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Comparison of Formulas Obtained for Analytical and LQ Idea Approaches to Determine the Optimal Actuator Location in Active Multimodal Beam Vibration Reduction

Elżbieta ŻOŁOPA[(1)], Adam BRAŃSKI[(2)]

[(1)] Halszki 31/12, 30-611 Kraków, Poland

[(2)] *Laboratory of Acoustics, Department of Electrical and Computer Engineering, Rzeszów University of Technology*
Powstańców Warszawy 12, 35-959 Rzeszów, Poland; e-mail: abranski@prz.rzeszow.pl

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In this paper an active multimodal beam vibration reduction via one actuator is considered. The optimal actuator distribution is analyzed with two methods: an exact mathematical principles and the LQ problem idea. It turned out that the same mathematical expressions are derived. Thus, these methods are equivalent.

Keywords: beam, actuator, active vibration reduction, LQ problem, system control, control of vibration.

1. Introduction

There are several kinds of reduction: a-reduction ([a]ctive) – a special case of a-reduction is p-reduction ([p]rotection) – 0-reduction (lack of reduction) and i-reduction ([i]ntensify vibration) (Brański, Lipiński, 2011). There are also many structures of which the vibrations are reduced via actuators, from the simple beam (Augustyn *et al.*, 2014; Brański, 2011), through the plate (Wiciak, 2007) to the very complicated structure (Kozień, Wiciak, 2008).

The vibration reduction of the beam can be performed inter alia in an active manner. The a-reduction, via actuators, is described for example in (de Silva, 2000; Hansen, Snyder, 1997; Fuller *et al.*, 1997; Wiciak, 2008). It depends on many factors enumerated in (Bruant, 2010; Brański, Lipiński, 2011; Brański, 2011). It seems that the most important factor is the distribution of the actuator on the structure, because once bonded actuator cannot change the place. In the recent years, a great number of papers has been published on this subject, see for example (Guney, Eskinat, 2007; Bruant, Proslier, 2005; Gupta *et al.*, 2010; Qiu *et al.*, 2007; Żołopa, Brański, 2014a); a survey is given in (Bruant *et al.*, 2010; Brański, 2011) and references cited therein.

It has been proved, in own papers, that for separate modes the most effective distribution of actuators is in the sub-areas with the largest curvatures. Moreover, the p-reduction of the vibration is possible (Brański, Lipiński, 2011; Brański *et al.*, 2010). From mathematical point of view, this distribution is called an optimal one (O-distribution). In (Brański, 2013) it is proved that the maximum efficiency of p-reduction is obtained for the O-distribution of the single actuator.

All above considerations are based on the exact mathematical principles. But it is proved in (Żołopa, Brański, 2014a) that the same results are obtained if the LQ problem idea is used. The LQ problem is formulated in the control theory (Bruant *et al.*, 2010) and repeated in (Żołopa, Brański, 2014a) and references cited therein. In this theory the boundary problem is formulated in an abstract Hilbert space and the point control and the distributed output are considered. As a criterion minimization of beam deflection at any point is chosen. In this case it means the optimization LQ problem.

Focusing on the problem of the optimal distribution of one actuator, the exact mathematical attitude is applied to the (multimodal) beam vibration, i.e. not only to one mode but to the sum ones. In this case the a-reduction is possible (not p-reduction). The preliminary results are presented in (Żołopa, Brański, 2014b).

It is very interesting – from a cognitive point of view – and useful in the future to demonstrate that the results obtained via the exact mathematics confirm the LQ problem idea. The aim of the paper is the comparison of these two methods. To do this, one defines the criterion which allows to find an optimal location actuator and which does not depend on the time. It turns out that this optimization problem can be solved using only numerical method.

The comparison of the methods is on the equations and expressions level. The numerical results are depicted in (Żołopa, Brański, 2014b). To the best authors' knowledge, the above mentioned problem treated via the LQ problem idea is not known.

All assumptions are based on (Żołopa, Brański, 2014b). The research object is an uniform, straight and rectangular cross-section beam simple supported at the ends. For simplicity, the dynamic effects of the actuator and glue added to the research set are omitted. The vibration excitation is assumed as the point harmonic force. The force acts with the frequency between natural frequencies. All results are achieved for steady state, i.e. for $t = 0$.

