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Bayesian DEJD Model and Detection of Asymmetry in Jump Sizes

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Abstract

News might trigger jump arrivals in financial time series. The "bad" news and "good" news seem to have distinct impact. In the research, a double exponential jump distribution is applied to model downward and upward jumps. Bayesian double exponential jump-diffusion model is proposed. Theorems stated in the paper enable estimation of the model's parameters, detection of jumps and analysis of jump frequency. The methodology, founded upon the idea of latent variables, is illustrated with simulated data.

Keywords: double exponential jump diffusion model, Kou model, Bernoulli jump-diffusion model, MCMC methods, latent variables

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1 Introduction

News concerning the companies, macroeconomic releases, cataclysms or wars has a huge impact on prices of shares, derivative securities, yields, commodities etc. (Milgrom 1981). Markets often react in a spontaneous way on flowing news. The reactions manifest themselves as jumps in time series.

There are models where jumps and small changes of values are considered simultaneously. Examples of such specifications include the jump-diffusion models and their discretizations (e.g. Ball and Torous (1983), Honore (1998), Ramezani and Zeng (1998), Kou (2002), Lin and Huang (2002), Hanson andWestman (2002), Hanson, Westman, and Zhu (2004), Barndorff-Nielsen and Shephard (2004), Piazzesi (2005), Barndorff-Nielsen and Shephard (2006b), Barndorff-Nielsen and Shephard (2006a), Yu (2007), Ramezani and Zeng (2007), Synowiec (2008), Weron (2008), Rifo and Torres (2009), Ane and Metais (2010), Ait-Sahalia and Jacod (2012), Lee 2012, Frame and Ramezani (2012), Kostrzewski (2012a), Kostrzewski (2012b), Kostrzewski (2014a)). One of the best known jump-diffusion model is the Merton model (Merton (1976)). In the Merton model, the jumps appear at random moments of time governed by the exponential distribution, whereas the number of jumps and their magnitudes are driven by the Poisson process and the normal distribution, respectively. The process of prices is continuous between jumps – just as in the Black-Scholes model (Black and Scholes (1973)).

It is generally known that the investors' reaction on "bad" and "good" news is different (crashophobia, Jackwerth and Rubinstein (1996)). In modelling time series it is a common way to account for this by employing distinct distributions for the negative and the positive jumps. An example of such an approach is to apply a double exponential distribution. In this case the negative and the positive jump distributions are exponential with some (distinct) parameters. In the Merton model, jump values are modeled via a normal distribution. However, if we replace the normal distribution with the double exponential one, we get a specification in which the negative and the positive jumps are handled separately. In this paper, we concentrate on discrete version of such constructions. In the jump-diffusion framework, the distribution of logarithmic returns is given by an infinite mixture of normal distributions. In practice, estimation of this model's parameters is conducted for some model approximation given by finite mixtures. The most famous approximation of the Merton model is the Bernoulli jump-diffusion model (Ball and Torous (1983)), which allows for at most a single jump per a unit of time (e.g. a day). The same idea is applied to the jump-diffusion model with the double exponential jump distribution. Such specification was considered by Kou (2002) in the context of pricing derivative securities and it is known as the Kou model. Moreover, it was analysed by Ramezani and Zeng (2007). This model is a special case of the Pareto-Beta jump-diffusion specification proposed by Ramezani and Zeng (1998), where two Poisson processes govern the arrival rate of "bad" and "good" information.

In this paper, we consider discretization of the double exponential jump-diffusion model, called the DEJD model. It is equivalent (under an appropriate parametrizations) to the model considered by Kou (2002), Ramezani and Zeng (2007) and Frame and Ramezani (2012). Under the DEJD specification, a single Bernoulli process controls jumps arrivals in returns, whereas the magnitudes of the upward and the downward jumps are generated by the double exponential distribution. The aim of the paper is to develop a Bayesian framework for the DEJD model under (some) proper priors. The idea underlying the statistical model is based on introducing latent variables. The technique of introducing the latent variables, in the framework of Bernoulli jump-diffusion model with a normal distribution of jump value, was proposed by Lin and Huang (2002). Moreover, we give a recipe how to conduct the Bayesian inference in practice, providing schemes of relevant numerical algorithms. Frame and Ramezani (2012) proposed the Bayesian specification for the equivalent mathematical model. They considered non-informative prior specifications with an exception of the jump intensity parameter. Bayesian framework for models with normal jump values is considered by Rifo and Torres (2009), Lin and Huang (2002), Kostrzewski (2012b) and Kostrzewski (2014a). The Merton model, Kou model and DEJD model are used in portfolio choice, pricing derivative securities and risk analysis. From a practical point of view, a reliable method for estimation of this model is of utmost importance. Finally, let us clarify that we are preoccupied with detecting jumps rather than relating them with, e.g., macroeconomic releases. The latter has been attempted by, e.g., Piazzesi (2005), Lee (2012) and Błędowska-Sójka (2012). The remainder of the paper is organized as follows. In Section 2, the theoretical details of the DEJD model are presented. The Bayesian DEJD model is defined in Section 3. Moreover, we propose numerical algorithms based on MCMC methods, which make Bayesian inference possible to apply. In Section 4, computational results are reported. The paper ends with some brief conclusions. The proofs of the proposed theorems are provided in the Appendix.

2 The DEJD process

Consider a standard Wiener process $W = (W_t)_{t\geq 0}$, a Poisson process $N = (N_t)_{t\geq 0}$ with the intensity $\lambda > 0$, and independent random variables $\Lambda = (Q_j)_{j\geq 1}$ such that Q_j has a double exponential distribution with density

$$f_{Q_j}(x) = p_D \eta_D \exp(\eta_D x) \mathbb{I}_{(-\infty,0)}(x) + p_U \eta_U \exp(-\eta_U x) \mathbb{I}_{[0,\infty)}(x), \tag{1}$$

where $\eta_U > 0, \eta_D > 0$, $p_D > 0, p_U > 0$, $p_D + p_U = 1$. Let us assume that W, N and Λ are independent. Finally, $S = (S_t)_{t\geq 0}$ denotes the price process of some risky asset. The logarithm of S is governed by a jump-diffusion process that constitutes the solution of the equation:

$$d(\ln S_t) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t + d\left(\sum_{i=1}^{N_t} Q_i\right).$$

It might be shown that

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t + \sum_{i=1}^{N_t} Q_i\right),$$

$$\ln\left(\frac{S_{t+\Delta}}{S_t}\right) = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta + \sigma\left(W_{t+\Delta} - W_t\right) + \sum_{i=N_t+1}^{N_{t+\Delta}} Q_i,\ \Delta > 0.$$

The last equation defines the process of the log-returns over a time interval Δ. The process is built of two components: the (pure) diffusion part,

$$\left(\mu - \frac{1}{2}\sigma^2\right)\Delta + \sigma\left(W_{t+\Delta} - W_t\right),$$

represent continuous variations, wheras the (pure) jump component,

$$\sum_{i=N_t+1}^{N_{t+\Delta}} Q_i,$$

reflects abnormal (extreme) movements in returns. There are three sources of randomness: W, N and Λ, affecting S. The (continuous) price behaviour between jumps is described by the geometric Brownian motion, W. The arrival rate of jumps is described by the Poisson process, N, and the jump magnitudes – by Λ. The process S depends on six unknown parameters: $\mu, \sigma, \lambda, p_U, \eta_U$ and η_D.
Before the Bayesian framework for the DEJD model is discussed (see Section 4), we provide some basics underlying the very specification of the model in question. The density of logarithmic rates of return, $\ln\left(\frac{S_{t+\Delta}}{S_t}\right)$, is an infinite mixture:

$$\sum_{k=0}^{\infty} \exp\left(-\lambda\Delta\right)\frac{\left(\lambda\Delta\right)^k}{k!} f_k\left(x\right), \tag{2}$$

where $\{f_k\}_{k=0}^{\infty}$ are densities related to distributions of W and Λ (Ramezani and Zeng (2007)). Because the series given by (2) is infinite, the density is intractable. Consider an approximation

$$\sum_{k=0}^{\infty} \exp\left(-\lambda\Delta\right)\frac{\left(\lambda\Delta\right)^k}{k!} f_k \approx \sum_{k=0}^{M} \exp\left(-\lambda\Delta\right)\frac{\left(\lambda\Delta\right)^k}{k!} f_k \tag{3}$$

for some $M > 0$. The approximation restricts the number of jumps over any time interval Δ to at most M. The case of $M = 0$ indicates no jumps over interval Δ.
Let us restrict further considerations to the discrete time framework. Time series $(x_1, x_{2,}...)$ for

$$x_i = \ln\left(\frac{S_{t_{i+1}}}{S_{t_i}}\right)$$

is observed at $(t_1, t_2, \dots)$. Moreover, $\Delta \equiv t_{i+1} - t_i > 0$ is a fixed time interval between following observations. Denote the vector of parameters as $\theta = (\mu, \sigma, \lambda, p_U, \eta_U, \eta_D)$, where $\theta \in \mathbb{R} \times (0, \infty) \times (0, \infty) \times (0, 1) \times (0, \infty) \times (0, \infty)$. If we normalize the approximation given by (3), we obtain the conditional data density (given parameters, θ, and the number of jumps over any time interval Δ, M):

$$p\left(x|\,\theta; M\right) = \sum_{k=0}^{M} w_k f_k\left(x\right), \tag{4}$$

where:

$$w_k = \frac{(\lambda\Delta)^k}{k!}\left[\sum_{j=0}^{M} \frac{(\lambda\Delta)^j}{j!}\right]^{-1}$$

