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$M/\vec{G}/n/0$ Erlang queueing system with heterogeneous servers and non-homogeneous customers

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Abstract. In the present paper, we investigate a multi-server queueing system with heterogeneous servers, unlimited memory space, and non-homogeneous customers. The arriving customers appear according to a stationary Poisson process. Service time distribution functions may be different for every server. Customers are additionally characterized by some random volume. On every server, the service time of the customer depends on their volume. The number of customers distribution function is obtained in the classical model of the system. In the model with non-homogeneous customers, the stationary total volume distribution function is determined in the term of Laplace–Stieltjes transform. The stationary first and second moments of a total customers volume are calculated. An analysis of some special cases of the model and some numerical examples are also included.

Key words: multi-server queueing systems, queueing systems with non-homogeneous customers, queueing systems with heterogeneous servers, total volume distribution, Laplace–Stieltjes transform.

1. Introduction

The first analysis of the classical $M/M/n/0$ queueing system was made by Erlang in [3] and [4]. In this model, the author assumes that the investigated system is composed of n identical servers. More precisely, service time is exponentially distributed with the same parameter μ for every server. In addition, customers arrive at the system with Poisson entrance flow, which means that the time intervals between successive customers' arrivals are exponentially distributed with the same parameter a. There is no queue in this system, so an arriving customer is lost if they find that the system is full (i.e. every server is busy). The main characteristic obtained during the analysis of this system is the number of customers distributions in the stationary mode p_k. It can be easily proven [11] that the obtained formulae also remain valid for a more universal $M/G/n/0$ queueing system with identical servers if we replace the constant μ by $1/\beta_1$, where β_1 is the mean value of service time. On the other hand, queueing systems with heterogeneous servers are rarely analyzed. The first investigations of $M/\vec{M}/n/0$ queueing systems with heterogeneous servers appeared in [6, 7], and [8]. Interesting analyses of $M/\vec{M}/n/0$ queueing system with heterogeneous servers may also be found in [10] and [12]. Meanwhile, some investigations of $M/\vec{G}/n/0$ systems with heterogeneous servers may be found in [5] and [9].

If we additionally assume that arriving customers have some random volume, then we obtain a very interesting new area of research that is connected with the concept of total volume, which is the sum of the volumes of all of the customers present in the system. These queueing systems are called queueing systems with non-homogeneous customers and the main purpose, in this case, is to obtain the total volume characteristics or loss characteristics, at least in stationary mode. The results connected with $M/G/n/0$ queueing system with identical servers and non-homogeneous customers are presented in [11].

The present paper aims to analyze a $M/\vec{G}/n/0$ queueing system with heterogeneous servers and non-homogeneous customers. The rest of this paper is organized as follows. Section 2 contains the analysis of the classical $M/\vec{G}/n/0$ queueing system with heterogeneous servers. The main purpose of this part is to obtain the number of customers distribution functions in the stationary mode. Section 3 contains an analysis of a non-classical $M/\vec{G}/n/0$ queueing system with non-homogeneous customers. In this section, we obtain the total volume distribution characteristics in stationary mode. Section 4 will investigate some special cases of the model analyzed in Section 3 and it will show that the character of the service time and customer volume dependency has an influence on the total volume characteristics. Finally, Section 5 contains some concluding remarks.

2. The classical model analysis

This section will investigate the modification of the classical $M/G/n/0$ queueing system in which the service time characteristics may differ for each server. The customers choose free servers in a random way, which means that an arriving customer will be serviced by one of the l free servers with probability $1/l$. Our purpose is to obtain the number of customers distribution functions in the stationary mode. To do this, we will use the generalized method of the auxiliary variable [2]. In the analyzed model, the number of customers is limited by value n and all present customers in the system are being serviced (i.e. there is no queue).

We denote the parameter of an entrance Poisson flow as a; the service time distribution function for j-th server as $B_j(t)$;

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and its first moment as β_j, $j = \overline{1, n}$. Let $\eta(t)$ be the number of customers present in the system at time instant t. In addition, we assume that there exist service time densities $b_j(t)$. This assumption is technical because it allows us to use, during system behavior analysis, the service intensity function that is defined for j-th server by the formula $\mu_j(x) = \frac{b_j(x)}{1 - B_j(x)}$, although we may obtain the same results without this assumption. Let $A(t)$ be the set of busy servers at time instant t. We denote the length of the time interval from the beginning of the service of the customer (that is still serviced by j-th server at time t) to time t as $\xi_j^*(t)$, $j \in A(t)$.

It is easy to prove that process

$$\big(\eta(t), A(t), \xi_j^*(t), j \in A(t)\big) \tag{1}$$

is a Markovian process that describes the behavior of the system. In the case of empty system $\big(\eta(t) = 0\big)$, the process reduces to $\eta(t)$.

Now we introduce the following functions:

$$P_k(t) = P\{\eta(t) = k\}, k = \overline{0, n}; \tag{2}$$

$$\begin{aligned} &P_k^{\{i_1, i_2, \dots, i_k\}}(t) = P\{\eta(t) = k, A(t) = \{i_1, i_2, \dots, i_k\}\}, \\ &k = \overline{1, n}; \end{aligned} \tag{3}$$

$$\begin{aligned} &G_k^{\{i_1, i_2, \dots, i_k\}}(x_1, \dots, x_k, t)dx_1 \dots dx_k = \\ &= P\{\eta(t) = k, A(t) = \{i_1, i_2, \dots, i_k\}, \\ &\xi_j^*(t) \in [x_j, x_j + dx_j),\ j \in A(t)\},\ k = \overline{1, n}. \end{aligned} \tag{4}$$

If $\eta(t) = n$, then the function described in (4) may be denoted simply as $G_n(x_1, \dots, x_n, t)$.