2. Forced beam vibrations and their reduction via actuator

The beam simple supported at both ends is considered (Kozień, 2013; Żołopa, Brański, 2014b); geometrical data of the beam are: ℓ – length, S – surface of the rectangular cross-section, $f_e = f_e(x,t) = f_x(x)f_t(t)$ – external load of the beam, where $f_t = \exp(i\omega_q t)$. The beam vibration equation is given by

$$EJ\big(D_x^4 u + \mu D_x^4(D_t u)\big) + \rho S\, D_t^2 u = -f_e\,, \tag{1}$$

where $u = u(x,t)$ – beam (transverse) deflection at the point x in time t, E – Young's modulus, J – moment of inertia, ρ – density of material, μ – retardation time (damping factor), $D^4(...) = \partial^4(...)/\partial x^4$, $D_t(...) = \partial(...)/\partial t$.

Boundary conditions take the form

$$\begin{aligned} u(x=0,\,t) &= 0, \qquad & D_x^2 u(x=0,\,t) &= 0,\\ u(x=\ell,\,t) &= 0, \qquad & D_x^2 u(x=\ell,\,t) &= 0. \end{aligned} \tag{2}$$

Besides, initial conditions are assumed to be equal to zero, i.e. $\mathbf{u}_0 = \{u(0,\,x) = 0,\ D_t u(0,\,x) = 0\}$.

In the steady state, $f_x(x)$ is the sum of the concentrated excitation force f_q and the equivalent force between beam-actuator $f(x)$ (Fuller *et al.*, 1997; Kasprzyk, Wiciak, 2007; Kozień, 2013; Żołopa, Brański, 2014b) – see Fig. 1, hence

$$\begin{aligned} f_x(x) &= -f_q\,\delta(x-x_q) + f(x)\\ &= -f_q\,\delta(x-x_q) + \big(f_a\,\delta(x-x_{1a})\\ &\quad - 2f_a\,\delta(x-x_a) + f_a\,\delta(x+x_{2a})\big), \end{aligned} \tag{3}$$

where f_q – force amplitude, x_q – excitation point, $x_{1a} = x_a - \ell_a/2$, $x_{2a} = x_a + \ell_a/2$, x_a – location of the actuator center.

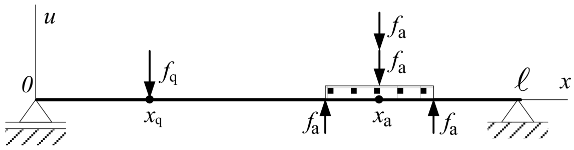

Fig. 1. Geometry of the beam and external forces.

The solution of the above problem for $\nu = 1, \ldots, \infty$ can be given by

$$\begin{aligned} u_e(x,\,t) &= u_q(x,\,t) + u_a(x,\,t)\\ &= \sum_\nu C^*_{e;\nu}\, I_{e;\nu}\, X_\nu(x)\exp(i\omega_q t)\\ &= \sum_\nu \overline{A}_{e;\nu}\, X_\nu(x)\exp(i\omega_q t), \end{aligned} \tag{4}$$

where

$$\begin{aligned} \overline{A}_{e;\nu} &= \overline{A}_{q;\nu} \pm \overline{A}_{a;\nu}\\ &= C^*_{e;\nu} I_{e;\nu} = C^*_{e;\nu}(I_{q;\nu} \pm I_{a;\nu}), \end{aligned} \tag{5}$$

$$C^*_\nu = \frac{1}{(1+i\mu\omega_q)\omega_\nu^2 - \omega_q^2}\,\frac{1}{\rho S}\,\frac{1}{\beta_\nu}\,, \tag{6}$$

$$\beta_\nu = \int_0^l X_\nu^2(x)\,\mathrm{d}x, \tag{7}$$

$$\begin{aligned} I_{q;\nu} &= -f_q\int_0^l \delta(x-x_q)X_\nu(x)\,\mathrm{d}x\\ &= -f_q\,X_\nu(x_q), \end{aligned} \tag{8}$$

$$I_{a;\nu} = \pm f_a \ell_a^2\,\kappa_\nu(x_a), \tag{9}$$

$$X_\nu(x) = K_2(\lambda_\nu\ell)\,K_2(\lambda_\nu x) - K_4(\lambda_\nu\ell)\,K_4(\lambda_\nu x) \tag{10}$$

and $\kappa_\nu(x_a)$ – curvature of $X_\nu(x)$ at the point $x = x_a$; the sign $\kappa_\nu(x_a)$ arises from the directions of the forces f_a (Brański, 2011), ω_q – excitation frequency, $K_\nu(.)$ – Krylov functions (Kaliski, 1986), λ_ν – they are the solution of the equation $K_2^2(\lambda_\nu\ell) - K_4^2(\lambda_\nu\ell) = 0$.