In the remainder of this research we assume $M = 1$, so that

$$p\left(x|\,\theta; M=1\right) = \frac{1}{1+\lambda\Delta} f_X\left(x\right) + \frac{\lambda\Delta}{1+\lambda\Delta} f_{X+Q}\left(x\right), \tag{5}$$

where $f_X = f_0$, $f_{X+Q} = f_1$, $X = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta + \sigma\Delta W_t$, $Q \sim f_Q$ for:

$$f_Q\left(x\right) = p_D \eta_D \exp\left(\eta_D x\right) \mathbb{I}_{(-\infty,0)}\left(x\right) + p_U \eta_U \exp\left(-\eta_U x\right) \mathbb{I}_{[0,\infty)}\left(x\right)$$

The first term on the right-hand side of (5) is referred to as the diffusion component, whereas the second one - the jump-diffusion component. The model in which logarithmic rates of return are assumed to follow the distribution given by (5) is further referred to as the DEJD model.
Note that for the Kou model (Kou (2002)) the density of logarithmic rates of return is given by:

$$p\left(x|\,\theta; M=1; \mathrm{Kou}\right) = \left(1 - \lambda\Delta\right) f_X\left(x\right) + \lambda\Delta f_{X+Q}\left(x\right),$$

for $\lambda\Delta < 1$. It is easy to see that

$$p\left(x \left| \left(\mu, \sigma, \frac{\lambda}{1+\lambda\Delta}, p_U, \eta_U, \eta_D\right); M=1; \mathrm{Kou}\right.\right) = p\left(x|\,\theta; M=1\right),$$

and for $\lambda\Delta < 1$

$$p\left(x|\,\theta; M=1; \mathrm{Kou}\right) = p\left(x \left| \left(\mu, \sigma, \frac{\lambda}{1-\lambda\Delta}, p_U, \eta_U, \eta_D\right); M=1\right.\right).$$

In other words, these two mathematical structures are equivalent (up to the parametrization). Ramezani and Zeng (2007) and Frame and Ramezani (2012) assume that the distribution of logarithmic returns is given by $p\left(x|\,\theta; M=1; \mathrm{Kou}\right)$. They denote such structure by DEJD. Due to the equivalence mentioned above, the process

and model described in my paper are also called DEJD. Note that the process is only the discrete version of the jump-diffusion process.
A weight ratio of the jump-diffusion and the diffusion weight:

$$\frac{\frac{\lambda\Delta}{1+\lambda\Delta}}{\frac{1}{1+\lambda\Delta}} = \lambda\Delta$$

equals the weight ratio:

$$\frac{\exp\left(-\lambda\Delta\right)\lambda\Delta}{\exp\left(-\lambda\Delta\right)} = \lambda\Delta$$

in the original model (2). Further considerations are limited only to the DEJD model. In what follows, for simplicity, density (5) is denoted as $p\left(\cdot\right|\theta)$ rather than $p\left(\cdot\right|\theta; M=1)$.

3 The Bayesian DEJD model

A Bayesian statistical model is defined by the joint density:

$$p\left(x,\theta\right) = p\left(x\left|\theta\right.\right)p\left(\theta\right),$$

where $x = (x_1, \ldots, x_n)$ is the observed data, θ is a vector of unknown parameters, $p\left(x\left|\theta\right.\right)$ is a sampling density and $p\left(\theta\right)$ is a prior density. The inference rests upon the posterior density $p\left(\theta\left|x\right.\right)$ of θ given data x (Bernardo and Smith (2002)). If $x_1, \ldots, x_n$ are mutually independent, then

$$p\left(\theta\left|x\right.\right) = \frac{p\left(x\left|\theta\right.\right)p\left(\theta\right)}{p\left(x\right)} = \frac{p\left(\theta\right)\prod\limits_{i=1}^{n}p\left(x_i\left|\theta\right.\right)}{\int_{\Theta}p\left(\theta\right)\prod\limits_{i=1}^{n}p\left(x_i\left|\theta\right.\right)d\theta}.$$

Given x, $p\left(x\left|\theta\right.\right)$ – as a function of θ – is called the likelihood function, whereas

$$p\left(x\right) = \int_{\Theta}p\left(\theta\right)\prod_{i=1}^{n}p\left(x_i\left|\theta\right.\right)d\theta$$

is the marginal data density, which is constant with respect to θ, so that

$$p\left(\theta\left|x\right.\right) \propto p\left(\theta\right)\prod_{i=1}^{n}p\left(x_i\left|\theta\right.\right).$$

In the present section we set the DEJD model in the Bayesian framework. To facilitate the process, we apply the following reparametrization: $\mu^{'} = \mu - \frac{1}{2}\sigma^2$,

$h = \frac{1}{\sigma^2}$, $L = \lambda\Delta$, so that $\theta = \left(\mu^{'}, h, L, p_U, \eta_U, \eta_D\right)$. When one analyses a time series which is (or, rather, is believed to be) a trajectory of a jump-diffusion process, then one does not actually know if a given data-point observation has been generated by the pure diffusion or the jump-diffusion component. In other words, one cannot determine which component of the series in (5), i.e. $f_X(x)$ or $f_{X+Q}(x)$ is "responsible for" the observation. To manage the problem let us introduce latent variables $\xi = (\xi_1, \ldots, \xi_n)$, where $\xi_i \in \{-1, 0, 1\}$ and $P(\xi_i = -1|\,\theta) = \frac{L}{1+L} p_D$, $P(\xi_i = 0|\,\theta) = \frac{1}{1+L}$, $P(\xi_i = 1|\,\theta) = \frac{L}{1+L} p_U$. The value $\xi_i = 0$ means no jump at $t = i\Delta$. The values $\xi_i = -1$ and $\xi_i = 1$ mean that jump occurs and its value is negative or positive, respectively. Moreover, it is convenient to introduce latent variables $J = (J_1, \ldots, J_n)$ corresponding to the jump value, where

$$\begin{aligned} p\left(J_i = j\,|\theta, \xi_i = -1\right) &= \eta_D \exp\left(\eta_D j\right) \mathbb{I}_{(-\infty,0)}(j), \\ p\left(J_i = j\,|\theta, \xi_i = 0\right) &= \delta_0(j), \\ p\left(J_i = j\,|\theta, \xi_i = 1\right) &= \eta_U \exp\left(-\eta_U j\right) \mathbb{I}_{(0,\infty)}(j), \end{aligned} \tag{6}$$

where $\delta_0(j) = \begin{cases} 1, & j = 0 \\ 0, & j \neq 0 \end{cases}$ is the Kronecker delta. Then, the $(2n+6)$-sized vector of all the unknown quantities is denoted by:

$$(\theta, \xi, J) = \left(\mu^{'}, h, L, p_U, \eta_U, \eta_D, \xi_1, ..., \xi_n, J_1, ..., J_n\right).$$

Moreover,

$$\begin{aligned} p\left(x_i\,|\theta, \xi_i, J_i\right) &= p\left(x_i\,|\theta, J_i\right) = \\ &= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{h}{\Delta}} \exp\left(-\frac{1}{2}\frac{h}{\Delta}\left(x_i - \mu^{'}\Delta - J_i\right)^2\right). \end{aligned} \tag{7}$$

The Bayesian model is given by:

$$\begin{aligned} p\left(x, \theta, \xi, J\right) &= p\left(x\,|\theta, \xi, J\right) p\left(\theta, \xi, J\right) = \\ &= p\left(x\,|\theta, J\right) p\left(\theta, \xi, J\right). \end{aligned}$$

Let the prior structure for (θ, ξ, J) be defined as:

$$\begin{aligned} p\left(\theta, \xi, J\right) = \quad & p\left(\mu^{'}\,|h\right) p(h)\, p(L)\, p(p_U)\, p(\eta_U)\, p(\eta_D) \cdot \\ & \cdot \prod_{i=1}^{n} p\left(J_i\,|\xi_i, \eta_D, \eta_U, L\right) \prod_{i=1}^{n} p\left(\xi_i\,|p_U, L\right), \end{aligned}$$

where

$p(h) = p_G(h; \nu_h, A_h)$, where $p_G(h; a, b) \propto h^{a-1} \exp(-hb)\, \mathbb{I}_{(0,\infty)}(h)$ is the density of a gamma distribution,

$p\left(\mu^{'}\left|h\right.\right)=\phi\left(\mu^{'};\mu_0,(hA_\mu)^{-1}\right)$, where $\phi\left(\mu^{'};m,v\right)$ is the density of a normal distribution with mean m and variance v,

$p\left(L\right)=p_{\chi^2(\nu)}\left(L\right)$, where $p_{\chi^2(\nu)}\left(L\right)\propto L^{\frac{\nu}{2}-1}\exp\left(-\frac{L}{2}\right)\mathbb{I}_{(0,\infty)}\left(L\right)$ is the density of a χ^2 distribution with ν degrees of freedom,

