Original research paper

Propagation of uncertainty by Monte Carlo simulations in case of basic geodetic computations

Patrycja Wyszkowska

University of Warmia and Mazury in Olsztyn
Institute of Geodesy
1 Oczapowskiego St., 10-719 Olsztyn, Poland
e-mail: pwyszkowska@wp.pl

https://orcid.org/0000-0002-0080-7897

Abstract: The determination of the accuracy of functions of measured or adjusted values may be a problem in geodetic computations. The general law of covariance propagation or in case of the uncorrelated observations the propagation of variance (or the Gaussian formula) are commonly used for that purpose. That approach is theoretically justified for the linear functions. In case of the non-linear functions, the first-order Taylor series expansion is usually used but that solution is affected by the expansion error. The aim of the study is to determine the applicability of the general variance propagation law in case of the non-linear functions used in basic geodetic computations. The paper presents errors which are a result of negligence of the higher-order expressions and it determines the range of such simplification. The basis of that analysis is the comparison of the results obtained by the law of propagation of variance and the probabilistic approach, namely Monte Carlo simulations. Both methods are used to determine the accuracy of the following geodetic computations: the Cartesian coordinates of unknown point in the three-point resection problem, azimuths and distances of the Cartesian coordinates, height differences in the trigonometric and the geometric levelling. These simulations and the analysis of the results confirm the possibility of applying the general law of variance propagation in basic geodetic computations even if the functions are non-linear. The only condition is the accuracy of observations, which cannot be too low. Generally, this is not a problem with using present geodetic instruments.

Keywords: variance propagation law, Monte Carlo simulations, non-linear functions

1. Introduction

The accuracy of measured or adjusted values is very important in surveying. The accuracy of observations or their functions may be expressed as a standard deviation. The determination of that accuracy may be a problem in some geodetic computations. Generally, an application of the law of covariance propagation is the most common way for that purpose but for the uncorrelated variables may be used the Gaussian

formula (or a special case of the covariance propagation law – the law of variance propagation). However, applying the law of propagation of variance is valid for the linear functions. In case of the non-linear functions, it may be used the Taylor series expansion limited to the first expression term (the linear one). That approach is unfortunately affected by the expansion error. On the other hand, it is possible to apply the Monte Carlo simulations (or in other words, the statistical sampling method), which can be used also for the non-linear functions. This method is very useful tool to solve many other problems of statistical data analysis, which may be applied in surveying and geodesy, e.g., to determine the accuracy of the Hodges-Lehmann estimates (Duchnowski and Wiśniewski, 2014; 2017), the influence of the leptokurtosis of the error distribution on the accuracy of several estimates (Duchnowski and Wyszkowska, 2017), the values of the mean success rates (Hekimoglu and Berber, 2003) or the subjective breakdown points and the probabilities of breakdown (Xu, 2005; Wyszkowska and Duchnowski, 2017). The simulation methods are also used to the other engineering problems, e.g., to consider the propagation of uncertainty in case of the mass calibration, the comparison loss in the microwave power meter calibration or the Gauge block calibration (JCGM 101:2008, 2008).

The foundation of the Monte Carlo method was Buffon's needle problem by Georges Louis Leclerc in the eighteenth century (see, e.g., Ramaley, 1969). The Monte Carlo simulations are well known since 1940s, when Stanisław Ulam, Nicholas Metropolis and John von Neumann participated in the Manhattan project (e.g., Metropolis and Ulam, 1949; Eckhardt, 1987; Metropolis, 1987). They also worked on the project of the hydrogen bomb. That method was used in diffusion and absorption of neutrons, which was hard to consider in any analytical way. The development of computers allowed us to carry out more complex simulations applied in many branches of science (e.g. Warnock, 1987). Simulated data sets are generated randomly but with definite probability distribution. Important part of these simulations is statistical analysis of the results obtained with certain accuracy. The Monte Carlo simulations are used for too complex processes, when it is hard or impossible to predict results in a traditional way. The Monte Carlo method has several variants, e.g. the Crude Monte Carlo method (CMC) (see, e.g., Fishman, 1986), the Sequential Monte Carlo method (see, e.g., Del Moral et al., 2006), the Quantum Monte Carlo method (see, e.g., Wang, 2011).

This paper presents using the simplest form of the Monte Carlo method – the CMC method to determine the estimates of the standard deviations of the following functions: linear – a simulated levelling line, non-linear – a height difference in the trigonometric levelling, a distance and an azimuth of the Cartesian coordinates, the Cartesian coordinates of unknown point in the three-point resection problem. It is assumed that the normal distributions are the stochastic model of measurement errors, which is well grounded from the theoretical point of view. The estimates of the standard deviation from the Monte Carlo simulations are compared with the standard deviations obtained by applying the law of propagation of variance. All computations are carried out in MathCad 15.0. Thus, the main objective of the paper is to determine

the applicability of the CMC method and the variance propagation law in case of the non-linear functions of basic geodetic computations.

