of nodal lines n.

6. Numerical results

The numerical results for the first ten dimensionless frequencies for different boundary conditions are presented in Tables 2–8. The nodal diameters are considered for the values between 0 and 7. The Neumann power series Eq. (30) were expanded only for $\eta=25$. Poisson ratio is taken as $\nu=0.3$ for all considered cases. The numerical results for free, clamped and simply supported circular plates are presented in Tables 2–4 with comparison to Refs. [1, 6, 9, 10, 16]. The numerical results for plates with sliding supports are shown in Table 5 with comparison to Refs. [6, 15]. The eigenvalues for plates with elastic supports and different values of elastic parameters on the edges are presented in Tables 6–8 with comparison to Refs. [6, 15].

Table 2

The first ten lower dimensionless frequencies λ of free vibration of clamped circular plates

Dimensionless frequency, λ		The number of nodal diameters, n							
		0	1	2	3	4	5	6	7
λ_0	GF	10.216	21.260	34.877	51.030	69.665	90.739	114.213	140.056
	Ref. [1]	10.215	21.260	34.88	51.04	69.665	90.739	114.212	140.056
	Ref. [6]	10.216	21.260	34.877	51.030	69.666	90.739	-	-
	Ref. [9]	10.215	21.260	34.877	51.03	69.665	90.738	-	-
	Ref. [16]	10.215	21.260	34.877	-	-	-	-	-
λ_1	GF	39.771	60.828	84.582	111.021	140.108	171.803	206.071	242.878
	Ref. [1]	39.771	60.82	84.58	111.01	140.107	171.802	206.070	242.878
	Ref. [6]	39.771	60.829	84.583	111.021	140.108	171.803	-	-
	Ref. [9]	39.771	60.828	84.582	111.021	140.107	171.802	-	-
	Ref. [16]	39.771	60.828	84.582	-	-	-	-	-
λ_2	GF	89.104	120.079	153.815	190.304	229.519	271.428	316.002	363.210
	Ref. [1]	89.104	120.08	153.81	190.30	229.518	271.428	316.001	363.209
	Ref. [6]	89.104	120.079	153.815	190.304	229.519	271.428	-	-
	Ref. [9]	89.104	120.079	153.815	190.303	229.518	271.428	-	-
	Ref. [16]	89.104	120.079	153.815	-	-	-	-	-
λ_3	GF	158.184	199.053	242.721	289.180	338.411	390.389	445.089	502.483
	Ref. [1]	158.183	199.06	242.71	289.17	338.411	390.389	-	-
	Ref. [6]	158.184	199.053	242.721	289.180	338.411	390.389	-	-
	Ref. [9]	158.184	199.053	242.720	289.179	338.411	390.389	-	-
	Ref. [16]	158.184	199.053	242.721	-	-	-	-	-
λ_4	GF	247.006	297.760	351.336	407.730	466.925	528.902	593.639	661.112
	Ref. [1]	247.005	297.77	351.38	407.72	-	-	-	-
	Ref. [6]	247.006	297.760	351.336	407.730	466.925	528.902	-	-
	Ref. [9]	247.006	297.76	351.335	407.729	466.925	528.902	-	-
	Ref. [16]	247.007	297.761	351.337	-	-	-	-	-
λ_5	GF	355.569	416.203	479.675	545.983	615.114	687.051	761.776	839.268
	Ref. [1]	355.568	416.20	479.65	545.97	-	-	-	-
	Ref. [6]	355.569	416.203	479.675	545.983	615.114	687.051	-	-
λ_6	GF	483.872	554.382	627.744	703.955	783.004	864.877	949.558	1037.03
	Ref. [1]	483.872	554.37	627.75	703.95	-	-	-	-
λ_7	GF	631.915	712.300	795.546	881.652	970.607	1062.40	1157.02	1254.45
	Ref. [1]	631.914	712.30	795.52	881.67	-	-	-	-
λ_8	GF	799.697	889.956	983.084	1079.08	1177.93	1279.64	1384.18	1491.55
	Ref. [1]	799.762	889.95	983.07	1079.0	-	-	-	-
λ_9	GF	987.219	1087.35	1190.36	1296.24	1404.98	1516.52	1629.73	1734.10
	Ref. [1]	987.216	1087.4	1190.4	1296.2	-	-	-	-