$p\left(\eta_U\right)=p_G\left(\eta_U;\nu_{U,\eta},A_{U,\eta}\right)$, $p\left(\eta_D\right)=p_G\left(\eta_D;\nu_{D,\eta},A_{D,\eta}\right)$,

$p\left(p_U\right)=p_B\left(p_U;a_U,b_U\right)$, where $p_B\left(p_U;\alpha,\beta\right)\propto p_U^{\alpha-1}\left(1-p_U\right)^{\beta-1}$ is the density of a beta distribution,

$P\left(\xi=(l_1,\ldots,l_n)\left|\theta\right.\right)=\Pi_{j\in\{-1,0,1\}}w_j^{n_j}$,
where $n_j=\#\left\{i\in\{1,2,\ldots,n\}:l_i=j\right\}$, $w_{-1}=\frac{L}{1+L}p_D$, $w_0=\frac{1}{1+L}$, $w_1=\frac{L}{1+L}p_U$,

$p\left(J_i=x_i\left|\theta,\xi_i=-1\right.\right)=p_G\left(-x_i;1,\eta_D\right)\propto\exp\left(x_i\eta_D\right)\mathbb{I}_{(-\infty,0)}\left(x_i\right)$,

$p\left(J_i=x_i\left|\theta,\xi_i=1\right.\right)=p_G\left(x_i;1,\eta_U\right)$,

$p\left(J_i=x_i\left|\theta,\xi_i=0\right.\right)=\delta_0$.

Posterior characteristics of the unknown quantities are calculated via the Markov Chain Monte Carlo (MCMC) methods (Gamerman and Lopes (2006)), combining the Gibbs sampler, the independence and the sequential Metropolis-Hastings algorithms, as well as the acceptance-rejection sampling (Chib and Greenberg (1995)). The theorems below make the algorithm ready to use.

Theorem 1 *Under the above assumptions:*

1. $$p\left(\mu^{'},h\left|x,\theta_{\setminus\{\mu^{'},h\}},\xi,J\right.\right)\propto$$
$$p_G\left(h;n/2+\nu_h,\frac{1}{2}\frac{ns}{\Delta}+A_h+\frac{1}{2}\frac{A_\mu n\Delta\left(\mu_0-\frac{\overline{x}-\overline{J}}{\Delta}\right)^2}{A_\mu+n\Delta}\right)\cdot$$
$$\cdot\phi\left(\mu^{'};\frac{\mu_0A_\mu+\left(\overline{x}-\overline{J}\right)n}{A_\mu+n\Delta},\frac{1}{h\left(A_\mu+n\Delta\right)}\right)$$

2. $p\left(L\left|x,\theta_{\setminus L},\xi,J\right.\right)\propto L^{N+\frac{\gamma_L}{2}-1}\exp\left(-\frac{L}{2}\right)\frac{1}{(1+L)^n}$,
where $N=n_{-1}+n_1$, $n_j=\#\left\{i\in\{1,2,...,n\}:l_i=j\right\}$

3. $p\left(p_U\left|x,\theta_{\setminus p_U},\xi,J\right.\right)=p_B\left(p_U;n_1+a_U,n_{-1}+b_U\right)$

4. $p\left((\eta_D,\eta_U)\left|x,\theta_{\setminus(\eta_D,\eta_U)},\xi,J\right.\right)\propto$
$p_G\left(\eta_D;\left(n_{D,\xi}+\nu_{D,\eta}\right),\left(A_{D,\eta}-N_{D,J}\right)\right)\cdot$
$\cdot p_G\left(\eta_U;\left(n_{U,\xi}+\nu_{U,\eta}\right),\left(A_{U,\eta}+N_{U,J}\right)\right)$,
where $N_{D,J}=\sum_{i=1}^n J_i\mathbb{I}_{(-\infty,0)}\left(J_i\right)$, $N_{U,J}=\sum_{i=1}^n J_i\mathbb{I}_{(0,\infty)}\left(J_i\right)$,

5. $p(\xi, J\,|x, \theta) = \prod_{i=1}^{n} p(J_i\,|x_i, \theta, \xi_i)\, p(\xi_i\,|x_i, \theta)$, *where*

   (a) $P(\xi_i = 0|x_i, \theta) = \frac{1}{G}\dfrac{1}{\sigma\sqrt{\Delta}}\phi\left(\dfrac{x_i - \mu^{'}\Delta}{\sigma\sqrt{\Delta}}; 0, 1\right)$

   (b) $P(\xi_i = -1|x_i, \theta) = \dfrac{1}{G}\eta_D \exp\left(\eta_D x_i - \mu^{'}\Delta\eta_D + \frac{1}{2}\sigma^2\Delta\eta_D^2\right)\cdot$
   $\cdot\Phi\left(-\dfrac{x_i - \left(\mu^{'}\Delta - \sigma^2\Delta\eta_D\right)}{\sigma\sqrt{\Delta}}; 0, 1\right) Lp_D,$
   where $\Phi(\cdot; m, v)$ *is the distribution function of a normal distribution with a mean* m *and variance* v.

   (c) $P(\xi_i = 1|x_i, \theta) = \frac{1}{G}\eta_U \exp\left(-\eta_U x_i + \mu^{'}\Delta\eta_U + \frac{1}{2}\sigma^2\Delta\eta_U^2\right)\cdot$
   $\cdot\Phi\left(\dfrac{x_i - \left(\mu^{'}\Delta + \sigma^2\Delta\eta_U\right)}{\sigma\sqrt{\Delta}}; 0, 1\right) Lp_U,$

   (d) $p(J_i = j\,|x_i, \theta, \xi_i = 0) = \delta_0(j),$

   (e) $p(J_i = j\,|x_i, \theta, \xi_i = -1) \propto \phi\left(j; x_i - \mu^{'}\Delta + \frac{\Delta}{h}\eta_D, \frac{\Delta}{h}\right)\mathbb{I}_{(-\infty,0)}(j),$

   (f) $p(J_i = j\,|x_i, \theta, \xi_i = 1) \propto \phi\left(j; x_i - \mu^{'}\Delta - \frac{\Delta}{h}\eta_U, \frac{\Delta}{h}\right)\mathbb{I}_{(0,\infty)}(j)$

   and

   $$G := \frac{1}{\sigma\sqrt{\Delta}}\phi\left(\frac{x_i - \mu^{'}\Delta}{\sigma\sqrt{\Delta}}; 0, 1\right) + \eta_D \exp\left(\eta_D x_i - \mu^{'}\Delta\eta_D + \tfrac{1}{2}\sigma^2\Delta\eta_D^2\right)\cdot$$
   $$\cdot\Phi\left(-\frac{x_i - \left(\mu^{'}\Delta - \sigma^2\Delta\eta_D\right)}{\sigma\sqrt{\Delta}}; 0, 1\right)\cdot Lp_D +$$
   $$+\eta_U \exp\left(-\eta_U x_i + \mu^{'}\Delta\eta_U + \tfrac{1}{2}\sigma^2\Delta\eta_U^2\right)\Phi\left(\frac{x_i - \left(\mu^{'}\Delta + \sigma^2\Delta\eta_U\right)}{\sigma\sqrt{\Delta}}; 0, 1\right)\cdot$$
   $$Lp_U.$$

The Gibbs algorithm rests upon sampling from the full conditional distributions. Since $p\left(\mu^{'}, h\,\middle|x, \theta_{\setminus\{\mu^{'}, h\}}, \xi, J\right)$, $p\left(\eta_D, \eta_U\,|x, \theta_{\setminus(\eta_D, \eta_U)}, \xi, J\right)$ and $p\left(p_U\,|x, \theta_{\setminus p_U}, \xi, J\right)$ are densities of the gamma-normal, gamma and beta distributions, sampling $\mu^{'}$,h, p_U, η_D and η_U is straightforward. Generating latent variabes ξ_i for $i = 1, ..., n$ does not pose a challenge either, as for each i variable ξ_i (given x_i and θ) has a discrete distribution with probabilities given in Theorem 1. Also generating J_i under given x_i, θ and $\xi_i = -1$ or $\xi_i = 1$, is easy because the distributions are truncated normal distributions. Note that if $\xi_i = 0$, then $J_i \equiv 0$. Sampling from

$p\left(L\left|x,\theta_{\setminus L},\xi,J\right.\right)$ is managed according to the following alternative propositions.

Proposition 2 *1. The independent Metropolis-Hastings algorithm with the candidate-generating density* $(2n+1)\,L \sim \chi^2_{2N+\nu_L}$ *and the transition probability:*

$$\min\left\{\exp\left(n\left(L^{(m+1)}-L^{(m)}\right)\right)\left[1+L^{(m+1)}\right]^{-n}\left[1+L^{(m)}\right]^{n},1\right\},$$

from a state $L^{(m)}$ *to* $L^{(m+1)}$ *can be used to sample from* $p\left(L\left|x,\theta_{\setminus L},\xi,J\right.\right)$.