In the stationary mode (which exists if $a\beta_j < \infty$, $j = \overline{1, n}$) we can introduce analogies that are independent of the time variable t:

$$p_k = \lim_{t \to \infty} P_k(t),\ \ k = \overline{0, n}\ ; \tag{5}$$

$$p_k^{\{i_1, i_2, \dots, i_k\}} = \lim_{t \to \infty} P_k^{\{i_1, i_2, \dots, i_k\}}(t),\ \ k = \overline{1, n}; \tag{6}$$

$$\begin{aligned} g_k^{\{i_1, i_2, \dots, i_k\}}(x_1, \dots, x_k) = \lim_{t \to \infty} G_k^{\{i_1, i_2, \dots, i_k\}}(x_1, \dots, x_k, t), \\ k = \overline{1, n}. \end{aligned} \tag{7}$$

If $k = n$, then we may simply denote $g_n(x_1, \dots, x_n)$ instead of $g_n^{\{i_1, i_2, \dots, i_n\}}(x_1, \dots, x_n)$.

The functions introduced in (4) and (7) are not symmetric considering all of the permutations of the variables x_j, $j \in A(t)$, as it is in classical $M/G/n/0$ Erlang system with identical servers.

It is clear that

$$\begin{aligned} p_k^{\{i_1, i_2, \dots, i_k\}} = \int_0^\infty \dots \int_0^\infty g_k^{\{i_1, i_2, \dots, i_k\}}(x_1, \dots, x_k) dx_1 \dots dx_k, \\ k = \overline{1, n}. \end{aligned} \tag{8}$$

If we analyze, for simplicity, the system behavior in the special case ($M/\vec{G}/2/0$ system), then we can write down the following equations:

$$\begin{aligned} P_0(t + \Delta t) = P_0(t)(1 - a\Delta t) + \Delta t\Big(\int_0^t G_1^{\{1\}}(x, t)\mu_1(x)dx + \\ + \int_0^t G_1^{\{2\}}(x, t)\mu_2(x)dx\Big) + o(\Delta t); \end{aligned} \tag{9}$$

$$\begin{aligned} G_1^{\{1\}}(x + \Delta t, t + \Delta t) = G_1^{\{1\}}(x, t)\big[1 - \big(a + \mu_1(x)\big)\Delta t\big] + \\ + \Delta t \int_0^t G_2(x, u, t)\mu_2(u)du + o(\Delta t); \end{aligned} \tag{10}$$

$$\begin{aligned} G_1^{\{2\}}(x + \Delta t, t + \Delta t) = G_1^{\{2\}}(x, t)\big[1 - \big(a + \mu_2(x)\big)\Delta t\big] + \\ + \Delta t \int_0^t G_2(u, x, t)\mu_1(u)du + o(\Delta t); \end{aligned} \tag{11}$$

$$\begin{aligned} G_2(x_1 + \Delta t, x_2 + \Delta t, t + \Delta t) = \\ = G_2(x_1, x_2, t)\big[1 - \big(\mu_1(x_1) + \mu_2(x_2)\big)\Delta t\big] + o(\Delta t); \end{aligned} \tag{12}$$

$$\int_0^{\Delta t} G_1^{\{1\}}(u, t + \Delta t)du = \frac{a}{2} P_0(t)\Delta(t) + o(\Delta t); \tag{13}$$

$$\int_0^{\Delta t} G_1^{\{2\}}(u, t + \Delta t)du = \frac{a}{2} P_0(t)\Delta(t) + o(\Delta t); \tag{14}$$

$$\int_0^{\Delta t} G_2(x + \Delta t, u, t + \Delta t)du = aG_1^{\{1\}}(x, t)\Delta t + o(\Delta t); \tag{15}$$

$$\int_0^{\Delta t} G_2(u, x + \Delta t, t + \Delta t)du = aG_1^{\{2\}}(x, t)\Delta t + o(\Delta t). \tag{16}$$

If $\Delta t \to 0$ then from equations (9–16) we obtain the following equations:

$$\begin{aligned} \frac{dP_0(t)}{dt} = -aP_0(t) + \int_0^t G_1^{\{1\}}(x, t)\mu_1(x)dx + \\ + \int_0^t G_1^{\{2\}}(x, t)\mu_2(x)dx; \end{aligned} \tag{17}$$

$$\begin{aligned} \frac{\partial G_1^{\{1\}}(x, t)}{\partial t} + \frac{\partial G_1^{\{1\}}(x, t)}{\partial x} = -\big(a + \mu_1(x)\big)G_1^{\{1\}}(x, t) + \\ + \int_0^t G_2(x, u, t)\mu_2(u)du; \end{aligned} \tag{18}$$

$$\begin{aligned} \frac{\partial G_1^{\{2\}}(x, t)}{\partial t} + \frac{\partial G_1^{\{2\}}(x, t)}{\partial x} = -\big(a + \mu_2(x)\big)G_1^{\{2\}}(x, t) + \\ + \int_0^t G_2(u, x, t)\mu_1(u)du; \end{aligned} \tag{19}$$