2. Propagation of variance

Let us consider a vector of random variables $\mathbf{X} = [X_1\ X_2\ \cdots\ X_n]^T$ and a vector of their functions $\mathbf{Y} = [Y_1\ Y_2\ \cdots\ Y_m]^T$, here $\mathbf{Y} = \mathbf{F(X)}$ is a transformation of the random vector $\mathbf{X}$. Let each Y_i be linear function of $\mathbf{X}$, hence all Y_i are differentiable, and the respective derivatives create the matrix $\mathbf{D}$. If the covariance matrix $\mathbf{C_X}$ is known, then we can write the well-known form of the law of propagation of covariance (see, e.g., Mikhail and Ackermann, 1976)

$$\mathbf{C_Y} = \mathbf{D C_X D}^T \tag{1}$$

If Y_i are not linear, one can apply the Taylor series expansion, which is usually limited to the first-order terms in order to obtain the linear approximation of Y_i (note, that there are some special non-linear functions for which one can derive the direct expression for variance of Y_i). Considering both vectors $\mathbf{Y}$ and $\mathbf{X}$, the Taylor series expansion requires computation of the Jacobian matrix $\mathbf{J}$, which is based on the approximate values $\mathbf{X}_0$. Taking $\mathbf{D} = \mathbf{J}$, one can apply the formula of Equation (1) to compute the covariance matrix $\mathbf{C_Y}$. Note, that such an approximation (linearization of Y_i) causes that the computed variances of Y_i are biased. Such a bias depends on the nature of linearized functions and in some cases, it might be unacceptable high. For example, function $\log x$, the natural logarithm, can be approximated with the expansion $x + 1$ only for small x. Thus, the presented linear approximation can be assumed as good enough only in a close neighbourhood of $\mathbf{X}_0$.

It is also possible to apply the variance propagation formula using the second-order Taylor series expansion. Such an approach requires a knowledge of the higher-order central moments. However, when the variables X_i are normally distributed, then the odd moments μ_k (k is odd) are equal to zero and we can also use the substitution $\mu_4 = 3\sigma^4$. Additionally, when Y is a function of the uncorrelated variables, then the variance σ_Y^2 can be assessed by the following formula (see, e.g., Anderson and Mattson, 2012).

$$\sigma_Y^2 \approx \sum_{i=1}^{n}\left(\frac{\partial Y}{\partial X_i}\right)^2 \sigma_{X_i}^2 + \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\left(\frac{\partial^2 Y}{\partial X_i \partial X_j}\right)^2 \sigma_{X_i}^2 \sigma_{X_j}^2 \tag{2}$$

Another solution of the problem how to propagate the covariance in case of the non-linear function is to apply probabilistic approaches to the uncertainty propagation (see, e.g., Lee and Chen, 2009). One of the most popular and simplest methods is an application of the Monte Carlo simulations. In general, we should assume the stochastic model of $\mathbf{X}$, then simulate realizations of such a random vector, by applying

F(X) we can obtain the set of realizations of **Y**, which is the basis for computing empirical variances and covariances.

The problem can also be solved by applying the delta method, which is classical technique of approximating the moments of a function of one or more random variables (see, e.g., Oehlert, 1992). That method is usually based on the polynomial approximation which is often a truncated Taylor series expansion limited to the first-order terms and the sample moments. If one assumes that the variables are approximately normally distributed and their standard deviations are small, then in fact the delta method leads to Equation (1) (see, e.g. Kass et al., 2014).

3. Numerical examples

The Crude Monte Carlo method applied in this section is based on the simulations of observations in a very basic way. We simulate the observation errors by generating the Gaussian random numbers with the procedure *rnorm* of MathCad 15.0. Such simulated errors are then added to the assumed "true" values of observations, respectively. The obtained observation sets are transformed to the sets of the function values which are the basis for computing the CMC estimators of the function standard deviations. Similar simulations can also be carried out by applying other software, e.g. GoldSim, MatLab, Oracle Crystal Ball, @Risk, XLSTAT, MonteCarlito, ModelRisk. It is worth noting that quality of the random number generator can influence the final results; however, such a problem is out of our interest within the paper.

In the next subsections we will apply two formulas for the variance propagation law (VPL), namely Equation (1) or Equation (2). For the sake clarity, such approaches will be denoted as VPL (I) and VPL (II), respectively.

3.1. Simulated levelling line

Let us consider a simulated levelling line, which consists of four points but only a height of the point A, H_A, is known. There are 3 observations (the height differences) between points A–1, h_1, 1–2, h_2, and 2–3, h_3. The following values of variances of the height differences are assumed $V(h_1) = 1$ [mm^2], $V(h_2) = 2$ [mm^2], $V(h_3) = 3$ [mm^2] additionally we assume that all observations are uncorrelated. If $H_A = 0$ [m], the height differences are equal to the respective heights of points 1, 2 and 3, hence

$$H_1 = h_1 \tag{3}$$

$$H_2 = h_1 + h_2 \tag{4}$$

$$H_3 = h_1 + h_2 + h_3 \tag{5}$$

Note that such functions are linear, so an application of the law of variance propagation must give the exact results. On the other hand, it is also possible to use the CMC method and compare the results of both approaches. Let the errors of the height differences in the levelling, ε_{h_1}, ε_{h_2}, ε_{h_3}, be normally distributed with the expected values equal to zero and known standard deviations, σ_{h_1}, σ_{h_2}, σ_{h_3}, i.e., $\varepsilon_{h_1} \sim N[0, \sigma_{h_1}]$, $\varepsilon_{h_2} \sim N[0, \sigma_{h_2}]$, $\varepsilon_{h_3} \sim N[0, \sigma_{h_3}]$. The standard deviations of the point heights, σ_{H_1}, σ_{H_2}, σ_{H_3} (from the variance propagation law) and the CMC estimates, $\hat{\sigma}_{H_1}$, $\hat{\sigma}_{H_2}$, $\hat{\sigma}_{H_3}$ (from the Monte Carlo method for chosen number of simulations n) are given in Table 1.