Table 3

The first ten lower dimensionless frequencies λ of free vibration of free circular plates

Dimensionless frequency, λ		The number of nodal diameters, n							
		0	1	2	3	4	5	6	7
λ_0	GF	9.003	20.474	5.358	12.439	21.835	33.494	47.378	63.455
	Ref. [1]	9.084	20.52	5.253	12.23	21.6	33.1	46.2	-
	Ref. [6]	9.003	20.475	5.358	12.439	21.835	33.495	-	-
	Ref. [9]	9.003	20.474	5.358	12.439	21.835	33.494	-	-
	Ref. [16]	9.003	20.474	5.358	-	-	-	-	-
λ_1	GF	38.443	59.811	35.260	53.007	73.542	96.755	122.57	150.928
	Ref. [1]	38.55	59.86	35.25	52.91	73.1	95.8	121.0	-
	Ref. [6]	38.443	59.812	35.260	53.008	73.543	96.755	-	-
	Ref. [9]	38.443	59.811	35.260	53.007	73.542	96.755	-	-
	Ref. [16]	38.443	59.812	35.260	-	-	-	-	-
λ_2	GF	87.750	118.957	84.366	111.945	142.431	175.735	211.789	250.535
	Ref. [1]	87.80	119.0	83.9	111.3	142.8	175.0	210.3	-
	Ref. [6]	87.750	118.957	84.366	111.945	142.431	175.735	-	-
	Ref. [9]	87.750	118.957	84.366	111.945	142.431	175.735	-	-
	Ref. [16]	87.753	118.961	84.361	-	-	-	-	-
λ_3	GF	156.818	197.872	153.306	190.692	231.03	274.252	320.299	369.121
	Ref. [1]	157.0	198.2	154.0	192.1	232.3	274.6	319.7	-
	Ref. [6]	156.818	197.872	153.306	190.692	231.03	274.252	-	-
	Ref. [9]	156.818	197.871	153.306	190.692	231.03	274.252	-	-
	Ref. [16]	156.826	197.883	153.310	-	-	-	-	-
λ_4	GF	245.634	296.54	242.036	289.238	339.413	392.505	448.467	507.254
	Ref. [1]	245.9	296.9	242.7	290.7	340.4	392.4	447.3	-
	Ref. [6]	245.634	296.540	242.036	289.238	339.413	392.505	-	-
	Ref. [9]	245.633	296.54	242.036	289.238	339.413	392.505	-	-
	Ref. [16]	245.651	296.564	242.049	-	-	-	-	-
λ_5	GF	354.192	414.956	350.534	407.562	467.573	530.521	596.365	665.069
	Ref. [1]	354.6	415.3	350.8	408.4	467.9	529.5	593.9	-
	Ref. [6]	-	-	350.534	407.562	467.573	530.521	-	-
λ_6	GF	482.491	553.115	478.79	545.651	615.501	688.3	764.014	842.609
	Ref. [1]	483.1	553.0	479.2	546.2	615.0	686.4	760.1	-
λ_7	GF	630.532	711.017	626.795	703.497	783.189	865.835	951.398	1039.89
	Ref. [1]	631.0	711.3	627.0	703.3	781.8	864.4	952.3	-
λ_8	GF	798.312	888.664	794.551	881.105	970.627	1063.2	1159.11	1257.52
	Ref. [1]	798.6	888.6	794.7	880.3	968.5	1061.0	1158.7	-
λ_9	GF	985.823	1086.19	981.983	1078.4	1178.33	1278.9	1373.23	1471.19
	Ref. [1]	986.0	1086.0	981.6	1076.0	1175.0	1277.0	1384.0	-