2. If $n-N-\frac{\nu_L}{2}>0$*, then the acceptance-rejection sampling with a proposition density of the gamma-gamma distribution (Bernardo and Smith (2002)):*

$$p(L)=p_{Gg}\left(L;n-N-\frac{\nu_L}{2},1,N+\frac{\nu_L}{2}\right),$$

and the acceptance probability $e^{-L/2}$ *can be used to sample from* $p\left(L\left|x,\theta_{\setminus L},\xi,J\right.\right)$*, where*

$$p_{Gg}(L;\alpha,\beta,n)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}\frac{\Gamma(\alpha+n)}{\Gamma(n)}\frac{L^{n-1}}{(\beta+L)^{\alpha+n}}\mathbb{I}_{(0,\infty)}(L)$$

Γ *is the gamma function,* $\alpha>0,\beta>0,n\in\{1,2,...\}$.

In practice, the condition $n-N-\frac{\nu_L}{2}>0$ is often satisfied.

4 Examples

In this section, we illustrate the methodology developed above. First, the estimation results of the DEJD model parameters for a simulated time series are presented. Subsequently, a few comments on applications of the DEJD structure for the real-world dataset of logarithmic rates of return are made.

All the calculations are performed in R. Numerical algorithms applied in the research require monitoring convergence of the generated chain to its limiting stationary distribution. Convergence of all the MCMC samplers exploited in our research is confirmed by visual inspection of the ergodic means, standard deviations and CUMSUM statistics plots (Yu and Mykland (1998)). The results seem to be robust to the choice of the starting point for the MCMC procedure.

In what follows, two different prior structures are considered, with the hyperparameters of each being displayed in Table 1. Formally, each prior specification defines a different Bayesian model, what yields the two DEJD$_I$, DEJD$_{II}$.

Table 1: Priors structures

Priors	μ'	A_μ	ν_h	A_h	a_U	b_U	$\nu_{U,\eta}$	$A_{U,\eta}$	$\nu_{D,\eta}$	$A_{D,\eta}$	ν_L
I	0.1	1	5	1	1	1	2.56	0.00576	2.56	0.00576	10Δ
II	0	1	5	1	1	1	0.5	1	0.5	1	10Δ

4.1 Simulation case studies

A series of $n = 10,000$ data points generated from the DEJD process is under consideration. Table 2 presents posterior means and standard deviations along with the true values of the parameters. The presented results are based on $100,000$ MCMC draws, preceeded by $100,000$ burn-in cycles.

Table 2: Posterior means and standard deviations for simulation data and the true parameters of the model

Model	DEJD_I		DEJD_{II}		True
θ	$E(\cdot\|x)$	$D(\cdot\|x)$	$E(\cdot\|x)$	$D(\cdot\|x)$	θ
μ	0.3262	0.0973	0.4632	0.0781	0.25
σ	0.3972	0.0043	0.4039	0.0039	0.4
p_U	0.4835	0.0666	0.3055	0.0526	0.5
η_D	5.3202	0.2778	5.1647	0.2721	5
η_U	30.6779	3.7369	19.2997	2.6807	30
λ	30.6419	4.8556	21.3400	2.0318	30

The posterior means of DEJD_I parameters are close to the true values. The posterior expectations of μ, η_U and λ, calculated under DEJD_{II}, differ substantially from the prespecified values. Values $E\left(\frac{1}{\eta_U}|x\right)$ and $E\left(\frac{1}{\eta_D}|x\right)$ are the posterior means of negative and positive jumps values, respectively. The value of $E\left(\frac{1}{\eta_U}|x\right)$ calculated under prior II is greater than the one obtained under prior I, so the positive jumps are (on average) greater under prior II. Note that the probability of positive jumps p_U and the jump intensity λ are greater for DEJD_I. Hence, one can expect that the number of detected positive jumps is lower for prior II, so the role of the jump component, under the DEJD_{II} framework, is smaller than in the case of prior I. It seems to be supported by a greater value of the trend parameter, μ, in DEJD_{II}. Figure 1 displays the marginal posteriors of parameters in the DEJD_I model, along with the prior densities. The prior distributions of η_D and η_U allow for large values of parameters. The data move the posterior probability to the left of the prior mode and the posterior means of the parameters stay close to the true values specified for η_D and η_U.

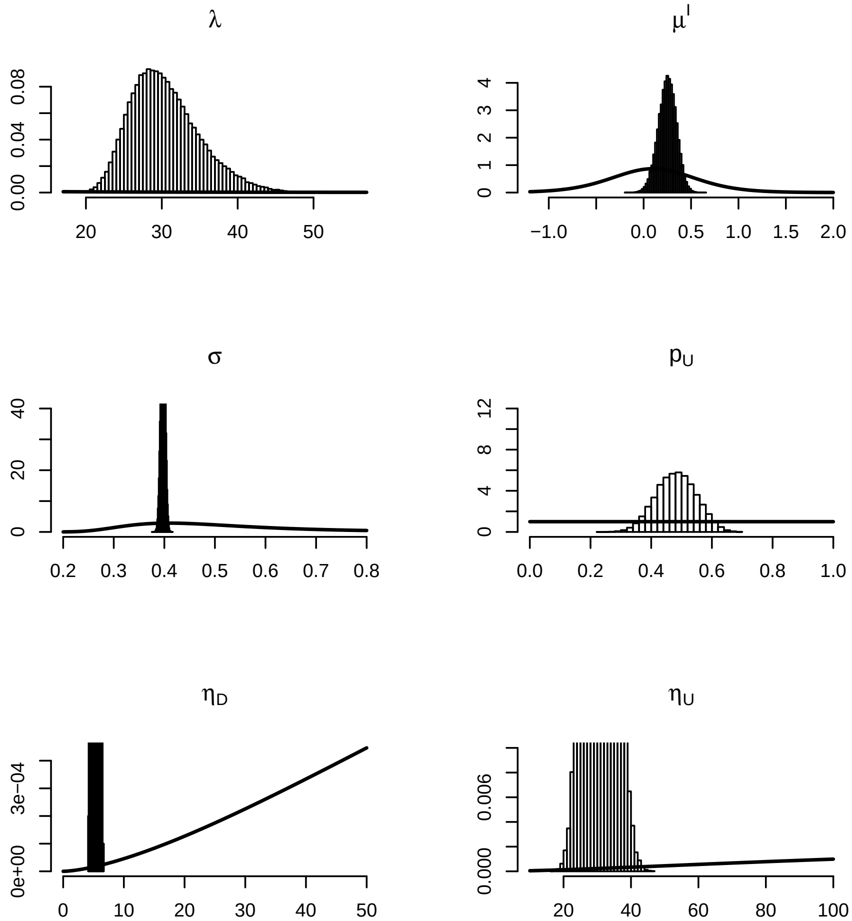
Figure 2 displays the marginal posteriors of parameters in the DEJD_{II} model, along with the prior densities. The plots reveal a considerable contribution of the data to

the shape of marginal posteriors. The prior specifications for η_D and η_U support lower values of the parameters. However, the data move the posterior distributions to the right (into the right tails) towards the true values.
Only in the case of prior I, the data were strong enough to move the posteriors of η_D and η_U close to the true values. We observe the impact of the inverse gamma prior distribution parameters upon the posterior distribution. The parameter prior impact on the posterior results was also observed in the case of a normal jump distribution (the JD(M)J model, Kostrzewski (2014a)).