$$\begin{gathered}\frac{\partial G_2(x_1, x_2, t)}{\partial t} + \frac{\partial G_2(x_1, x_2, t)}{\partial x_1} + \frac{\partial G_2(x_1, x_2, t)}{\partial x_2} = \\ = -\big(\mu_1(x_1) + \mu_2(x_2)\big)G_2(x_1, x_2, t);\end{gathered} \tag{20}$$

$$G_1^{\{1\}}(0, t) = \frac{a}{2}P_0(t); \tag{21}$$

$$G_1^{\{2\}}(0, t) = \frac{a}{2}P_0(t); \tag{22}$$

$$G_2(x, 0, t) = aG_1^{\{1\}}(x, t); \tag{23}$$

$$G_2(0, x, t) = aG_1^{\{2\}}(x, t). \tag{24}$$

In the stationary mode ($t \to \infty$) from (17–24), we easily obtain:

$$0 = -ap_0 + \int_0^\infty g_1^{\{1\}}(x)\mu_1(x)dx + \int_0^\infty g_1^{\{2\}}(x)\mu_2(x)dx; \tag{25}$$

$$\frac{\partial g_1^{\{1\}}(x)}{\partial x} = -\big(a + \mu_1(x)\big)g_1^{\{1\}}(x) + \int_0^\infty g_2(x, u)\mu_2(u)du; \tag{26}$$

$$\frac{\partial g_1^{\{2\}}(x)}{\partial x} = -\big(a + \mu_2(x)\big)g_1^{\{2\}}(x) + \int_0^\infty g_2(u, x)\mu_1(u)du; \tag{27}$$

$$\frac{\partial g_2(x_1, x_2)}{\partial x_1} + \frac{\partial g_2(x_1, x_2)}{\partial x_2} = -\big(\mu_1(x_1) + \mu_2(x_2)\big)g_2(x_1, x_2); \tag{28}$$

$$g_1^{\{1\}}(0) = \frac{a}{2}p_0; \tag{29}$$

$$g_1^{\{2\}}(0) = \frac{a}{2}p_0; \tag{30}$$

$$g_2(x, 0) = ag_1^{\{1\}}(x); \tag{31}$$

$$g_2(0, x) = ag_1^{\{2\}}(x). \tag{32}$$

Now we add the normalization condition:

$$\begin{gathered}p_0 + \int_0^\infty g_1^{\{1\}}(x)dx + \int_0^\infty g_1^{\{2\}}(x)dx + \\ + \int_0^\infty\int_0^\infty g_2(x_1, x_2)dx_1dx_2 = 1.\end{gathered} \tag{33}$$

By direct substitution, we can check that the solutions of equations (25–32) have the form:

$$g_1^{\{1\}}(x) = \frac{ap_0}{2}\big(1 - B_1(x)\big); \tag{34}$$

$$g_1^{\{2\}}(x) = \frac{ap_0}{2}\big(1 - B_2(x)\big); \tag{35}$$

$$g_2(x_1, x_2) = \frac{a^2p_0}{2}\big(1 - B_1(x_1)\big)\big(1 - B_2(x_2)\big). \tag{36}$$

Using (8) and the well-known formula $\beta_j = \int_0^\infty[1 - B_j(u)]du$, we finally obtain:

$$p_1^{\{1\}} = \int_0^\infty \frac{ap_0}{2}\big(1 - B_1(x)\big)dx = \frac{a\beta_1p_0}{2}; \tag{37}$$

$$p_1^{\{2\}} = \int_0^\infty \frac{ap_0}{2}\big(1 - B_2(x)\big)dx = \frac{a\beta_2p_0}{2}; \tag{38}$$

$$p_1 = p_1^{\{1\}} + p_1^{\{2\}} = \frac{ap_0}{2}(\beta_1 + \beta_2); \tag{39}$$

$$\begin{gathered}p_2 = \int_0^\infty\int_0^\infty \frac{a^2p_0}{2}\big(1 - B_1(x_1)\big)\big(1 - B_2(x_2)\big)dx_1dx_2 = \\ = \frac{a^2p_0}{2}\int_0^\infty\big(1 - B_1(x_1)\big)dx_1 \cdot \int_0^\infty\big(1 - B_2(x_2)\big)dx_2 = \\ = \frac{a^2p_0\beta_1\beta_2}{2}.\end{gathered} \tag{40}$$

And by the help of (33):

$$p_0 = \left[1 + \frac{a}{2}(\beta_1 + \beta_2) + \frac{a^2}{2}\beta_1\beta_2\right]^{-1}. \tag{41}$$

In the same way, we can analyze the $M/\vec{G}/n/0$ system with heterogeneous servers for the arbitrary n. The problem is that the number of equations describing the system behavior increases exponentially together with the increasing value of n, so in this case we use some set notations. Let $\{C_k^n\}$ denote the set of all of the k-element combinations of the n-element set. The equations in the stationary mode can be presented in the following form:

$$0 = -ap_0 + \sum_{j=1}^n \int_0^\infty g_1^{\{j\}}(x)\mu_j(x)dx; \tag{42}$$

$$\begin{gathered}\sum_{j\in\{i_1,\dots,i_k\}} \frac{\partial g_k^{\{i_1,\dots,i_k\}}(x_1,\dots,x_k)}{\partial x_j} = \\ = -\Big(a + \sum_{j\in\{i_1,\dots,i_k\}} \mu_j(x_j)\Big)g_k^{\{i_1,\dots,i_k\}}(x_1,\dots,x_k) + \\ + \sum_{j\notin\{i_1,\dots,i_k\}} \int_0^\infty g_{k+1}^{\{i_1,\dots,i_k,j\}}(x_1,\dots,x_{j-1},u,x_j,\dots,x_k)\mu_j(u)du, \\ \{i_1,\dots,i_k\} \in \{C_k^n\},\ k = \overline{1, n-1};\end{gathered} \tag{43}$$