Hereunder the steady state of the problem is considered for $t = 0$. Because of harmonic type of excitation for steady state instead of Eq. (4) is

$$u_e(x) = \sum_\nu \left|C^*_{e,\nu}\right|\, I_{e,\nu}\, X_\nu(x)\sin(\varphi_\nu), \tag{11}$$

where φ_ν is solution of the equation

$$\cos\varphi_\nu = \frac{\omega_\nu^2 - \omega_q^2}{\big((\omega_\nu^2-\omega_q^2)^2 + (\mu\,\omega_q\,\omega_\nu^2)^2\big)^{1/2}}\,. \tag{12}$$

In the future consideration, the curvature of the beam $\kappa(x)$ plays a major part; it is defined as $\kappa(x) = \pm D^2 u_q(x)$ (Brański, Lipiński, 2011; Brański, 2011) and references cited therein and it is the result of the excited force f_q only.

3. Optimal actuator location, mathematical attitude

The task is to find such location of the actuator, described with x_a, which minimize the efficiency reduction coefficient $E(x)$, i.e. $\min_{\{x_a\}} E(x) = \min_{\{x_a\}} u_e(x)/f_\Sigma$, $4f_a = f_\Sigma$ (Żołopa, Brański, 2014a). This coefficient is a goal function of the optimal problem. The minimization of $E(x)$ means that the a-reduction is the most effective. If the energy added to the actuator is constant, the f_Σ is constant too and the optimal problem takes the form

$$\min_{\{x_a\}} u_e(x) = u_{e;\min}(x) \tag{13}$$

or in explicit form via Eq. (11)

$$\min_{\{x_a\}} u_e(x) = \min_{\{x_a\}} \sum_\nu \left|C^*_{e,\nu}\right| \Big(-f_q X_\nu(x_q) + f_a\,\ell_a^2\,\kappa_\nu(x_a)\Big) X_\nu(x)\sin(\varphi_\nu), \tag{14}$$

where the real part of $u_e(x)$ is taken into account.

The necessary condition of the minimization is

$$D_a u_e(x) = \sum_\nu \left|C^*_{e,\nu}\right| f_a\,\ell_a^2\,D_a\,\kappa_\nu(x_a) \cdot X_\nu(x)\sin(\varphi_\nu) = 0, \tag{15}$$

where $D_a(...) = D_{x_a}(...)$.

The sufficient condition is

$$D_a^2 u_e(x) = \sum_\nu \left|C^*_{e,\nu}\right| f_a\,\ell_a^2\,D_a^2\kappa_\nu(x_a) \cdot X_\nu(x)\sin(\varphi_\nu) > 0. \tag{16}$$

The above problem is solved and the results are presented in (Żołopa, Brański, 2014b).

4. Optimal actuator location based on the idea of LQ problem

The optimal position of the actuator which depends on the time is not acceptable from technical point of view. So it is necessary to formulate another criterion. For this purpose let us introduce new labels and variables, i.e.

$$z_1 = u_e(x,\,t), \qquad z_2 = D_t u_e(x,\,t), \tag{17}$$

hence $z_1 = z_1(x,t)$, $z_2 = z_2(x,t)$.

Substituting Eq. (17) into Eq. (1) one obtains an equations in a matrix form

$$\begin{bmatrix} D_t z_1 \\ D_t z_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\gamma\,D_x^4 & -\gamma\,\mu\,D_x^4 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/(\rho\,S)\,f_x(x) \end{bmatrix} f_t(t), \tag{18}$$

where I is an identity operator and $\gamma = (E\,J)/(\rho\,S)$.

To formally formulate the LQ problem, the boundary problem described in the above section should be described with abstract model. For this purpose one introduces an abstract Hilbert state space $\mathsf{H} := D(A_0^{1/2}) \times L^2(0,\,\ell)$, where the operators are defined as

$$\begin{aligned} A_0 &= \gamma\,D_x^4, \\ D(A_0) &= \big\{z \in L^2(0,\ell):\ D_x z,\ D_x^2 z \in L^2(0,\ell), \\ &\qquad D_x^3 z,\ D_x^4 z \in L^2(0,\ell),\ z(0) = 0, \\ &\qquad z(\ell) = 0,\ D_x^2 z(0) = 0,\ D_x^2 z(\ell) = 0\big\}. \end{aligned} \tag{19}$$

In this space the scalar product is defined $\langle z,\,w\rangle_{\mathsf{H}} = \langle z_1,\,w_1\rangle_{L^2(0,\ell)} + \langle A_0^{1/2} z_2,\,A_0^{1/2} w_2\rangle_{L^2(0,\ell)}$, where

$$w = [w_1,\,w_2]^{\mathrm{T}}. \tag{20}$$

For fixed time $t \geq 0$, the state vector

$$z(t) = [z_1(\cdot,t),\ z_2(\cdot,t)]^{\mathrm{T}} \tag{21}$$

belongs to the space H. So, the set Eq. (18) may be formulated in an abstract form as

$$D_t z(t) = A\,z(t) + B\,w(t), \qquad y(t) = C\,z(t). \tag{22}$$

The state operator $A\colon (D(A) \subset \mathsf{H}) \to \mathsf{H}$ is defined as follows:

$$A\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ -\gamma\,D_x^4 z_1 - \gamma\,\mu\,D_x^4 z_2 \end{bmatrix}. \tag{23}$$

the control vector B takes the form

$$B = \begin{bmatrix} 0 \\ -1/(\rho\,S)\,f_x(x) \end{bmatrix} \notin \mathsf{H}. \tag{24}$$