4.2 Discussion on applications of the DEJD model

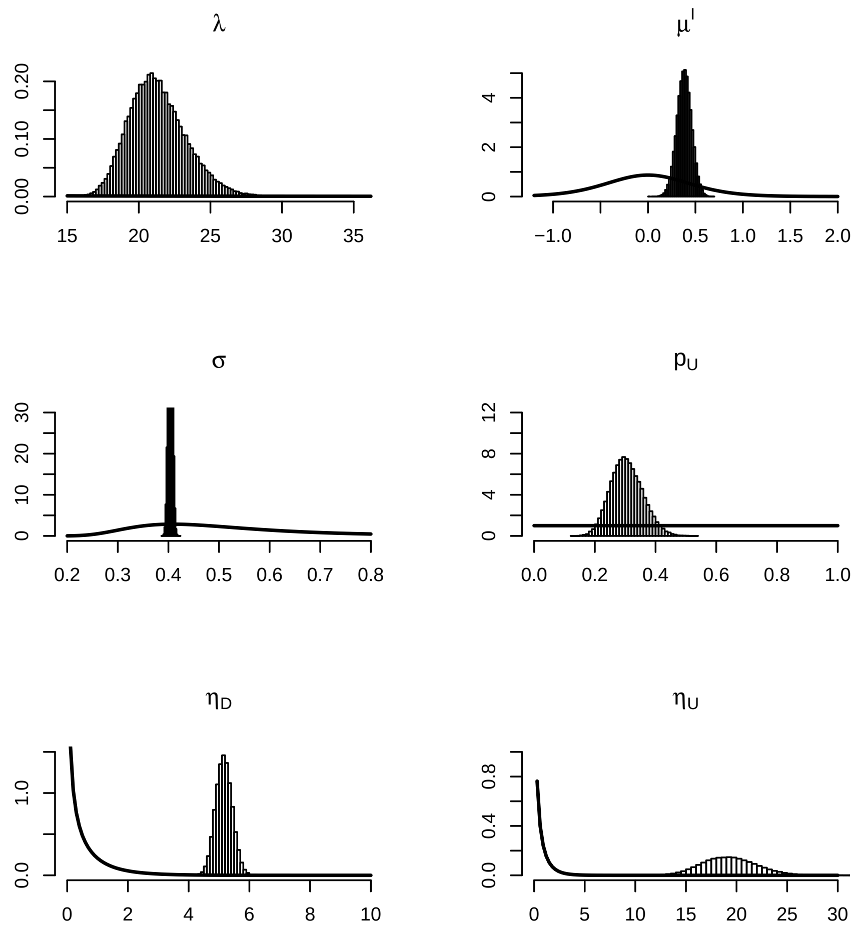
Jumps in time series are often defined as the values exceeding some arbitrarily choosen thresholds. Different thresholds lead to various number of jumps (Weron (2008)). Thresholds are commonly set symmetrically either around zero or the sample mean, and are defined as a multiply of the sample standard deviation. If the empirical distribution of time series features e.g. a negative skewness, then symmetric thresholds do not seem valid.
The latent variables ξ_i might be used to identify data points with a jump. Formally, an event of jump appearance is equivalent to $\xi_i = -1$ or $\xi_i = 1$. Unfortunately, one does not observe ξ_i, but we can assess the posterior probability of a jump: $P(\xi_i \neq 0 \,|x)$ for each day $i = 1, \ldots, n$. Let us assume that a jump occurs at the i-th period if probability $P(\xi_i \neq 0 \,|x)$ exceeds an arbitrarily choosen value of 0.5, which corresponds to the aforementioned thresholds. However, the problem of asymmetry or symmetry is not a matter here.
Note that independence of observations is assumed under the DEJD framework. This assumption is difficult to accept in the case of logarithmic returns of financial data. Regardless of not fulfilling the assumption the model was applied to a few time series of logarithmic rates of return for which the independence was rejected. The following observations and conclusions based on detecting and investigating jumps correspond to some expectations and intuitions. It was observed that the values found as jumps were values with the highest absolute values, so they might be treated as extreme values. The frequency of jumps was low. This observation conforms with the expectation that (Poisson's) jumps are sporadic events. Moreover, it was noted that higher posterior probabilities of jumps went along with higher volatility of the time series.
A time series of the daily logarithmic growth rates of the KGHM quotations on the Warsaw Stock Exchange from January 23, 2006 to February 22, 2010 was also considered. KGHM is a copper producer and one of the largest Polish exporters. In the case of this series it was noticed that thresholds, which distinguish between "small" movements and jumps, were symmetric neither around zero nor the sample mean. There were more negative jumps than the positive ones, which might correspond with a fatter left-hand side tail of the sample distribution.
Moreover, the series of daily logarithmic rates of return on the S&P100 Index over the

Figure 1: Marginal posterior (bars) and prior densities (solid line) of parameters in the DEJD$_I$ model



period from March 5, 1984 through July 8, 1997, and the daily logarithmic rates of return on the closing prices of the ICE ECX future contracts expiring on 16 December, 2013, over the period 3 January, 2011 through 1 October, 2013 were also considered (Kostrzewski (2014c)). In both cases periods of no jumps alternated with the ones of frequent jumps, which hinted at the existence of jump clustering (the same conclusion was drawn under the JD(M)J specifications in Kostrzewski (2012b), Kostrzewski (2014c), Kostrzewski (2014b)). The very term "jump clustering" is analogous to the

Figure 2: Marginal posterior (bars) and prior densities (solid line) of parameters in the DEJD$_{II}$ model



one of "volatility clustering" and means that waiting times between two consecutive jumps tend to cluster.

The results presented above appear quite reasonable and ecourage one to think about whether they might be acceptable also for dependent observations. If the assumption of independence of logarithmic returns is rejected, then we can often find a permutation of the time series for which the hypothesis of independence is not rejected. Let us assume that we have the series of independent observations which is

the permutation of the original time series. Then, we might apply the Bayesian DEJD model for the new time series according to the theory defined above. We could also look at the DEJD model as the mixture model defined by (5) and applied for the set of data rather than the time series. Note that the technique of inference is the same for any permutation of the data and is a consequence of the commutative property of multiplication (the likelihood function is the product of densities). Therefore, formally, the method defined above is able to divide the permuted data into two main parts which might be labelled as "small" values and "jumps". The "jumps" are realizations of f_{X+Q}. Then, the "jumps" are further divided into "negative jumps" and "positive jumps". Finally, if we reorder data to the original order we might observe clusters of jumps. The mathematical model considered here does not feature any structure to capture phenomeon of jump clustering explicitly. Nevertheless, the data still might be strong enough to display this phenomeon.
However, it is crucial to test whether inference on the number, size and asymmetry of jumps is really robust with regard to different forms of dependence, characterizing real-world return data and modelled using mainly GARCH or SV processes.
According to the above argumentation, the DEJD structure seems to be still useful in practice to preliminary investigation of jumps occurence in financial time series.

5 Conclusions

In the paper, the Bayesian DEJD model is developed. To employ the model in practice, numerical techniques based on the MCMC methods are proposed. The Bayesian statistics equipped with the MCMC mehods gives us an easy way of estimating parameters of the DEJD model. The methodology is illustrated with a simulation experiment. Latent variables enable detection of negative and positive jumps and analysis of their frequency and distributions. Unfortunately, the results might hinge on the prior assumptions. This feature is commonly observed in models based on mixture distributions (Frühwirth-Schnatter (2006), Johannes and Polson (2010)).
In the Merton model, the JD(M)J model and the Bernoulli jump-diffusion model the jump value distributions are normal. If a mean of normal distribution is not equal zero, than the distribution is asymmetric with respect to zero, so the jump-diffusion model with a double exponential distribution and its discrete approximations, such as the DEJD model, constitute only alternative tools of modeling asymmetric jumps. The focus of future research will be placed upon specifications with stochastic jump intensity, which would enhance the model structure so as to account for jump clustering explicitly. Moreover, further studies will concentrate on jump clustering under stochastic volatility framework in order to capture the dependence of observed returns.

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References

[1] Ait-Sahalia Y. and Jacod J. (2012): Analyzing the Spectrum of Asset Returns: Jump and Volatility Components in High Frequency Data, *Journal of Economic Literature*, 50, 1007–1050.

[2] Ane T. and Metais C. (2010): Jump Distribution Characteristics: Evidence from European Stock Markets, *International Journal of Business and Economics*, 9(1), 1–22.

[3] Ball C. and Torous W. (1983): A simplified Jump Process for Common Stock Returns, *Journal of Financial and Quantitative Analysis*, 1(18), 53–65.

[4] Barndorff-Nielsen O. and Shephard N. (2004): Power and Bipower Variation with Stochastic Volatility and Jumps, *Journal of Financial Econometrics*, 2, 1–37.

[5] Barndorff-Nielsen O. and Shephard N. (2006a): Econometrics of Testing for Jumps in Financial Economics Using Bipower Variation, *Journal of Financial Econometrics*, 4, 1–30.

[6] Barndorff-Nielsen O. and Shephard N. (2006b): Impact of Jumps on Returns and Realised Variances: Econometric Analysis of Time-Deformed Levy Processes, *Journal of Econometrics*, 131, 217–252.

[7] Bernardo J. and Smith A. (2002): *Bayesian Theory*. Wiley Series in Probability and Statistics.

[8] Będowska-Sójka B. (2012): Jumps in Stock Returns: Evidence from the Polish Stock Exchangepp. 121–135, no. 10 [in:] *FindEcon Monograph Series: Advances in Financial Market Analysis*. Lodz University Press.

[9] Black F. and Scholes M. (1973): The Pricing of Options and Corporate Liabilities, *Journal of Political Economy*, 81(3), 637–654.

[10] Chib S. and Greenberg E. (1995): Understanding the Metropolis-Hastings Algorithm, *The American Statistician*, 49, 327–335.

[11] Frame S. and Ramezani C. (2012): Bayesian Estimation of Asymmetric Jump-Diffusion Processes, *Working paper*. http://cyrus.cob.calpoly.edu/MCMC/Frame- Ramezani-Bayesian%20Estimation %20of%20Asymmetric%20Jump-Diffusion%20Processes%20.pdf.

[12] Frühwirth-Schnatter S. (2006): *Finite Mixtures and Markov Switching Models*, Springer Series in Statistics. New York/Berlin/Heidelberg.

[13] Gamerman D. and Lopes H. (2006): *Markov Chain Monte Carlo. Stochastic Simulation for Bayesian Inference*. Chapman & Hall/CRC.

[14] Hanson F. and Westman J. (2002): Stochastic Analysis of Jump- Diffusions for Financial Log-Return Processes, [in:] *Stochastic Theory and Control: Proceedings of a Workshop held in Lawrence, Kansas*, [ed.] Pasik-Duncan B., vol. 280 of Lecture Notes in Control and Information Sciences, pp. 169–184. Springer-Verlag, New York.