Table 4

The first ten lower dimensionless frequencies λ of free vibration of simply supported circular plates

Dimensionless frequency, λ		The number of nodal diameters, n							
		0	1	2	3	4	5	6	7
λ_0	GF	4.935	13.898	25.613	39.957	56.841	76.203	97.944	122.179
	Ref. [6]	4.935	13.898	25.613	39.957	56.842	76.203	-	-
	Ref. [9]	4.935	13.898	25.613	39.957	56.841	76.203	-	-
	Ref. [10]	4.935	13.898	25.614	39.957	56.842	76.203	-	-
	Ref. [16]	4.935	13.898	25.613	-	-	-	-	-
λ_1	GF	29.72	48.478	70.117	94.549	121.702	151.518	183.948	218.951
	Ref. [6]	29.720	48.479	70.117	94.549	121.702	151.518	-	-
	Ref. [9]	29.72	48.478	70.117	94.549	121.702	151.518	-	-
	Ref. [10]	29.71	48.478	70.116	94.549	121.702	151.518	-	-
	Ref. [16]	29.72	48.478	70.117	-	-	-	-	-
λ_2	GF	74.156	102.773	134.298	168.675	205.851	245.778	288.414	333.721
	Ref. [6]	74.156	102.772	134.298	168.675	205.851	245.778	-	-
	Ref. [9]	74.156	102.773	134.297	168.675	205.851	245.778	-	-
	Ref. [10]	74.15	102.773	134.297	168.674	205.852	245.778	-	-
	Ref. [16]	74.155	102.773	134.297	-	-	-	-	-
λ_3	GF	138.318	176.801	218.203	262.485	309.607	359.532	412.221	467.644
	Ref. [6]	138.318	176.801	218.203	262.485	309.607	359.532	-	-
	Ref. [9]	138.318	176.801	218.202	262.484	309.607	359.532	-	-
	Ref. [10]	138.320	176.801	218.202	262.485	309.607	359.535	-	-
	Ref. [16]	138.317	176.800	218.202	-	-	-	-	-
λ_4	GF	222.215	270.566	321.841	376.012	433.049	492.919	555.592	621.04
	Ref. [6]	222.215	270.566	321.841	376.012	433.049	492.919	-	-
	Ref. [9]	222.215	270.566	321.840	376.012	433.049	492.919	-	-
	Ref. [10]	222.215	270.567	321.841	376.012	433.048	492.918	-	-
	Ref. [16]	222.213	270.564	321.838	-	-	-	-	-
λ_5	GF	325.849	384.069	445.215	509.268	576.203	645.992	718.611	794.034
	Ref. [6]	325.849	384.069	445.216	509.268	576.203	645.992	-	-
λ_6	GF	449.222	517.31	588.328	662.258	739.081	818.774	901.312	986.672
λ_7	GF	592.333	670.29	751.18	834.984	921.690	1011.28	1103.81	1199.27
λ_8	GF	755.182	843.008	933.744	1027.46	1123.98	1223.59	1324.62	1427.13
λ_9	GF	937.764	1035.54	1136.62	1238.65	1346.82	1450.60	1560.95	1742.34

Table 5

The first ten lower dimensionless frequencies λ of free vibration of circular plates with sliding edge