Table 1. Standard deviations of heights σ_{H_1}, σ_{H_2}, σ_{H_3} and CMC estimates $\hat{\sigma}_{H_1}$, $\hat{\sigma}_{H_2}$, $\hat{\sigma}_{H_3}$ for different number of simulations n

Crude Monte Carlo method				Variance propagation law		
n	$\hat{\sigma}_{H_1}$ [mm]	$\hat{\sigma}_{H_2}$ [mm]	$\hat{\sigma}_{H_3}$ [mm]	σ_{H_1} [mm]	σ_{H_2} [mm]	σ_{H_3} [mm]
1000	0.97542	1.69348	2.41906			
10000	1.00419	1.73926	2.46086			
25000	1.00103	1.74134	2.44893	1.000	1.732	2.449
50000	0.99634	1.73236	2.45045			
100000	0.99578	1.73513	2.44849			

Thanks to the values in Table 1, it is possible to determine the accuracy of the estimate of the standard deviation for the linear function. The values of the CMC estimates are presented with 5 decimal places on the purpose to notice the differences between obtained results. In case of the smallest number of simulations $n = 1000$ there are the biggest differences (up to 3%) between σ_{H_1}, σ_{H_2}, σ_{H_3} and $\hat{\sigma}_{H_1}$, $\hat{\sigma}_{H_2}$, $\hat{\sigma}_{H_3}$. When the number of simulations increases, the differences are smaller than 1% which can be assumed as the accuracy of the CMC estimates here. The values of $\hat{\sigma}_{H_1}$, $\hat{\sigma}_{H_2}$, $\hat{\sigma}_{H_3}$ are close to σ_{H_1}, σ_{H_2}, σ_{H_3} for $n \geq 25000$. In fact, Table 1 presents results of a single experiment, for other experiments we can obtain slightly different results. Because the simulation time is not very long, we decided to assume the biggest considered number of simulations, thus in the all next numerical experiments $n = 100000$.

3.2. Trigonometric levelling

The next example concerns a height difference in the trigonometric levelling, Δh, which is a non-linear function of a vertical angle, α, and a slope distance, d'. That test is to determine differences between the standard deviations of the height difference in the trigonometric levelling, $\sigma_{\Delta h}$, and the CMC estimates, $\hat{\sigma}_{\Delta h}$, when the standard deviations of a vertical angle, σ_α, and a slope distance, $\sigma_{d'}$, are doubled in the

subsequent variants. The assumed expected values of the vertical angle $E(\alpha) = 10$ [ᵍ] and the slope distance $E(d') = 100$ [m]; these variables are uncorrelated. The errors of the vertical angle, ε_α, and the slope distance, $\varepsilon_{d'}$, have normal distributions, i.e., $\varepsilon_\alpha \sim N[0,\sigma_\alpha]$ and $\varepsilon_{d'} \sim N[0,\sigma_{d'}]$. For the first variant: $\sigma_\alpha = 0.005$ [ᵍ], $\sigma_{d'} = 0.005$ [m]. For the next variants, the standard deviations of the vertical angle, σ_α, and the slope distance, $\sigma_{d'}$, are doubled, namely $\sigma_\alpha \to 2\sigma_\alpha$ and $\sigma_{d'} \to 2\sigma_{d'}$. The computations are made for 15 variants. Figure 1 presents the standard deviations of the height difference in the trigonometric levelling, $\sigma_{\Delta h}$, from the variance propagation law and the estimates of the standard deviations of the height difference in the trigonometric levelling, $\hat{\sigma}_{\Delta h}$, based on the Crude Monte Carlo method.
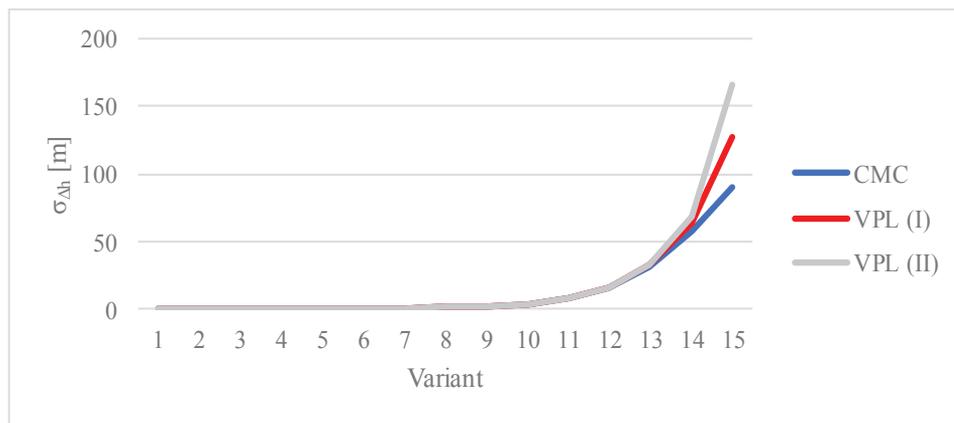

Fig. 1. Standard deviation of height difference in trigonometric levelling $\sigma_{\Delta h}$ computed by VPL (VPL (I) concerns the first-order Taylor series expansion and VPL (II) – the second-order Taylor series expansion) and CMC estimate $\hat{\sigma}_{\Delta h}$