[15] Hanson F., Westman J. and Zhu Z. (2004): Multinomial maximum likelihood estimation of market parameters for stock jump-diffusion models, [in:] *Mathematics of Finance. Proceedings of the 2003 AMS-IMS-SIAM Joint Summer Research Seminar in Applied Mathematics*, [ed.] Yin G. and Zhang Q., vol. 351 of Contemporary Mathematics, pp. 155–169. American Mathematical Society, Providence, RI.

[16] Honore P. (1998): Pitfalls in Estimating Jump-Diffusion Models, *Working Paper* Series no 18. The Aarhus School of Business. http://www.stat.purdue.edu/figueroa/Teaching/Stat598F/HonoreJmpDffsEst.pdf

[17] Jackwerth J. and Rubinstein M. (1996): Recovering Probability Distributions from Contemporary Security Prices, *Journal of Finance*, 51, 347–369.

[18] Johannes M. and Polson N. (2010): MCMC Methods for Continuous- Time Financial Econometrics, [in:] *Handbook of Financial Econometrics*, [ed.] Ait-Sahalia Y. and Hansen L., vol. 2, pp. 1–72. North Holland.

[19] Kostrzewski M. (2012a): Bayesian Pricing of the Optimal-Replication Strategy for European Option in the JD(M)J Model, *Dynamic Econometric Models*, 12, 53–71.

[20] Kostrzewski M. (2012b): On the Existence of Jumps in Financial Time Series, *Acta Physica Polonica B*, 43, 2001–2019, DOI: 10.5506/APhysPolB.43.2001.

[21] Kostrzewski M. (2014a): Bayesian Inference for the Jump-Diffusion Model with M Jumps, *Communications in Statistics – Theory and Methods*, 43, 3955–3985, DOI: 10.1080/03610926.2012.755202.

[22] Kostrzewski M. (2014b): The Hawkes Process and Time-Varying Jump Intensity in Financial Time Series, [in:] *8-th International Days of Statistics and Economics*, [ed.] Loster T. and Pavelka T., pp. 743–754, Prague, Czech Republic. Melandrium, http://msed.vse.cz/msed 2014/article/324-Kostrzewski-Maciej -paper.pdf

[23] Kostrzewski M. (2014c): Some Method of Detecting the Jump Clustering Phenomenon in Financial Time Series, [in:] *17-th International Scientific Conference Applications of Mathematics and Statistics in Economics*, [ed.] Rusnak Z. and Zmyślona B., pp. 141–150, Wrocław, Poland. Wrocław University of Economics, DOI: 10.15611/amse.2014.17.15.

[24] Kou S. (2002): A jump-diffusion model for option pricing. *Management Science*, 48, 1086–1101.

[25] Lee S. (2012): Jumps and Information Flow in Financial Markets, *Review of Financial Studies*, 25, 311–344.

[26] Lin S. and Huang M. (2002): Estimating Jump-Diffusion Models Using the MCMC Simulation, *Working paper 0215E*. National Tsing Hua University Department of Economics NTHU Working Paper Series.

[27] Merton R. (1976): Option pricing when underlying stock return rates are discontinuous, *Journal of Financial Economics*, 3, 125–144.

[28] Milgrom P. (1981): Good news and bad news: Representation theorems and applications, *Bell Journal of Economics*, 12, 380–391.

[29] Piazzesi M. (2005): Bond Yields and the Federal Reserve, *Journal of Political Economy*, 113, 311–344.

[30] Ramezani C. and Zeng Y. (2007): Maximum likelihood estimation of the double exponential jump-diffusion process, *Annals of Finance*, 3, 487–507.

[31] Ramezani C. and Zeng Y. (1998): Maximum Likelihood Estimation of Asymmetric Jump-Diffusion Models, *Working paper* http://cyrus.cob.calpoly.edu/JUMPMLE/Ramezani-Zeng-PBJD.pdf

[32] Rifo L. and Torres S. (2009): Full Bayesian Analysis for a Class of Jump-Diffusion Models, *Communications in Statistics – Theory and Methods*, 38, 1262–1271.

[33] Synowiec D. (2008): Jump-diffusion models with constant parameters for financial log-return processes, *Computers and Mathematics with Applications*, 56, 2120–2127.

[34] Weron R. (2008): Market price of risk implied by Asian-style electricity options and futures, *Energy Economics*, 30, 1098–1115.

[35] Yu B. and Mykland P. (1998): Looking at Markov Samplers Through Cumsum Path Plots: A Simple Diagnostic Idea, *Statistics and Computing*, 8, 275–286.

[36] Yu J. (2007): Closed-form likelihood approximation and estimation of jump-diffusions with an application to the realignment risk of the Chinese yuan, *Journal of Econometrics*, 141, 1245–1280.

Appendix

Lemma 3 *Under the conditions stated in the paper, the likelihood function is given by*

$$p\left(x\left|\theta,\xi,J\right.\right)=h^{n/2}\exp\left(-\frac{1}{2}h\left\{\frac{ns}{\Delta}+n\Delta\left(\frac{\overline{x}-\overline{J}}{\Delta}-\mu^{'}\right)^{2}\right\}\right),\tag{8}$$

where

$$s=\frac{1}{n}\sum_{i=1}^{n}\left(x_i-J_i-\left(\overline{x}-\overline{J}\right)\right)^2,$$

$$\overline{x}=\frac{1}{n}\sum_{i=1}^{n}x_i,\ \overline{J}=\frac{1}{n}\sum_{i=1}^{n}J_i.$$

Proof 4 *From the independence of x_i's and (7) it may be concluded that*

$$p\left(x\left|\theta,\xi,J\right.\right)=\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}}\sqrt{\frac{h}{\Delta_i}}\exp\left(-\frac{1}{2}\frac{h}{\Delta}\left(x_i-\mu^{'}\Delta-J_i\right)^2\right)$$
$$\propto h^{n/2}\exp\left(-\frac{1}{2}\frac{h}{\Delta}\sum_{i=1}^{n}\left(x_i-J_i-\mu^{'}\Delta\right)^2\right).$$

Some algebraic calculations lead to the following formula:

$$\sum_{i=1}^{n}\left(x_i-J_i-\mu^{'}\Delta\right)^2=ns+n\left(\left(\overline{x}-\overline{J}\right)-\mu^{'}\Delta\right)^2.$$

It follows that

$$\prod_{i=1}^{n}p\left(x_i\left|\theta,J_i\right.\right)=h^{n/2}\exp\left(-\frac{1}{2}h\left\{\frac{ns}{\Delta}+n\Delta\left(\frac{\overline{x}-\overline{J}}{\Delta}-\mu^{'}\right)^{2}\right\}\right).$$

Lemma 5 *Under the conditions stated in the paper,*

1.
$$A_\mu\left(\mu_0-\mu^{'}\right)^2+n\Delta\left(\frac{\overline{x}-\overline{J}}{\Delta}-\mu^{'}\right)^2=\frac{A_\mu n\Delta\left(\mu_0-\frac{\overline{x}-\overline{J}}{\Delta}\right)^2}{A_\mu+n\Delta}.\tag{9}$$

2.
$$\int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{\Delta}}\phi\left(\frac{z-\mu^{'}\Delta}{\sigma\sqrt{\Delta}};0,1\right)\eta_D\exp\left(\eta_D\left(x-z\right)\right)\mathbb{I}_{\left(-\infty,0\right)}\left(x-z\right)dz=\tag{10}$$
$$=\eta_D\exp\left(\eta_D x-\mu^{'}\Delta\eta_D+\frac{1}{2}\sigma^2\Delta\eta_D^2\right)\Phi\left(-\frac{x-\left(\mu^{'}\Delta-\sigma^2\Delta\eta_D\right)}{\sigma\sqrt{\Delta}};0,1\right),$$

where $\phi(\cdot; m, v)$ and $\Phi(\cdot; m, v)$ are the density and the cumulative distribution of the normal distribution $N(m, v)$, respectively.

3.

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{\Delta}} \phi\left(\frac{z - \mu^{'}\Delta}{\sigma\sqrt{\Delta}}; 0, 1\right) \eta_U \exp\left(-\eta_U (x - z)\right) \mathbb{I}_{[0,\infty)}(x - z)\, dz = \tag{11}$$

$$= \eta_U \exp\left(-\eta_U x\right) \exp\left(\mu^{'}\Delta\eta_U + \frac{1}{2}\sigma^2\Delta\eta_U^2\right) \Phi\left(\frac{x - \left(\mu^{'}\Delta + \sigma^2\Delta\eta_U\right)}{\sigma\sqrt{\Delta}}; 0, 1\right).$$

4.

$$\sqrt{\frac{h}{\Delta}} \phi\left(\sqrt{\frac{h}{\Delta}}\left(x_i - \mu^{'}\Delta - j\right); 0, 1\right) \eta_D \exp\left(\eta_D j\right) \mathbb{I}_{(-\infty,0)}(j) = \tag{12}$$

$$= C \exp\left(-\frac{1}{2}\frac{h}{\Delta}\left(j - \left[\left(x_i - \mu^{'}\Delta\right) + \frac{\Delta}{h}\eta_D\right]\right)^2\right) \mathbb{I}_{(-\infty,0)}(j),$$

where C does not depend on j.