Dimensionless frequency, λ		The number of nodal diameters, n							
		0	1	2	3	4	5	6	7
λ_0	GF	14.682	3.082	8.784	16.902	27.343	40.055	55.003	72.160
	Ref. [6]	14.682	3.082	8.785	16.902	27.343	40.056	-	-
	Ref.[15]	14.682	-	8.785	-	-	-	-	-
λ_1	GF	49.218	28.398	44.904	64.130	86.004	110.464	137.462	166.958
	Ref. [6]	49.218	28.399	44.904	64.130	86.004	110.464	-	-
	Ref. [15]	49.218	28.399	44.904	-	-	-	-	-
λ_2	GF	103.499	72.859	99.361	128.677	160.754	195.539	232.990	273.067
	Ref. [6]	103.499	72.859	99.361	128.677	160.754	195.539	-	-
	Ref. [15]	103.500	72.860	99.359	-	-	-	-	-
λ_3	GF	177.521	137.025	173.442	212.716	254.806	299.671	347.273	397.579
	Ref. [6]	177.521	137.025	173.442	212.716	254.806	299.671	-	-
	Ref. [15]	-	137.009	173.564	-	-	-	-	-
λ_4	GF	271.282	220.923	267.231	316.419	368.456	423.308	480.944	541.333
	Ref. [6]	271.282	220.923	267.231	316.419	368.456	423.308	-	-
λ_5	GF	384.782	324.577	380.746	439.830	501.783	566.578	634.188	704.587
	Ref. [6]	-	324.577	380.746	439.830	501.783	566.578	-	-
λ_6	GF	518.021	447.929	513.995	582.965	654.818	729.532	807.082	887.445
λ_7	GF	671.0	591.039	666.980	745.830	827.573	912.191	999.671	1090.01
λ_8	GF	843.718	753.888	839.699	928.440	1020.12	1114.57	1212.05	1312.14
λ_9	GF	1036.21	936.488	1032.24	1130.51	1231.13	1338.58	1435.74	1534.41

Table 6

The first ten lower dimensionless frequencies λ of free vibration of circular plates with elastic supports ($\varphi = 0.1$; $\psi = 100$)

Dimensionless frequency, λ		The number of nodal diameters, n							
		0	1	2	3	4	5	6	7
λ_0	GF	4.854	12.139	19.079	25.685	33.427	43.294	55.625	70.451
	Ref. [6]	4.854	12.140	19.080	25.686	33.427	43.295	-	-
	Ref.[15]	4.854	12.140	19.080	-	-	-	-	-
λ_1	GF	22.097	31.019	42.420	57.965	77.227	99.656	124.95	152.939
	Ref. [6]	22.098	31.020	42.421	57.965	77.227	99.657	-	-
	Ref. [15]	22.098	31.020	42.420	-	-	-	-	-
λ_2	GF	44.938	63.924	87.285	114.196	144.263	177.279	213.124	251.717
	Ref. [6]	44.938	63.925	87.285	114.196	144.263	177.279	-	-
	Ref. [15]	44.938	63.925	87.285	-	-	-	-	-
λ_3	GF	90.469	120.973	154.912	192.032	232.184	275.267	321.207	369.944
	Ref. [6]	90.469	120.973	154.912	192.032	232.184	275.267	-	-
λ_4	GF	158.359	199.127	243.102	290.168	340.243	393.257	449.155	507.890
	Ref. [6]	158.359	199.127	243.102	290.168	340.243	393.257	-	-
λ_5	GF	246.673	297.434	351.324	408.273	468.222	531.121	596.924	665.594
	Ref. [6]	246.673	297.434	351.324	408.273	468.222	531.121	-	-
λ_6	GF	354.968	415.648	479.418	546.229	616.038	688.804	764.490	843.059
λ_7	GF	483.113	553.683	627.321	703.989	783.653	866.277	951.818	1040.36
λ_8	GF	631.051	711.503	795.003	881.537	971.030	1063.60	1159.48	1255.58
λ_9	GF	798.765	889.108	982.443	1078.60	1178.52	1278.32	1373.61	1799.22

Table 7
The first ten lower dimensionless frequencies λ of free vibration of circular plates with elastic supports ($\varphi = 10$; $\psi = 100$)