An output operator is $Cz = z_1$. Because of that one observes the first state variable, viz. the beam deflection at the point x and the fixed t. The LQ problem, with the finite time horizon T, lies in minimization of the quadratic cost functional

$$J(\mathbf{u}_0, w) = \|w\|^2_{L^2(0,T;C)} + \|y\|^2_{L^2(0,T;C)} \tag{25}$$

over trajectories of Eq. (22).

This functional is not standard for LQ problem with finite time horizon; it is sligthly modified. For standard LQ problem, the control theory gives the answer how to construct optimal regulator to ensure an asymptotic stability closed loop system and also is known form of optimal control. It will be the subject of the future research.

Next, the control is assumed in the form $f_t = \exp(i\omega_f t) \in L^2_{\mathrm{loc}}(0,\infty;C)$, C is the complex number set; note that f_t is not optimal. For such f_t the value of the functional is calculated. This value depends, among others, on actuators distribution. Minimizing the value of $J(\mathbf{u}_0, f_t)$ one finds the actuators distribution.

The criterion which allows to find the optimal location of actuator (not depending on time) is to minimize the following function (the value of the cost functional for particular control f_t)

$$J(x_a) = \int_0^T |\exp(i\omega_q t)|^2 \, \mathrm{d}t + \int_0^T |u_e(x,t)|^2 \, \mathrm{d}t = T + \int_0^T |u_e(x,t)|^2 \, \mathrm{d}t. \qquad (26)$$

As can be seen, the optimal location of actuator does not depend on the time t and it can be solved using numerical method. In explicit form $J(x_a)$ takes the form

$$J(x_a) = T + |u_e(x)|^2 \, \|1\|_{L^2} = T + \left| \sum_\nu |C^*_{e,\nu}| \left(-f_q X_\nu(x_q) + f_a \, \ell_a^2 \, \kappa_\nu(x_a)\right) X_\nu(x) \sin(\varphi_\nu) \right|^2. \qquad (27)$$

To find x_a, which minimize J, the necessary condition is

$$D_a J(x_a) = 2u_e(x) \left(\sum_\nu |C^*_{e,\nu}| \, f_a \, \ell_a^2 \cdot D_a \, \kappa_\nu(x_a) \, X_\nu(x) \sin(\varphi_\nu) \right) = 0, \qquad (28)$$

while the sufficient condition is

$$D_a^2 J(x_a) = 2u_e(x) \left(\sum_\nu |C^*_{e,\nu}| \, f_a \, \ell_a^2 \, D_a^2 \, \kappa_\nu(x_a) \cdot X_\nu(x) \sin(\varphi_\nu) \right) + 2 \left(\sum_\nu |C^*_{e,\nu}| \, f_a \, \ell_a^2 \, D_a \kappa_\nu(x_a) \cdot X_\nu(x) \sin(\varphi_\nu) \right)^2 > 0. \qquad (29)$$

If for some x_a, $D_a J(x_a) = 0$, and at this point the second derivative of J is greater than zero, then the J takes the local minimum. This case is correct from mathematical point of view and the necessary condition Eq. (28) is equivalent to the condition Eq. (15). Substituting these conditions in Eq. (29) one obtains Eq. (16). It means that the LQ attempt can be applied instead of the analytical one.

5. Summary and conclusions

Up to now it is proved for one mode that the optimal distribution of one actuator is in the sub area with the largest curvatures. The considerations are performed twofold: first, based on the exact mathematical principles and next, based on the LQ problem idea. As a result, both methods give the same mathematical expressions. So, the numerical calculations are also the same.

In this paper, the above mentioned methods are compared in the case of the multimodal beam vibrations. One actuator can be also applied to reduction multimodal case of vibration. Using formulated criterions, see (14), (26), it is possible to determinate the best location of actuator. Formula (15) or equivalent (28) provides the necessary condition. Formula (16) or equivalent (29) gives the sufficient condition. The same mathematical expressions are obtained . Thus, both methods can be applied alternatively. But it seems that the LQ method is more useful and must not to be assume the moment of time, however the pure mathematical method is more handy.

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