5.

$$\sqrt{\frac{h}{\Delta}} \phi\left(\sqrt{\frac{h}{\Delta}}\left(x_i - \mu^{'}\Delta - j\right); 0, 1\right) \eta_U \exp\left(-\eta_U j\right) \mathbb{I}_{(0,\infty)}(j) = \tag{13}$$

$$= C \exp\left(-\frac{1}{2}\frac{h}{\Delta}\left[\left(j - \left[\left(x_i - \mu^{'}\Delta\right) - \frac{\Delta}{h}\eta_U\right]\right)^2\right]\right) \mathbb{I}_{(0,\infty)}(j),$$

where C does not depend on j.

Proof 6 *Tedious, but simple calculations lead to the claims.*

Theorem 7 *Under the conditions stated in the paper,*

1. $p\left(\mu^{'}, h \,\middle|\, x, \theta_{\setminus\{\mu^{'},h\}}, \xi, J\right) \propto$

$$p_G\left(h; n/2 + \nu_h, \tfrac{1}{2}\frac{ns}{\Delta} + A_h + \frac{1}{2}\frac{A_\mu n\Delta\left(\mu_0 - \frac{\overline{x} - \overline{J}}{\Delta}\right)^2}{A_\mu + n\Delta}\right)\cdot$$

$$\cdot\phi\left(\mu^{'}; \frac{\mu_0 A_\mu + \left(\overline{x} - \overline{J}\right) n}{A_\mu + n\Delta}, \frac{1}{h\left(A_\mu + n\Delta\right)}\right)$$

2. $p\left(L\left|x,\theta_{\setminus L},\xi,J\right.\right)\propto L^{N+\frac{\gamma_L}{2}-1}\exp\left(-\frac{L}{2}\right)\frac{1}{(1+L)^n}$,
where $N=n_{-1}+n_1$, $n_j=\#\left\{i\in\{1,2,...,n\}:l_i=j\right\}$

3. $p\left(p_U\left|x,\theta_{\setminus p_U},\xi,J\right.\right)=p_B\left(p_U;n_1+a_U,n_{-1}+b_U\right)$

4. $p\left((\eta_D,\eta_U)\left|x,\theta_{\setminus(\eta_D,\eta_U)},\xi,J\right.\right)\propto$
$p_G\left(\eta_D;\left(n_{D,\xi}+\nu_{D,\eta}\right),\left(A_{D,\eta}-N_{D,J}\right)\right)\cdot$
$\cdot p_G\left(\eta_U;\left(n_{U,\xi}+\nu_{U,\eta}\right),\left(A_{U,\eta}+N_{U,J}\right)\right)$,
where $N_{D,J}=\sum_{i=1}^n J_i\mathbb{I}_{(-\infty,0)}\left(J_i\right)$, $N_{U,J}=\sum_{i=1}^n J_i\mathbb{I}_{(0,\infty)}\left(J_i\right)$,

5. $p\left(\xi,J\left|x,\theta\right.\right)=\prod_{i=1}^n p\left(J_i\left|x_i,\theta,\xi_i\right.\right)p\left(\xi_i\left|x_i,\theta\right.\right)$,
where

   (a) $P\left(\xi_i=0|x_i,\theta\right)=\frac{1}{G}\frac{1}{\sigma\sqrt{\Delta}}\phi\left(\frac{x_i-\mu^{'}\Delta}{\sigma\sqrt{\Delta}};0,1\right)$

   (b) $P\left(\xi_i=-1|x_i,\theta\right)=\frac{1}{G}\eta_D\exp\left(\eta_D x_i-\mu^{'}\Delta\eta_D+\frac{1}{2}\sigma^2\Delta\eta_D^2\right)\cdot$
   $\cdot\Phi\left(-\frac{x_i-\left(\mu^{'}\Delta-\sigma^2\Delta\eta_D\right)}{\sigma\sqrt{\Delta}};0,1\right)Lp_D$,
   where $\Phi\left(\cdot;m,v\right)$ *is the distribution function of a normal distribution with a mean* m *and variance* v.

   (c) $P\left(\xi_i=1|x_i,\theta\right)=\frac{1}{G}\eta_U\exp\left(-\eta_U x_i+\mu^{'}\Delta\eta_U+\frac{1}{2}\sigma^2\Delta\eta_U^2\right)\cdot$
   $\cdot\Phi\left(\frac{x_i-\left(\mu^{'}\Delta+\sigma^2\Delta\eta_U\right)}{\sigma\sqrt{\Delta}};0,1\right)Lp_U$,

   (d) $p\left(J_i=j\left|x_i,\theta,\xi_i=0\right.\right)=\delta_0\left(j\right)$,

   (e) $p\left(J_i=j\left|x_i,\theta,\xi_i=-1\right.\right)\propto\phi\left(j;x_i-\mu^{'}\Delta+\frac{\Delta}{h}\eta_D,\frac{\Delta}{h}\right)\mathbb{I}_{(-\infty,0)}\left(j\right)$,

   (f) $p\left(J_i=j\left|x_i,\theta,\xi_i=1\right.\right)\propto\phi\left(j;x_i-\mu^{'}\Delta-\frac{\Delta}{h}\eta_U,\frac{\Delta}{h}\right)\mathbb{I}_{(0,\infty)}\left(j\right)$

   and

$$G:=\frac{1}{\sigma\sqrt{\Delta}}\phi\left(\frac{x_i-\mu^{'}\Delta}{\sigma\sqrt{\Delta}};0,1\right)+\eta_D\exp\left(\eta_D x_i-\mu^{'}\Delta\eta_D+\frac{1}{2}\sigma^2\Delta\eta_D^2\right)\cdot$$
$$\cdot\Phi\left(-\frac{x_i-\left(\mu^{'}\Delta-\sigma^2\Delta\eta_D\right)}{\sigma\sqrt{\Delta}};0,1\right)\cdot Lp_D+$$
$$+\eta_U\exp\left(-\eta_U x_i+\mu^{'}\Delta\eta_U+\frac{1}{2}\sigma^2\Delta\eta_U^2\right)\Phi\left(\frac{x_i-\left(\mu^{'}\Delta+\sigma^2\Delta\eta_U\right)}{\sigma\sqrt{\Delta}};0,1\right)\cdot$$
$$Lp_U.$$

Proof 8 *The aim is to calculate the full conditional posteriors of unknown parameters.*

1) By the prior assumptions:

$$\begin{aligned} p\left(\mu^{'}, h\right) &= p\left(\mu^{'} \left| h\right.\right) p\left(h\right) \\ &\propto \frac{1}{\sqrt{2\pi}}\sqrt{hA_\mu}\exp\left(-\frac{1}{2}hA_\mu\left(\mu-\mu_0\right)^2\right)h^{\nu_h-1}\exp\left(-hA_h\right) \end{aligned}$$

The full conditional distribution of $\left(\mu^{'}, h\right)$ *is given by:*

$$p\left(\mu^{'}, h\left|x, \theta_{\setminus\{\mu^{'},h\}}, \xi, J\right.\right) \propto p\left(x\left|\theta, J\right.\right) p\left(\mu^{'}, h\right).$$

From (8) it follows that

$$\begin{aligned} p\left(\mu^{'}, h\left|x, \theta_{\setminus\{\mu^{'},h\}}, \xi, J\right.\right) &\propto h^{n/2+\nu_h-1/2}\exp\left(-h\left\{\frac{1}{2}\frac{ns}{\Delta}+A_h\right\}\right) \\ &\cdot\exp\left(-\frac{1}{2}h\left[A_\mu\left(\mu_0-\mu^{'}\right)^2+n\Delta\left(\frac{\overline{x}-\overline{J}}{\Delta}-\mu^{'}\right)^2\right]\right). \end{aligned}$$

From (9) it may be concluded that

$$\begin{aligned} p\left(\mu^{'}, h\left|x, \theta_{\setminus\{\mu^{'},h\}}, \xi, J\right.\right) &\propto h^{n/2+\nu_h-1}\exp\left(-h\left\{\frac{1}{2}\frac{ns}{\Delta}+A_h+\frac{1}{2}\frac{A_\mu n\Delta\left(\mu_0-\frac{\overline{x}-\overline{J}}{\Delta}\right)^2}{A_\mu+n\Delta}\right\}\right) \\ &\cdot h^{1/2}\exp\left(-\frac{1}{2}h\left(A_\mu+n\Delta\right)\left[\mu^{'}-\frac{\mu_0A_\mu+\left(\overline{x}-\overline{J}\right)n}{A_\mu+n\Delta}\right]^2\right). \end{aligned}$$

2) From the formulas of marginal distributions:

$$\begin{aligned} p\left(L\left|x, \theta_{\setminus L}, \xi, J\right.\right) &= \frac{p\left(x\left|\theta, \xi, J\right.\right)p\left(L\left|\theta_{\setminus L}, \xi, J\right.\right)}{p\left(x\left|\theta_{\setminus L}, \xi, J\right.\right)} \\ &\propto p\left(x\left|\theta, \xi, J\right.\right)p\left(L\left|\theta_{\setminus L}, \xi, J\right.\right) \\ &\propto p\left(x\left|\theta, \xi, J\right.\right)p\left(J\left|\theta, \xi\right.\right)p\left(L\left|\theta_{\setminus L}, \xi\right.\right) \\ &\propto p\left(J\left|\theta, \xi\right.\right)p\left(L\left|\theta_{\setminus L}, \xi\right.\right) \end{aligned}$$

$$p\left(L\left|\theta_{\setminus L}, \xi=\left(l_1, ..., l_n\right)\right.\right) \propto p\left(L\right)\left[1+L\right]^{-n}\Pi_{j=-1}^{1}L^{|j|n_j}.$$