Dimensionless frequency, λ		The number of nodal diameters, n							
		0	1	2	3	4	5	6	7
λ_0	GF	7.790	13.845	19.352	25.740	34.090	44.808	57.962	73.504
	Ref. [6]	7.790	13.845	19.352	25.741	34.090	44.809	-	-
	Ref.[15]	7.790	13.845	19.352	-	-	-	-	-
λ_1	GF	22.128	32.098	45.801	63.015	83.282	106.348	132.075	160.377
	Ref. [6]	22.128	32.098	45.802	63.016	83.282	106.348	-	-
	Ref. [15]	22.128	32.098	45.802	-	-	-	-	-
λ_2	GF	49.253	70.468	95.064	122.728	153.294	186.663	222.764	261.542
	Ref. [6]	49.254	70.468	95.065	122.728	153.294	186.663	-	-
	Ref. [15]	49.254	70.468	95.064	-	-	-	-	-
λ_3	GF	98.741	130.265	164.874	202.466	242.962	286.302	332.436	381.320
	Ref. [6]	98.741	130.265	164.874	202.466	242.962	286.302	-	-
λ_4	GF	168.599	210.017	254.463	301.883	352.226	405.447	461.505	520.364
	Ref. [6]	168.599	210.017	254.463	301.883	352.226	405.447	-	-
λ_5	GF	258.213	309.453	363.707	420.938	481.108	544.179	610.119	678.895
	Ref. [6]	258.213	309.453	363.707	420.938	481.108	544.180	-	-
λ_6	GF	367.477	428.531	492.594	559.638	629.632	702.545	778.349	857.013
λ_7	GF	496.382	567.255	641.135	717.998	797.818	880.574	966.209	1054.78
λ_8	GF	644.938	725.639	809.342	896.037	985.712	1078.17	1174.70	1272.15
λ_9	GF	813.16	903.712	997.277	1093.80	1193.01	1300.43	1379.91	1497.66

Table 8
The first ten lower dimensionless frequencies λ of free vibration of circular plates with elastic supports ($\varphi = 1000$; $\psi = 100$)

Dimensionless frequency, λ		The number of nodal diameters, n							
		0	1	2	3	4	5	6	7
λ_0	GF	8.809	14.552	19.475	25.769	34.480	45.828	59.746	76.123
	Ref. [6]	8.809	14.552	19.475	25.769	34.481	45.828	-	-
	Ref.[15]	8.809	14.552	19.475	-	-	-	-	-
λ_1	GF	22.142	32.628	47.554	65.992	87.405	111.561	138.339	167.666
	Ref. [6]	22.143	32.629	47.554	65.993	87.406	111.561	-	-
	Ref. [15]	22.142	32.629	47.554	-	-	-	-	-
λ_2	GF	51.441	74.280	100.364	129.418	161.309	195.953	233.288	273.267
	Ref. [6]	51.442	74.281	100.364	129.418	161.309	195.953	-	-
	Ref. [15]	51.442	74.280	100.366	-	-	-	-	-
λ_3	GF	104.413	137.654	173.880	213.012	254.988	299.755	347.270	397.496
	Ref. [6]	104.413	137.654	173.880	213.012	254.988	299.755	-	-
λ_4	GF	177.926	221.172	267.357	316.442	368.387	423.155	480.711	541.024
	Ref. [6]	177.926	221.172	267.357	316.442	368.387	423.155	-	-
λ_5	GF	271.391	324.553	380.643	439.636	501.504	566.217	633.747	704.066
	Ref. [6]	271.391	324.553	380.643	439.636	501.504	566.217	-	-
λ_6	GF	384.668	447.716	513.690	582.572	654.339	728.968	806.434	886.713
λ_7	GF	517.708	590.630	666.479	745.238	826.891	911.421	998.782	1089.06
λ_8	GF	670.492	753.282	838.999	927.644	1019.22	1113.55	1211.74	1310.34
λ_9	GF	843.007	935.670	1031.21	1129.53	1229.99	1337.24	1424.49	1540.84

7. Conclusions

In this paper, the Green's functions have been employed to solve natural vibration of circular thin plates with different boundary conditions. The universal Green's function for different nodal diameters is defined in a closed form. The ten lower natural frequencies are calculated for eight different natural modes. The limited independent solutions of differential Euler equations were expanded into the Neumann power series using the method of successive approximation. This approach allows to obtain the analytical frequency equations as power series rapidly convergent to exact eigenvalues for a different number of nodal diameters. The obtained results are in good agreement with results obtained by other methods presented in literature. The obtained numerical results can be used to validate the accuracy of other numerical methods as benchmark values. The calculations are evaluated with the help of Mathematica v10, which is a symbolic calculation software.