First, let us analyse the results of VPL (I). The values of $\sigma_{\Delta h}$ and $\hat{\sigma}_{\Delta h}$ in Figure 1 are similar for the first twelve variants, their differences are up to 1%. Thus, they are smaller (or similar) than the accuracy of the estimate of the standard deviation computed by the Monte Carlo method, which is set at 1%. More significant difference (about 3%) between $\sigma_{\Delta h}$ and $\hat{\sigma}_{\Delta h}$ occurs in variant 13, when $\sigma_\alpha \approx 20.5$ [ᵍ] and $\sigma_{d'} \approx 20.5$ [m]. The differences are even bigger in the next variants. For variants 13–15 the differences between $\sigma_{\Delta h}$ and $\hat{\sigma}_{\Delta h}$ are larger than the accuracy of the estimate (1%), so the values of the standard deviations computed by the propagation of variance are not reliable. Now, let us consider the application of VPL (II). For the first ten variants the difference between VPL (I) and VPL (II) is lower than 0.1%. For the next variants it grows up to 30%. Thus generally, for the observations of high accuracy the difference is not significant, and for the observations of low accuracy the results obtained by applying the Equation (2) are even worse.

3.3. Distance

Another example is a distance, d, calculated of the Cartesian coordinates of the points A and B. The assumed expected values of the coordinates of the points A and B are as follow: $E(X_A) = 0$ [m], $E(Y_A) = 0$ [m] and $E(X_B) = 100$ [m], $E(Y_B) = 100$ [m]. Let us consider the case of the uncorrelated coordinates. The distributions of the errors of these coordinates, ε_{X_A}, ε_{Y_A}, ε_{X_B}, ε_{Y_B} are normal, i.e., $\varepsilon_{X_A} \sim N[0,\sigma_{X_A}]$, $\varepsilon_{Y_A} \sim N[0,\sigma_{Y_A}]$ and $\varepsilon_{X_B} \sim N[0,\sigma_{X_B}]$, $\varepsilon_{Y_B} \sim N[0,\sigma_{Y_B}]$. The assumed values for first variant: $\sigma_{X_A} = 0.01$ [m], $\sigma_{Y_A} = 0.01$ [m], $\sigma_{X_B} = 0.01$ [m], and $\sigma_{Y_B} = 0.01$ [m]; for the next variants, the standard deviations of the coordinates of the point B, σ_{X_B} and σ_{Y_B}, are doubled $\sigma_{X_B} \to 2\sigma_{X_B}$ and $\sigma_{Y_B} \to 2\sigma_{Y_B}$. The computations are made for 15 variants. In case of the correlated coordinates, we use the following formulas to simulate the coordinates of the points A, X_A, Y_A and B, X_B, Y_B

$$X_A = E(X_A) + t_A + u_{X_A} \tag{6}$$

$$Y_A = E(Y_A) + t_A + u_{Y_A} \tag{7}$$

$$X_B = E(X_B) + t_B + u_{X_B} \tag{8}$$

$$Y_B = E(Y_B) + t_B + u_{Y_B} \tag{9}$$

where: t_A – common variable for the coordinates of the point A; u_{X_A}, u_{Y_A} – different variables for the coordinates of the point A; t_B – common variable for the coordinates of the point B; u_{X_B}, u_{Y_B} – different variables for the coordinates of the point B. These variables have normal distributions, i.e., $t_A \sim N[0,\sigma_{t_A}]$, $u_{X_A} \sim N[0,\sigma_{u_{X_A}}]$, $u_{Y_A} \sim N[0,\sigma_{u_{Y_A}}]$ and $t_B \sim N[0,\sigma_{t_B}]$, $u_{X_B} \sim N[0,\sigma_{u_{X_B}}]$, $u_{Y_B} \sim N[0,\sigma_{u_{Y_B}}]$. The variances and the covariances of the coordinates of the points A and B are the functions of the variances of the common and different variables for the coordinates of the points A and B, for example

$$V(X_A) = V(t_A) + V(u_{X_A}) \tag{10}$$

$$cov(X_A,Y_A) = cov\ (Y_A,X_A) = V(t_A) \tag{11}$$

The values of the first variant: $V(t_A) = cov(X_A,Y_A) = cov(Y_A,X_A) = 0.00005$ [m^2], $V(u_{X_A}) = V(u_{Y_A}) = 0.00005$ [m^2], $V(t_B) = cov(X_B,Y_B) = cov(Y_B,X_B) = 0.00005$ [m^2], $V(u_{X_B}) = V(u_{Y_B}) = 0.00005$ [m^2]. For the subsequent variants, the variances of the variables of the point B, $V(t_B)$, $V(u_{X_B})$, $V(u_{Y_B})$, increases four times, i.e. $V(t_B) \to 4V(t_B)$, $V(u_{X_B}) \to 4V(u_{X_B})$, $V(u_{Y_B}) \to 4V(u_{Y_B})$. The coordinates of the points A and B are correlated in all variants – the correlation coefficient of the coordinates of the point A $\rho_{X_A,Y_A} = 0.5$, the same as for the point B $\rho_{X_B,Y_B} = 0.5$. Note, that application of the second-order Taylor series expansion and VPL of Equation (2) is not possible for the correlated variables.