$$\sum_{j=1}^n \frac{\partial g_n^{\{i_1,\dots,i_n\}}(x_1,\dots,x_n)}{\partial x_j} = -\sum_{j=1}^n \mu_j(x_j)g_n^{\{i_1,\dots,i_n\}}(x_1,\dots,x_n); \tag{44}$$

$$g_1^{\{j\}}(0) = \frac{a}{n}\,p_0,\; j = \overline{1,\, n}; \tag{45}$$

$$g_k^{\{i_1,\,\dots,\,i_k\}}(x_1,\,\dots,\,x_{j-1},\,0,\,x_{j+1},\,\dots,\,x_k) =$$
$$= \frac{a}{n-k+1} g_{k-1}^{\{i_1,\,\dots,\,i_{j-1},\,i_{j+1},\,\dots,\,i_k\}}(x_1,\,\dots,\,x_{j-1},\,x_{j+1},\,\dots,\,x_k), \tag{46}$$
$$j = \overline{1,\,k},\; k = \overline{2,\,n}.$$

By direct substitution, we may obtain the solutions in the following form:

$$g_k^{\{i_1,\,i_2,\,\dots,\,i_k\}}(x_1,\,x_2,\,\dots,\,x_k) =$$
$$= \frac{a^k(n-k)!\,p_0}{n!} \prod_{j\in\{i_1,\,\dots,\,i_k\}} \big(1 - B_j(x_j)\big);\; k = \overline{1,\,n}. \tag{47}$$

Using (8) in an analogous way to (40), we obtain:

$$p_k^{\{i_1,\,\dots,\,i_k\}} = \frac{a^k(n-k)!\,p_0}{n!} \prod_{j\in\{i_1,\,\dots,\,i_k\}} \beta_j,\; k = \overline{1,\,n}; \tag{48}$$

Thus:

$$p_k = \frac{a^k(n-k)!\,p_0}{n!} \sum_{i\in\{C_k^n\}} \prod_{j\in i} \beta_j,\; k = \overline{1,\,n}. \tag{49}$$

From the normalization condition, we obtain:

$$p_0 = \left[1 + \frac{1}{n!}\sum_{k=1}^{n} a^k(n-k)! \sum_{i\in\{C_k^n\}} \prod_{j\in i} \beta_j\right]^{-1}. \tag{50}$$

If we introduce the notation: $y_j = a\beta_j$ then formulae (48–50) take the form:

$$p_k^{\{i_1,\,\dots,\,i_k\}} = \frac{(n-k)!\,p_0}{n!} \prod_{j\in\{i_1,\,\dots,\,i_k\}} y_j,\; k = \overline{1,\,n}; \tag{51}$$

$$p_k = \frac{(n-k)!\,p_0}{n!} \sum_{i\in\{C_k^n\}} \prod_{j\in i} y_j,\; k = \overline{1,\,n}; \tag{52}$$

$$p_0 = \left[1 + \frac{1}{n!}\sum_{k=1}^{n} (n-k)! \sum_{i\in\{C_k^n\}} \prod_{j\in i} y_j\right]^{-1}. \tag{53}$$

As was investigated, the number of customers distribution function in the stationary mode depends only on the first moments of functions $B_j(x)$ and does not depend on the formulae that define them. Formulae (51–53) are interesting not only from the theoretical point of view. On the basis of (52) and (53), we can calculate some very practical characteristics, including the mean value of the number of customers present in the system in the stationary mode $\big(E\eta = \sum_{k=0}^{n} kp_k\big)$ and the loss probability (p_n). On the other hand, we may also investigate the usage of each server. For example, if the analyzed system is composed of three servers, then we may compute the usage of each server q_i, $i = \overline{1,\,3}$ as follows:

$$q_1 = p_1^{\{1\}} + p_2^{\{1,2\}} + p_2^{\{1,3\}} + p_3^{\{1,2,3\}}.$$
$$q_2 = p_1^{\{2\}} + p_2^{\{1,2\}} + p_2^{\{2,3\}} + p_3^{\{1,2,3\}}.$$
$$q_3 = p_1^{\{3\}} + p_2^{\{1,3\}} + p_2^{\{2,3\}} + p_3^{\{1,2,3\}}.$$

Let us now consider the following numerical example. Assume that we deal with $M/\vec{G}/3/0$ queueing system with heterogeneous servers. The main values of service time on each server are equal to: $\beta_1 = 2$, $\beta_2 = 3$, $\beta_3 = 4$ consequently and $a = 2$. Then, by using (51–53), we may obtain the numerical results connected with the number of customers distribution function. We present them in the first column of Table 1, together with the results obtained by simulation for three distributions of service time. In the next columns, we present the results for exponential distribution (service time is exponentially distributed for each server with parameters $\mu_1 = 1/2$, $\mu_2 = 1/3$, $\mu_3 = 1/4$), uniform distribution (on the interval [1, 3] for first server, [2, 4] for second server and [3, 5] for third server), and constant distribution (service time is constant for each server with parameters $t_1 = 2$, $t_2 = 3$, $t_3 = 4$). The results show that the number of customers distribution function depends only on the mean values of service time on each server, and does not depend on the formulae of the service time distribution functions.