REFERENCES

[1] A.W. Leissa, *Vibration of Plates*, National Aeronautics and Space Administration, Washington, 1969.

[2] H.F. Bauer and W. Eidel, "Determination of the lower natural frequencies of circular plates with mixed boundary conditions", *J. Sound and Vibration* 292, 742–764 (2006).

[3] S. Chakraverty and M. Petyt, "Natural frequencies for free vibration of nonhomogeneous elliptic and circular plates using two-dimensional orthogonal polynomials", *Applied Mathematical Modelling* 21 (7), 399–417 (1997).

[4] R.B. Bhat, S. Chakraverty, and I. Stiharu, "Recurrence scheme for the generation of two-dimensional boundary characteristic orthogonal polynomials to study vibration of plates", *J. Sound and Vibration* 216 (2), 321–327 (1998).

[5] T.Y. Wu and G.R. Liu, "Free vibration analysis of circular plates with variable thickness by the generalized differential quadrature rule", *Int. J. Solids and Structures* 38, 7967–7980 (2001).

[6] T.Y. Wu, Y.Y. Wang, and G.R. Liu, "Free vibration analysis of circular plates using generalized differential quadrature rule",

Computer Methods in Applied Mechanics and Engineering 191, 5365–5380 (2002).

[7] J. Jaroszewicz and L. Zoryj, "The method of partial discretization in free vibration problems of circular plates with variable distribution of parameters", *Int. Applied Mechanics* 42 (3), 364–373 (2006).

[8] F. Ebrahimi and A. Rastgo, "An analytical study on the free vibration of smart circular thin FGM plate based on classical plate theory", *Thin-Walled Structures* 46 (12), 1402–1408 (2008).

[9] H.S. Yalcin, A. Arikoglu, and I. Ozkol, "Free vibration analysis of circular plates by differential transformation method", *Applied Mathematics and Computation* 212, 377–386 (2009).

[10] Z.H. Zhou, K.W. Wong, X.S. Xu, and A.Y.T. Leung, "Natural vibration of circular and annular thin plates by Hamiltonian approach", *J. Sound and Vibration* 330, 1005–1017 (2011).

[11] S. Kukla and M. Szewczyk, "Frequency analysis of annular plates with elastic concentric supports by Green's function method", *J. Sound and Vibration* 300, 387–393 (2007).

[12] S.V. Sorokin and N. Peake, "Vibrations of sandwich plates with concentrated masses and spring – like inclusions", *J. Sound and Vibration* 237 (2), 203–222 (2000).

[13] S. Kukla, *Green's Functions and Their Properties*, Czestochowa University of Technology, Czestochowa, 2009.

[14] J. Jaroszewicz and L. Zoryj, *Methods of Free Axisymmetric Vibration Analysis of Circular Plates Using by Influence Functions*, Bialystok University of Technology, Bialystok, 2005.

[15] S. Azimi, "Free vibration of circular plates with elastic edge supports using receptance method", *J. Sound and Vibration* 120, 37–52 (1988).

[16] G. Duan, X. Wang, and Ch. Jin, "Free vibration analysis of circular thin plates with stepped thickness by the DSC element method", *Thin-Walled Structures* 85, 25–33 (2014).

[17] F.G. Tricomi, *Integral Equations*, Dover Publications, New York, 1985.

[18] Y.V. Shestopalov and Y.G. Smirnov, *Integral Equations*, Karlstad University, Karlstad, 2002.