Figure 2 presents the standard deviations of the distance, σ_d, obtained from the variance propagation law and the estimates of the standard deviations of the distance, $\hat{\sigma}_d$, from the CMC method for the uncorrelated coordinates, and Figure 3 – for the correlated coordinates ($\rho_{X_A,Y_A} = 0.5$, $\rho_{X_B,Y_B} = 0.5$).
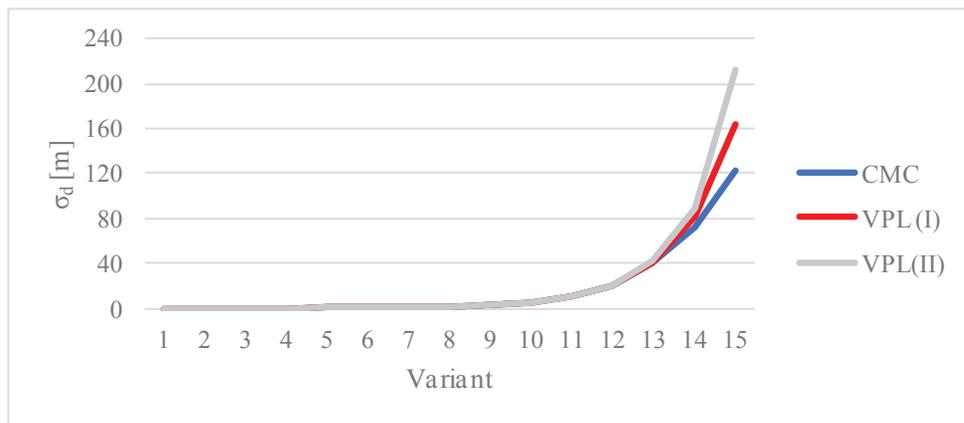

Fig. 2. Standard deviation of distance σ_d computed by VPL and CMC estimate $\hat{\sigma}_d$ for uncorrelated coordinates
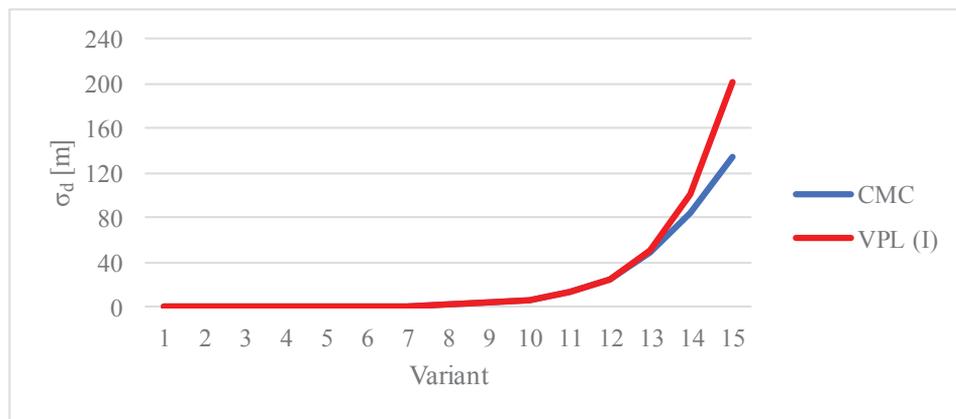

Fig. 3. Standard deviation of distance σ_d computed by VPL (I) and CMC estimate $\hat{\sigma}_d$ for correlated coordinates ($\rho_{X_A,Y_A} = 0.5$, $\rho_{X_B,Y_B} = 0.5$)

Firstly, let us consider the CMC method and VPL (I). Figures 2 and 3 show that the differences between the first twelve values of σ_d and $\hat{\sigma}_d$ are smaller than 1%, which is similar relation to that in Figure 1. For the rest variants, the differences between σ_d and $\hat{\sigma}_d$ are again larger than obtained accuracy of the estimate of the standard deviation computed by the Monte Carlo method (1%). It happens when

$\sigma_{X_A} = \sigma_{Y_A} = 0.01$ [m] and $\sigma_{X_B} = \sigma_{Y_B} \geq 41$ [m] (the values from variant 13). Once more, for these variants the values of σ_d are not reliable. Furthermore, the differences between σ_d and $\hat{\sigma}_d$ for the variants 13–15 are larger in case of the correlated coordinates. As for the application of VPL (II), the difference between VPL (I) and VPL (II) in Figure 2 is lower than 0.1% for the first ten variants and it increases to 30% in the last variant, which is analogous to the previous example.