Table 1
The number of customers distribution function in the $M/\vec{G}/3/0$ system

p_k	theoret.	sim.-exponential	sim.-uniform	sim.-constant
p_0	0.017751	0.017652	0.017674	0.017685
p_1	0.106509	0.106639	0.106439	0.106212
p_2	0.307692	0.307517	0.307813	0.307646
p_3	0.568047	0.568192	0.568074	0.568458

3. The model with non-homogeneous customers

This section will analyze the modification of the classical $M/\vec{G}/n/0$ queueing system with heterogeneous servers in which the arriving customers additionally have, independently of the other customers, some random volume ζ that is a non-negative random variable. In general, service time on j-th server ξ_j depends on customer volume ζ. In other words, for each server we have the following distribution function: $F_j(x,t) = P\{\zeta < x,\, \xi_j < t\}$, $j = \overline{1,\,n}$. The aim of our investigation is to obtain the characteristics of the total volume $\sigma(t)$, which is the sum of the volumes of all customers present in the system in time instant t. In the steady state, the process $\sigma(t)$ converges to a random variable σ. We assume that the total volume is unlimited.

Let us introduce the following notations: $D(x) = P\{\sigma < x\}$ is the total volume distribution function in the steady state; $\delta(s) =$

$= \int_0^\infty e^{-sx} dD(x)$ is the Laplace–Stieltjes transform of the random variable σ; δ_i is its i-th moment; $\alpha_j(s,q) = \int_0^\infty \int_0^\infty e^{-sx-qt} dF_j(x,t)$, $j = \overline{1,n}$ denotes the double Laplace–Stieltjes transform of the random vector (ζ, ξ_j); and α_j^{lk} is the mixed moment of the $(l+k)$-th order of this vector.

To obtain the characteristics of the total volume distribution function in the steady state, we introduce some conditional distribution functions of the total volume, and we then use the total probability theorem. We introduce the conditional total volume distribution function as follows:

$$\begin{aligned} H_k^{\{i_1,\ldots,i_k\}}(x, y_1, \ldots, y_k) = P\{\sigma < x \mid \eta = k, \\ A = \{i_1, \ldots, i_k\},\ \xi^*_{i_1} = y_1, \ldots, \xi^*_{i_k} = y_k\}, \\ \{i_1, \ldots, i_k\} \in \{C_k^n\},\ k = \overline{1, n}, \end{aligned} \tag{54}$$

where η is the number of customers present in the system in stationary mode, A is the set of busy servers, and $\xi^*_j, j \in \{i_1, \ldots, i_k\}$ are the stationary analogies of the functions $\xi^*_j(t)$ (which were introduced in Section 2).

By using the total probability theorem, we obtain the following formula:

$$\begin{aligned} D(x) = p_0 + \sum_{k=1}^{n} \sum_{\{i_1,\ldots,i_k\} \in \{C_k^n\}} \int_0^\infty \ldots \int_0^\infty g_k^{\{i_1,\ldots,i_k\}}(y_1, \ldots, y_k) \times \\ \times H_k^{\{i_1,\ldots,i_k\}}(x, y_1, \ldots, y_k) dy_1 \ldots dy_k. \end{aligned} \tag{55}$$

Now we use Laplace–Stieltjes to transform to both sides of (55), obtaining:

$$\begin{aligned} \delta(s) = p_0 + \sum_{k=1}^{n} \sum_{\{i_1,\ldots,i_k\} \in \{C_k^n\}} \int_0^\infty \ldots \int_0^\infty g_k^{\{i_1,\ldots,i_k\}}(y_1, \ldots, y_k) \times \\ \times h_k^{\{i_1,\ldots,i_k\}}(s, y_1, \ldots, y_k) dy_1 \ldots dy_k. \end{aligned} \tag{56}$$

where

$$h_k^{\{i_1,\ldots,i_k\}}(s, y_1, \ldots, y_k) = \int_0^\infty e^{-sx} dH_k^{\{i_1,\ldots,i_k\}}(x, y_1, \ldots, y_k)$$

is the Laplace–Stieltjes transform of the distribution function that was defined in (54).

Now we find the formula for $h_k^{\{i_1,\ldots,i_k\}}(s, y_1, \ldots, y_k)$. We denote as $\chi_j, j \in A$ – the volume of the customers serviced by j-th server. Let $E_j(x) = P\{\chi_j < x \mid \xi^*_j = y_j\}, j \in A$ be the conditional distribution function of the random variable χ_j under assumption that its service lasts y_j time units.

Now we use well-known formula [11]:

$$dE_j(x) = [1 - B_j(y_j)]^{-1} \int_{u=y_j}^\infty dF_j(x, u). \tag{57}$$

If we introduce the Laplace–Stieltjes transform $e_j(s) = \int_0^\infty e^{-sx} dE_j(x)$, then transformation of (57) leads to the following result:

$$e_j(s) = [1 - B_j(y_j)]^{-1} \int_{x=0}^\infty e^{-sx} \int_{u=y_j}^\infty dF_j(x, u). \tag{58}$$

It is rather clear that if $A = \{i_1, \ldots, i_k\}$ then total volume σ is the sum of the independent random variables $\chi_j, j \in A$. The conditional distribution function $H_k^{\{i_1,\ldots,i_k\}}(x, y_1, \ldots, y_k)$ is the convolution of the distributions $E_j(x)$, namely:

$$H_k^{\{i_1,\ldots,i_k\}}(x, y_1, \ldots, y_k) = E_{i_1} * E_{i_2} * \ldots * E_{i_k}(x). \tag{59}$$