3.4. Azimuth

Determination of an azimuth, A, computed of the Cartesian coordinates of the points A and B is the next numerical example. The assumptions are the same as for the distance in the cases of the uncorrelated and the correlated coordinates of the points A and B. This time the computations are made for 18 variants in both cases. Figure 4 presents the standard deviations of the azimuth, σ_A, computed by the law of propagation of variance and the estimates of the standard deviations of the azimuth, $\hat{\sigma}_A$, by the CMC method, when the coordinates are uncorrelated; while the values for the correlated coordinates ($\rho_{X_A, Y_A} = 0.5$, $\rho_{X_B, Y_B} = 0.5$) are shown in Figure 5.
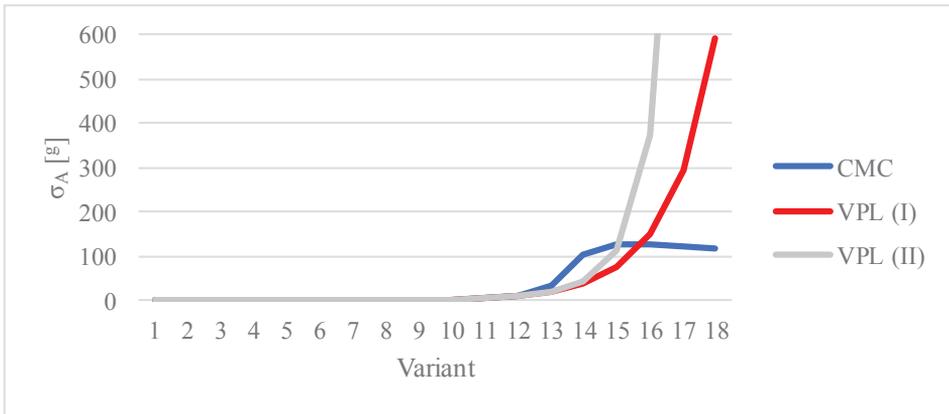

Fig. 4. Standard deviation of azimuth σ_A computed by VPL and CMC estimate $\hat{\sigma}_A$
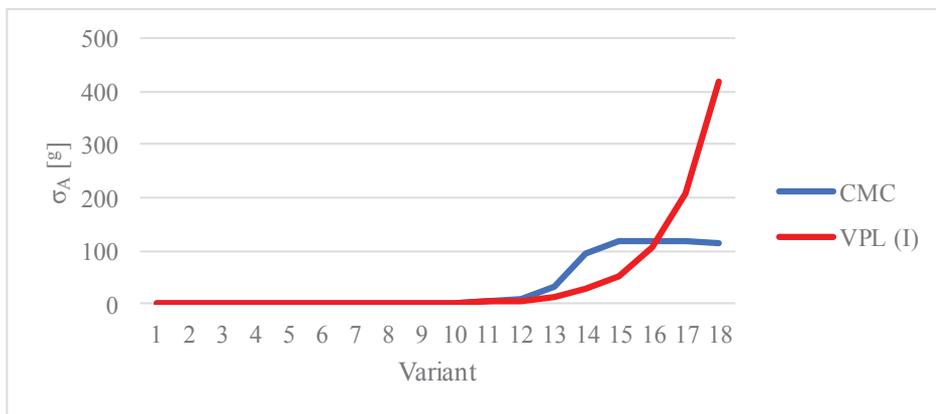for uncorrelated coordinates

Fig. 5. Standard deviation of azimuth σ_A computed by VPL (I) and CMC estimate $\hat{\sigma}_A$
for correlated coordinates ($\rho_{X_A,Y_A} = 0.5$, $\rho_{X_B,Y_B} = 0.5$)

First, let us compare results obtained by VPL (I) and the CMC method. Figures 4
and 5 present increasing σ_A for all variants, but in case of $\hat{\sigma}_A$ only for variants 1–15.
The values of $\hat{\sigma}_A$ are very similar for the rest of the variants. This time the differences
between σ_A and $\hat{\sigma}_A$ are smaller than 1% (the assumed accuracy of the estimate of the
standard deviation computed by the Monte Carlo method) for the first eleven variants in
Figure 4 ($\sigma_{X_B} = \sigma_{Y_B} \leq 10$ [m]) or the first ten variants in Figure 5 ($\sigma_{X_B} = \sigma_{Y_B} \leq 5$ [m]).
Moreover, for variants 12–15 in Figure 4 and variants 11–16 in Figure 5 σ_A are too low
in relation to $\hat{\sigma}_A$; for the next variants, the effect is opposite. Besides, the differences
between σ_A and $\hat{\sigma}_A$ are bigger for variants 12–15 for the correlated coordinates and for
variants 16–18 for the uncorrelated coordinates. The results obtained from VPL (II) in
Figure 4 are similar to the previous ones. For the ten first variants the difference between
VPL (I) and VPL (II) is negligible. For the next variants, results of VPL (II) grow more
rapidly than results of VPL (I), thus they become more distant from the results of the
CMC method.

Figure 6 presents the histograms of the azimuths computed by the Crude Monte
Carlo method ($n = 100000$) for the uncorrelated coordinates in case of three variants
of the values of the standard deviations of the coordinates of the point B (the
other assumptions are the same as earlier): variant 1 $\sigma_{X_B} = \sigma_{Y_B} = 200$ [m], variant
2 $\sigma_{X_B} = \sigma_{Y_B} = 500$ [m], variant 3 $\sigma_{X_B} = \sigma_{Y_B} = 2000$ [m]. Such values of σ_{X_B} and σ_{Y_B}
are irrational from the practical point of view; however, they help to analyse how the
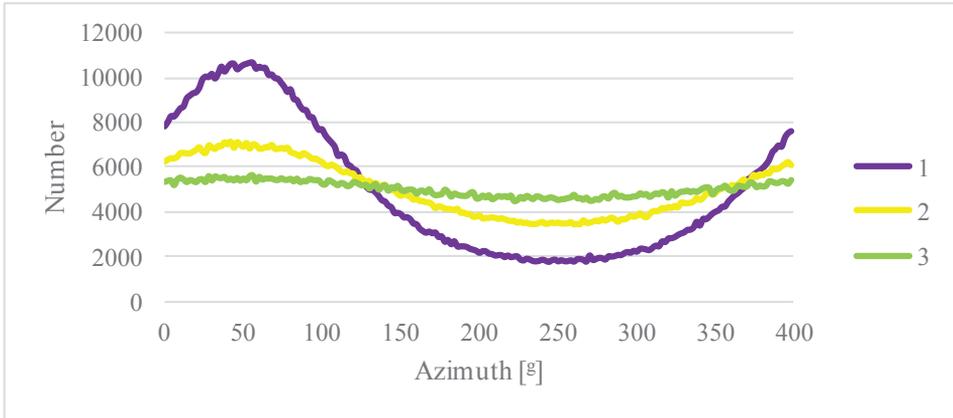obtained histograms change with decreasing accuracy of the coordinates.