On the basis of the properties of the Laplace–Stieltjes transform, we quickly obtain:

$$h_k^{\{i_1,\ldots,i_k\}}(s, y_1, \ldots, y_k) = \prod_{j \in \{i_1,\ldots,i_k\}} e_j(s). \tag{60}$$

From (56, 58, 60), and (47) we finally obtain:

$$\begin{aligned} \delta(s) = p_0 + \sum_{k=1}^{n} \sum_{i \in \{C_k^n\}} \frac{a^k (n-k)! p_0}{n!} \times \\ \times \int_0^\infty \ldots \int_0^\infty \left[\prod_{j \in i} \int_{x=0}^\infty e^{-sx} \int_{u=y_j}^\infty dF_j(x, u) \right] dy_1 \ldots dy_k. \end{aligned} \tag{61}$$

Given that k-dimensional integral present in (61) may be presented in the form of product of k one-dimensional integrals, then formula (61) may be rewritten in a simple form:

$$\begin{aligned} \delta(s) = p_0 + \sum_{k=1}^{n} \sum_{i \in \{C_k^n\}} \frac{a^k (n-k)! p_0}{n!} \times \\ \times \prod_{j \in i} \left[\int_{x=0}^\infty e^{-sx} \int_{z=0}^\infty dz \int_{u=z}^\infty dF_j(x, u) \right]. \end{aligned} \tag{62}$$

Now we compute the integral in (62):

$$\begin{aligned} \int_{x=0}^\infty e^{-sx} \int_{z=0}^\infty dz \int_{u=z}^\infty dF_j(x, u) = \\ = \int_{x=0}^\infty \int_{u=0}^\infty e^{-sx} dF_j(x, u) \int_{z=0}^u dz = \\ = \int_0^\infty \int_0^\infty u e^{-sx} dF_j(x, u) = \frac{\partial \alpha_j(s, q)}{\partial q} \Big|_{q=0}. \end{aligned} \tag{63}$$

Finally, using (63), we obtain the following result:

$$\delta(s) = p_0 \left(1 + \frac{1}{n!} \sum_{k=1}^{n} (n-k)! \sum_{i \in \{C_k^n\}} \prod_{j \in i} -a \frac{\partial \alpha_j(s, q)}{\partial q} \Big|_{q=0} \right). \tag{64}$$

The obtained formula let us compute the first two moments $\delta_1 = E\delta$, $\delta_2 = E\delta^2$ of the total volume in stationary mode. We use the well-known formulae that are connected with the Laplace–Stieltjes transform properties: $\delta_1 = -\delta'(0)$, $\delta_2 = \delta''(0)$. On the other hand, the mix moments of the $(l+k)$-th order of the random vector (ζ, ξ_j) can be computed using the properties of the double Laplace–Stieltjes transform $\alpha_j(s, q)$:

$$\alpha_j^{lk} = E\left(\zeta^l \xi_j^k\right) = (-1)^{l+k} \frac{\partial^{l+k} \alpha_j(s, q)}{\partial s^l \partial q^k} \Big|_{s=0,\, q=0}. \tag{65}$$

By using the above properties of single and double Laplace–Stieltjes transforms, and (64) we finally obtain:

$$\delta_1 = -\delta'(0) = p_0\left(\frac{1}{n!}\sum_{k=1}^{n}\alpha^k(n-k)!\sum_{i\in\{C_k^n\}}\prod_{j\in i}\alpha_j^{11}\prod_{l\in i\setminus\{j\}}\beta_l\right). \quad (66)$$

$$\begin{aligned}\delta_2 = \delta''(0) = p_0\Bigg(\frac{1}{n!}\sum_{k=1}^{n}\alpha^k(n-k)!\times \\ \times\sum_{i\in\{C_k^n\}}\Bigg[\sum_{j\in i}\alpha_j^{21}\prod_{l\in i\setminus\{j\}}\beta_l + 2\sum_{\{j,m\}\subset i}\alpha_j^{11}\alpha_m^{11}\prod_{l\in i\setminus\{j,m\}}\beta_l\Bigg]\Bigg).\end{aligned} \quad (67)$$

If the set of indexes of the sum or product symbol is empty, then we omit these sums or products in computation. In the process of computing formulae (66) and (67), we use the following properties of the derivatives:

$$\left(\prod_{i=1}^{n}f_i(x)\right)' = \sum_{i=1}^{n}f_i'(x)\prod_{l\in\{1,\ldots,n\}\setminus\{i\}}f_l(x). \quad (68)$$

$$\begin{aligned}\left(\prod_{i=1}^{n}f_i(x)\right)'' = \sum_{i=1}^{n}f_i''(x)\prod_{l\in\{1,\ldots,n\}\setminus\{i\}}f_l(x) + \\ + 2\prod_{\{i,j\}\subset\{1,\ldots,n\}}f_i'(x)f_j'(x)\prod_{l\in\{1,\ldots,n\}\setminus\{i,j\}}f_l(x).\end{aligned} \quad (69)$$

The obtained results can be used to design computer or communication systems of the analogous type but with limited memory space. For example, we can consider a more practical system $M/\vec{G}/n/(0, V)$ in which the volume of all customers is limited by the V value. This means that the arriving customer may also be lost when there are free servers in the system but their volume is too high (i.e. the sum of the volume of arriving customer and the volumes of other customers that are served is bigger than V value). This limitation leads to additional losses of customers. In the process of designing a computer system, we may choose the V value in such a way that minimizes these additional losses. Then, we use the (66, 67) formulae and, for example, choose the value of V from the interval $[\delta_1 - m\delta, \delta_1 + m\delta]$, where $\delta = \sqrt{\delta_2 - \delta_1^2}$ is the standard deviation of total customers volume in the stationary mode, and m constant, $m = (1, 2, 3, \ldots)$ may be chosen by the computer system designer. This choice has an influence on the loss probability in the $M/\vec{G}/n/(0, V)$ system. From the practical point of view, it is best to choose the following value: $V = \delta_1 + 3\delta$.