Fig. 6. Histograms of azimuths computed by Crude Monte Carlo method

Figure 6 shows that the histograms of the azimuths tend to the uniform distribution. Note, that the CMC estimate of the standard deviation, $\hat{\sigma}_A$, (Figures 4 and 5) tends to the value of 115.47 [^{g}] – which in fact is the standard deviation in the uniform distribution for such an interval. That fact also confirms the correctness of the results, which are obtained by the Monte Carlo simulations.

3.5. Three-point resection problem

The last example is computing the Cartesian coordinates X_D, Y_D of a point D by the resection solution presented by (Ligas, 2013). The assumed coordinates of the known points A, B and C are as follow: $X_A = 0$ [m], $Y_A = 0$ [m], $X_B = -50$ [m], $Y_B = 50$ [m] and $X_C = 0$ [m], $Y_C = 100$ [m]. To simplify the computations, we assume that the angles $\alpha = BDA$ and $\beta = CDB$ are equal to each other and close to 50 [^{g}], so the point D lies near the danger circle. Additionally, let the errors, ε_α, ε_β, be normally distributed, i.e., $\varepsilon_\alpha \sim N[0,\sigma_\alpha]$ and $\varepsilon_\beta \sim N[0,\sigma_\beta]$, where $\sigma_\alpha = \sigma_\beta = 0.001$ [^{g}]. We should realize that in surveying engineering, one should usually avoid the situation where the resection point lies very close to the danger circle. However, such a situation is plausible in navigation (especially when we have only three reference points). In such a case the assessment of the resection accuracy is essential and it may strongly influence the next computations, for example the prediction of the vehicle position. Therefore, let us assess such an accuracy by VPL (I) or the CMC method for $n = 100000$ simulations. The results are listed in Table 2.

Table 2. Standard deviations of coordinates σ_{X_D}, σ_{Y_D} and CMC estimates $\hat{\sigma}_{X_D}$, $\hat{\sigma}_{Y_D}$

$\alpha = \beta$ [g]	Crude Monte Carlo method		Variance propagation law	
	$\hat{\sigma}_{\hat{X}_D}$ [m]	$\hat{\sigma}_{\hat{Y}_D}$ [m]	σ_{X_D} [m]	σ_{Y_D} [m]
51	$1.078 \cdot 10^{-3}$	0.069	$1.077 \cdot 10^{-3}$	0.066
50.5	$1.128 \cdot 10^{-3}$	0.139	$1.094 \cdot 10^{-3}$	0.139
50.05	0.028	1.411	$1.109 \cdot 10^{-3}$	1.412
50.005	2.819	13.728	$1.111 \cdot 10^{-3}$	14.140
50.0005	34.955	35.286	$1.111 \cdot 10^{-3}$	141.419

The results of the VPL (I) and the CMC method are almost the same for the first variant. The difference between the methods increases significantly when the point D approaches the danger circle. One can note that the standard deviation σ_{X_D} remains almost the same while σ_{Y_D} increases rapidly when applying the variance propagation law. On the other hand, $\hat{\sigma}_{\hat{X}_D}$ and $\hat{\sigma}_{\hat{Y}_D}$ grow with the different rapidity but finally they both stabilize close to the value 35.35 [m]. Such a value is very close to the theoretical value of the standard deviation of the coordinate *X* (or *Y*), when one considers the set of points which are spread evenly on the danger circle. On the one hand, we should realize the fact results from the numerical problems of the resection solution (high risk of the erroneous solution for the resection point lying very close the danger circle), but on the other, it is a reason for which the CMC method should be regarded as more reliable in such a context.

To apply VPL (II) one should compute the values of the second-order derivatives. In the present case the first-order derivatives are so complex that the built-in procedures of MathCad 15.0 cannot deal with such a problem. One should realize that the solution surely exists; however, it is hard to get it. Thus, in this context the application of VPL (II) is just impossible. Considering results presented in the previous subsections, the question is whether such an effort is justified.

4. Conclusions

The Monte Carlo simulations are one of the basic methods of using random numbers which is applied in different branches of science. The results presented in the paper show one of the possible applications of the Monte Carlo method in surveying. The Monte Carlo simulations are good alternative for assessing accuracy by the variance propagation law. The main aim of the paper was to examine the applicability of the variance propagation law for the non-linear functions of basic geodetic computations. It is obvious, that the law of propagation of variance always gives exact results in case of the linear functions, which also gives us a possibility to assess the accuracy

of the estimate of the standard deviation computed by the Monte Carlo method (here 1%).