4. Special case analysis

This section will investigate two practical special cases of the analyzed model. In the first case, customer volume and service times for j-th server, $j = \overline{1, n}$ are independent. The second case presents a situation in which service time for j-th server is proportional to customer volume with coefficient c_j i.e. $\xi_j = c_j\zeta$. For these special cases, we obtain the formulae for the total volume distribution function in stationary mode and for its first two moments.

4.1. Service time and customer volume are independent. Assume that customer volume and service time are independent for every server. More precisely, every pair $(\zeta, \xi_j), j = \overline{1, n}$ present two independent random variables. Let $L(x) = P\{\zeta < x\}$ and $B_j(t) = P\{\xi_j < t\}$ be the distribution functions of the random variables ζ and ξ_j. We denote the first two moments of the random variable ζ as φ_1 and φ_2. In this case, we have the obvious formula:

$$\alpha_j(s, q) = \varphi(s)\beta_j(q), \quad (70)$$

where $\varphi(s) = \int_0^\infty e^{-sx}dL(x)$ and $\beta_j(q) = \int_0^\infty e^{-qt}dB_j(t)$ are the Laplace–Stieltjes transforms of the random variables ζ and ξ_j, respectively. Then, we have the equality:

$$\frac{\partial\alpha_j(s,q)}{\partial q}\Big|_{q=0} = -\varphi(s)\beta_j. \quad (71)$$

Now we substitute the obtained formula into (64) and we then obtain the following:

$$\delta(s) = p_0\left(1 + \frac{1}{n!}\sum_{k=1}^{n}(n-k)!\big(a\varphi(s)\big)^k\sum_{i\in\{C_k^n\}}\prod_{j\in i}\beta_j\right). \quad (72)$$

Thus:

$$\delta_1 = p_0\varphi_1\left(\frac{1}{n!}\sum_{k=1}^{n}ka^k(n-k)!\sum_{i\in\{C_k^n\}}\prod_{j\in i}\beta_j\right); \quad (73)$$

$$\delta_2 = p_0\left(\frac{1}{n!}\sum_{k=1}^{n}ka^k(n-k)!\left[\varphi_2 + (k-1)\varphi_1^2\right]\sum_{i\in\{C_k^n\}}\prod_{j\in i}\beta_j\right). \quad (74)$$

Also assume that customer volume is exponentially distributed with parameter f and the service times on each of n servers are also exponentially distributed with parameters $\mu_1, \ldots, \mu_n$. Then, we obtain the following formula:

$$\delta(s) = p_0\left(1 + \frac{1}{n!}\sum_{k=1}^{n}\left(\frac{af}{f+s}\right)^k(n-k)!\sum_{i\in\{C_k^n\}}\prod_{j\in i}\frac{1}{\mu_j}\right). \quad (75)$$

The formulae for first two moments can be obtained from (73) and (74) by making substitutions: $\varphi_1 = 1/f$, $\varphi_2 = 2/f^2$, $\beta_j = 1/\mu_j$. Using the Laplace transform inversion of the function $\delta(s)/s$, with the help of computer algebra systems [1], we may obtain formula for total volume distribution function in a form:

$$\begin{aligned}D(x) = p_0\Bigg(1 + \frac{1}{n!}\sum_{k=1}^{n}a^k(n-k)!\times \\ \times\left[1 - e^{-fx}\sum_{l=0}^{k-1}\frac{(fx)^l}{l!}\right]\sum_{i\in\{C_k^n\}}\prod_{j\in i}\frac{1}{\mu_j}\Bigg).\end{aligned} \quad (76)$$

The total volume distribution function is a linear combination of Erlang distributions with parameter f.

4.2. Service time is proportional to the customer volume. Assume now that the service time on j-th server is proportional to the customer volume with coefficient c_j, $j = \overline{1, n}$ i.e $\xi_j = c_j\zeta$. In this case, we obtain the formula:

$$\alpha_j(s, q) = \varphi(s + c_j q). \tag{77}$$

Thus:

$$\frac{\partial\alpha_j(s, q)}{\partial q}\Big|_{q=0} = c_j\varphi'(s). \tag{78}$$

If we substitute this formula into (64), we obtain:

$$\delta(s) = p_0\left(1 + \frac{1}{n!}\sum_{k=1}^{n}\left(-a\varphi'(s)\right)^k (n-k)! \sum_{i\in\{C_k^n\}}\prod_{j\in i} c_j\right). \tag{79}$$

The first two moments may be computed basing on the following formulae:

$$\delta_1 = ap_0\varphi_2\left(\frac{1}{n!}\sum_{k=1}^{n} k(n-k)!(a\varphi_1)^{k-1}\sum_{i\in\{C_k^n\}}\prod_{j\in i} c_j\right); \tag{80}$$

$$\begin{aligned}\delta_2 = p_0\Bigg(&\frac{1}{n!}\sum_{k=1}^{n} ka^k\varphi_1^{k-2}(n-k)!\times\\ &\times\left[\varphi_3\varphi_1 + (k-1)\varphi_2^2\right]\sum_{i\in\{C_k^n\}}\prod_{j\in i} c_j\Bigg),\end{aligned} \tag{81}$$

where φ_3 is the third moment of the customer volume.

Also assume that customer volume is exponentially distributed with parameter f and the service times on each of n servers are proportional to their volumes with coefficients c_j. Then, the service times are also exponentially distributed with parameters $f/c_1, \ldots, f/c_n$. Then, we obtain the following formula:

$$\delta(s) = p_0\left(1 + \frac{1}{n!}\sum_{k=1}^{n}\frac{(af)^k}{(f+s)^{2k}}(n-k)!\sum_{i\in\{C_k^n\}}\prod_{j\in i} c_j\right). \tag{82}$$

The formulae for the first two moments can be obtained from (80) and (81) by making substitutions: $\varphi_1 = 1/f$, $\varphi_2 = 2/f^2$, $\varphi_3 = 6/f^3$. Using the Laplace transform inversion, we finally obtain the total volume distribution function in a form:

$$\begin{aligned}D(x) = p_0\Bigg(&1 + \frac{1}{n!}\sum_{k=1}^{n} a^k(n-k)!\times\\ &\times\left[1 - e^{-fx}\sum_{l=0}^{2k-1}\frac{(fx)^l}{l!}\right]\sum_{i\in\{C_k^n\}}\prod_{j\in i} c_j\Bigg).\end{aligned} \tag{83}$$

This formula is similar to (76) but here we have more Erlang distributions in the sum. This means that even if we choose coefficients in such a way that from the classical point of view time service distribution functions are the same in these two situations then characteristics of the total volume distribution vary even on the level of first two moments. So, the character of the analyzed dependency has an influence on the total volume characteristics.

In fact, if in the first model we substitute $\mu_j = f/c_j$, $j = \overline{1, n}$, then from the classical point of view two models are equivalent; that is, we have, for example, the same service times distribution functions but the characteristics of total volume distribution are not the same. In the first model, after rather easy computations, we obtain:

$$\delta_1 = \frac{p_0}{f}\left(\frac{1}{n!}\sum_{k=1}^{n} k(n-k)!\left(\frac{a}{f}\right)^k\sum_{i\in\{C_k^n\}}\prod_{j\in i} c_j\right); \tag{84}$$

$$\delta_2 = \frac{p_0}{f^2}\left(\frac{1}{n!}\sum_{k=1}^{n} k(k+1)(n-k)!\left(\frac{a}{f}\right)^k\sum_{i\in\{C_k^n\}}\prod_{j\in i} c_j\right). \tag{85}$$

In the second model, we obtain:

$$\delta_1 = \frac{2p_0}{f}\left(\frac{1}{n!}\sum_{k=1}^{n} k(n-k)!\left(\frac{a}{f}\right)^k\sum_{i\in\{C_k^n\}}\prod_{j\in i} c_j\right); \tag{86}$$

$$\delta_2 = \frac{2p_0}{f^2}\left(\frac{1}{n!}\sum_{k=1}^{n} k(2k+1)(n-k)!\left(\frac{a}{f}\right)^k\sum_{i\in\{C_k^n\}}\prod_{j\in i} c_j\right). \tag{87}$$

Comparing the above formulae, we notice that the value of first moment is exactly two times greater in the second model and the value of the second moment is also greater in the second model but in this case the coefficient is not constant and depends on the number of servers.

We will now illustrate the theoretical results with some numerical examples. Let us consider $M/\vec{G}/3/0$ queueing system in two versions M1 and M2. In the first version of the system, service times are exponentially distributed with parameters $\mu_1 = 1/2$, $\mu_2 = 1/3$, $\mu_3 = 1/4$. Service times and customer volumes are independent and the customer volumes are exponentially distributed with parameter $f = 1$. In the second version, the customer volumes are also exponentially distributed with the same parameter $f = 1$ but service times are proportional to customer volumes with coefficients $c_1 = 2$, $c_2 = 3$, $c_3 = 4$. In Table 2, we present the results connected with the first two moments of the summary volume in stationary mode obtained using (84–87), together with the results obtained by simulation that confirm the theoretical results.

Table 2
Moments of the summary volume in $M/\vec{G}/3/0$ system

	M1-theoret.	M1-sim.	M2-theoret.	M2-sim.
δ_1	2.426036	2.424834	4.851368	4.852071
δ_2	8.875740	8.886603	30.650888	30.658217

5. Conclusions

In the present paper, we have investigated the modification of a $M/G/n/0$ queueing system with heterogeneous servers and non-homogeneous customers. In the beginning, we obtain a number of customers distribution functions in the stationary mode for the classical queueing system $M/\vec{G}/n/0$ in which service time characteristics may be different for every server. Both the theoretical and the simulation results show that the number of customers distribution function does not depend on the form of the service time distributions, but on their mean values. Later on, we analyzed a non-classical $M/\vec{G}/n/0$ queueing system with non-homogeneous customers in which service times and customer volumes are dependent and the total volume is unlimited. For this system, we obtain the formulae for the Laplace–Stieltjes transform of the total volume distribution function and its first two moments. Then, we investigate two special cases in which we obtain the formulae for the total volume distribution function in exact form. We then show that the character of the dependency of service time and customer volume has an influence on the total volume characteristics. Our simulation results also confirm this finding.

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