The results obtained by the application of the CMC method indicate that the variance propagation law for the non-linear functions does not yield good results in some special cases, for example for the low accuracy of the observations or numerical problems with computing correct solutions. On the other hand, the variance propagation law can be applied in the practice of geodetic calculations because the geodetic measurements have usually relatively small standard deviations. Such a high accuracy of the geodetic measurements means that the linear approximation by the Taylor series expansion is good enough in case of the basic surveying computations, and one can assume that $\mathbf{D} = \mathbf{J}$, hence also apply Equation (1) for the non-linear functions. On the other side, application of the second-order Taylor series expansion should generally yield better results. However, the results presented here show that in case of high accuracy measurements the difference between VPL (II) and VPL (I) is negligible. For the low accuracy of measurements, the quadratic approximation is even worse than the linear one. Sometimes the application of VPL (II) can be hard to carry out due to the complex expression of the second derivatives. Considering the fact that VPL (II) requires bigger effort and the results obtained in case of simple geodetic computations are not promising, VPL (I) seems to be a better choice. The other limitations of the method in question as well as pros and cons of application of VPL (II) can be found in (Anderson and Mattson, 2012).

The Monte Carlo method proves that it can be applied to check the traditional computations. Generally, the probabilistic methods for the uncertainty propagation, including the Monte Carlo simulations, should be applied to estimate the variance of the function of the measurements if the function itself is highly non-linear and/or the accuracy of measurements is very low.

Acknowledgments

References

Anderson, T.V. and Mattson, C.A. (2012). Propagating skewness and kurtosis through engineering models for low-cost, meaningful, nondeterministic design. *J Mech Design*, 134(10). DOI: 10.1115/1.4007389.

Del Moral, P., Doucet, A. and Jasra, A. (2006). Sequential Monte Carlo samplers. *J R Stat Soc B*, 68(3), 411-436. DOI: 10.1111/j.1467-9868.2006.00553.x.

Duchnowski, R. and Wiśniewski, Z. (2014). Comparison of two unconventional methods of estimation applied to determine network point displacement. *Surv Rev,* 46(339), 401-405. DOI: 10.1179/1752270614Y.0000000127.

Duchnowski, R. and Wiśniewski, Z. (2017). Accuracy of the Hodges-Lehmann estimates computed by applying Monte Carlo simulations. *Acta Geod Geophys*. DOI: 10.1007/s40328-016-0186-0.

Duchnowski, R. and Wyszkowska, P. (2017). Leptokurtosis of error distribution and its influence on estimation accuracy. The case of three estimates applied in adjustment of geodetic measurements. In: Baltic Geodetic Congress, June 22-25 2017, IEEE. DOI: 10.1109/BGC.Geomatics.2017.32.

Eckhardt, R. (1987). Stan Ulam, John von Neumann and the Monte Carlo method. *Los Alamos Sci*, 15, 131-137.

Fishman, G. S. (1986). A Monte Carlo sampling plan for estimating network reliability. *Oper Res*, 34(4), 581-594. DOI: 10.1287/opre.34.4.581.

Hekimoglu, S. and Berber, M. (2003). Effectiveness of robust methods in heterogeneous linear models. *J Geod,* 76(11), 706-713. DOI: 10.1007/s00190-002-0289-y.

JCGM 101:2008. (2008). *Evaluation of measurement data – Supplement 1 to the "Guide to the expression of uncertainty in measurement" – Propagation of distributions using a Monte Carlo method*. Geneva: Joint Committee for Guides in Metrology, International Organization for Standardization.

Kass, R.E., Eden, U. and Brown, E. (2014). *Analysis of Neural Data*. New York: Springer. DOI: 10.1007/978-1-4614-9602-1.

Lee, S. H. and Chen, W. (2009). A comparative study of uncertainty propagation methods for black-box-type problems. *Struct Multidiscip O*, 37(3), 239-253. DOI: 10.1007/s00158-008-0234-7.

Ligas, M. (2013). Simple solution to the three point resection problem. *J Surv Eng*, 139(3), 120-125. DOI: 10.1061/(ASCE)SU.1943-5428.0000104.

Mikhail, E.M. and Ackermann, F.E. (1976). *Observations and least squares*. New York, NY: Harper and Row.

Metropolis, N. (1987). The beginning of the Monte Carlo method. *Los Alamos Sci*, 15, 125-130.

Metropolis, N. and Ulam, S. (1949). The Monte Carlo method. *J Am Stat Assoc*, 44(247), 335-341. DOI: 10.1080/01621459.1949.10483310.

Oehlert, G.W. (1992). A note on the delta method. *Am Stat*, 46(1), 27-29. DOI: 10.1080/00031305.1992.10475842.

Ramaley, J. F. (1969). Buffon's noodle problem. *Am Math Mon*, 76(8), 916-918. DOI: 10.2307/2317945.

Xu, P. (2005). Sign-constrained robust least squares, subjective breakdown point and the effect of weights of observations on robustness. *J Geod,* 79(1), 146-159. DOI: 10.1007/s00190-005-0454-1.

Wang, Y. (2011). Quantum Monte Carlo simulation. *Ann Appl Stat* 5(2A), 669-683. DOI: 10.1214/10-AOAS406.

Warnock, T. (1987). Random-number generators. *Los Alamos Sci*, 15, 137-141.

Wyszkowska, P. and Duchnowski, R. (2017). Subjective breakdown points of R-estimators applied in deformation analysis. In: Environmental Engineering 10th International Conference, April 27-28 2017, (pp. 1-6). DOI: 10.3846/enviro.2017.250.