By the independence

$$p\left(L\left|x, \theta_{\setminus L}, \xi=\left(l_1, ..., l_n\right), J\right.\right) \propto \prod_{i=1}^{n}p\left(J_i\left|\theta, \xi_i\right.\right)p\left(L\right)\left[1+L\right]^{-n}\Pi_{j=-1}^{1}L^{|j|n_j}.$$

By the independence and from

$$p\left(J_i = j\left|\theta, \xi_i\right.\right) = = p\left(J_i = j\left|\eta_D, \eta_U, \xi_i\right.\right) \tag{14}$$

$$= \delta_0\left(j\right)\mathbb{I}_{\{0\}}\left(\xi_i\right) + \eta_D \exp\left(\eta_D j\right)\mathbb{I}_{(-\infty,0)}\left(j\right)\mathbb{I}_{\{-1\}}\left(\xi_i\right) + + \eta_U \exp\left(-\eta_U j\right)\mathbb{I}_{(0,\infty)}\left(j\right)\mathbb{I}_{\{1\}}\left(\xi_i\right) \tag{15}$$

it follows that

$$p\left(L\left|x, \theta_{\setminus L}, \xi = \left(l_1, ..., l_n\right), J\right.\right) \propto p\left(L\right)\frac{L^N}{\left(1+L\right)^n} \propto L^{N+\frac{\gamma_L}{2}-1}\exp\left(-\frac{L}{2}\right)\frac{1}{\left(1+L\right)^n}.$$

3) Combining (14) and the prior assumptions: $p\left(p_U\left|\theta_{\setminus p_U}\right.\right) = p\left(p_U\right)$ *and* $p\left(\xi\left|\theta\right.\right) = p\left(\xi\left|p_U, L\right.\right)$*, gives*

$$\begin{aligned} p\left(p_U\left|x, \theta_{\setminus p_U}, \xi, J\right.\right) &= p\left(p_U\left|\theta_{\setminus p_U}, \xi, J\right.\right) = \\ &= \frac{p\left(J\left|\theta, \xi\right.\right)p\left(p_U\left|\theta_{\setminus p_U}, \xi\right.\right)}{p\left(J\left|\theta_{\setminus p_U}, \xi\right.\right)} = \\ &= p\left(p_U\left|\theta_{\setminus p_U}, \xi\right.\right) = \\ &= \frac{p\left(\xi\left|\theta\right.\right)p\left(p_U\left|\theta_{\setminus p_U}\right.\right)}{p\left(\xi\left|\theta_{\setminus p_U}\right.\right)} \propto p\left(\xi\left|p_U, L\right.\right)p\left(p_U\right). \end{aligned}$$

Hence,

$$\begin{aligned} p\left(p_U\left|x, \theta_{\setminus p_U}, \xi = \left(l_1, ..., l_n\right), J\right.\right) &\propto p\left(p_U\right)\Pi_{j=-1}^{1} w_j^{n_j} \\ &\propto p\left(p_U\right)p_D^{n_{-1}}p_U^{n_1} \\ &\propto \left(1-p_U\right)^{n_{-1}+b_U-1}p_U^{n_1+a_U-1}. \end{aligned}$$

4) Note that

$$\begin{aligned} p\left(\eta_U, \eta_D\left|x, \theta_{\setminus(\eta_U,\eta_D)}, \xi, J\right.\right) &= \frac{p\left(\eta_U, \eta_D, x\left|\theta_{\setminus(\eta_U,\eta_D)}, \xi, J\right.\right)}{p\left(x\left|\theta_{\setminus(\eta_U,\eta_D)}, \xi, J\right.\right)} = \\ &= \frac{p\left(x\left|\theta, \xi, J\right.\right)p\left(\eta_U, \eta_D\left|\theta_{\setminus(\eta_U,\eta_D)}, \xi, J\right.\right)}{p\left(x\left|\theta_{\setminus(\eta_U,\eta_D)}, \xi, J\right.\right)} \end{aligned}$$

Because $p\left(x\left|\theta, \xi, J\right.\right) = p\left(x\left|\theta_{\setminus(\eta_U,\eta_D)}, \xi, J\right.\right)$*, therefore*

$$\begin{aligned} p\left(\eta_U, \eta_D\left|x, \theta_{\setminus(\eta_U,\eta_D)}, \xi, J\right.\right) &= p\left(\eta_U, \eta_D\left|\theta_{\setminus(\eta_U,\eta_D)}, \xi, J\right.\right) \\ &\propto p\left(J|\theta, \xi\right)p\left(\xi|\theta\right)p\left(\eta_U, \eta_D|\theta_{\setminus\eta_U,\eta_D}\right) \end{aligned}$$

By the prior assumption $p\left(\eta_U, \eta_D | \theta_{\setminus(\eta_U,\eta_D)}\right) = p\left(\eta_U, \eta_D\right)$ *and under* $\prod_{i\in\emptyset} := 1$ *it follows that*

$$\begin{aligned} p\left(\eta_U, \eta_D \left| x, \theta_{\setminus(\eta_U,\eta_D)}, \xi = (l_1, ..., l_n), J\right.\right) &\propto p\left(J|\theta, \xi = (l_1, ..., l_n)\right) \cdot \\ &\cdot p\left(\xi = (l_1, ..., l_n) |\theta\right) p\left(\eta_U, \eta_D\right) \\ &\propto p\left(\eta_U, \eta_D\right) \prod_{i=1}^{n} p\left(J_i|\theta, \xi_i = l_i\right) \\ &\propto p\left(\eta_U, \eta_D\right) \prod_{i:l_i=-1} p\left(J_i|\theta, \xi_i = -1\right) \cdot \\ &\cdot \prod_{i:l_i=1} p\left(J_i|\theta, \xi_i = 1\right). \end{aligned}$$

From (6) we get:

$$\begin{aligned} &p\left(\eta_U, \eta_D \left| x, \theta_{\setminus(\eta_U,\eta_D)}, \xi = (l_1, ..., l_n), J = (j_1, ..., j_n)\right.\right) \\ &\propto \eta_D^{n_{-1}+\nu_{D,\eta}-1} \exp\left(-\left(A_{D,\eta} - N_{D,J}\right)\eta_D\right) \mathbb{I}_{(0,\infty)}\left(\eta_D\right) \cdot \\ &\cdot \eta_U^{n_1+\nu_{U,\eta}-1} \exp\left(-\left(A_{U,\eta} + N_{U,J}\right)\eta_U\right) \mathbb{I}_{(0,\infty)}\left(\eta_U\right) \end{aligned}$$

5) From the independence

$$\begin{aligned} p\left(\xi, J \left| x, \theta\right.\right) &= p\left(J \left| x, \theta, \xi\right.\right) p\left(\xi \left| x, \theta\right.\right) = \\ &= \prod_{i=1}^{n} p\left(J_i \left| x_i, \theta, \xi_i\right.\right) p\left(\xi_i \left| x_i, \theta\right.\right) \end{aligned}$$

a) Each ξ_i *is a discrete random variable and* $\xi_i \in \{-1, 0, 1\}$. *From the prior assumptions*

$$\begin{aligned} P\left(\xi_i = l_i | x_i, \theta\right) &= p\left(x_i | \xi_i = l_i, \theta\right) P\left(\xi_i = l_i | \theta\right) \frac{p\left(\theta\right)}{p\left(x_i, \theta\right)} = \\ &= p\left(x_i | \xi_i = l_i, \theta\right) w_{l_i} \frac{p\left(\theta\right)}{p\left(x_i, \theta\right)}. \end{aligned}$$

If $l_i = 0$*, then a jump does not occur and*

$$P\left(\xi_i = 0 | x_i, \theta\right) = \frac{1}{\sigma\sqrt{\Delta}} \phi\left(\frac{x_i - \mu' \Delta}{\sigma\sqrt{\Delta}}; 0, 1\right) \frac{1}{1+L} \frac{p\left(\theta\right)}{p\left(x_i, \theta\right)}.$$

b) If $l_i = -1$*, then a negative jump occurs and the value of log interest rate is a sum of the pure diffusion component and the pure jump component. In other* *words, the density* $p\left(x_i|\xi_i=-1,\theta\right)$ *is a convolution of* $\frac{1}{\sigma\sqrt{\Delta}}\phi\left(\frac{y_i-\mu^{'}\Delta}{\sigma\sqrt{\Delta}};0,1\right)$ *and* $\eta_D\exp\left(\eta_D v_i\right)\mathbb{I}_{(-\infty,0)}\left(z_i\right)$. *From (10) we obtain*

$$p\left(x|\xi_i=-1,\theta\right)=\int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{\Delta}}\phi_{0,1}\left(\frac{z-\mu^{'}\Delta}{\sigma\sqrt{\Delta}}\right)\eta_D\exp\left(\eta_D\left(x-z\right)\right)\mathbb{I}_{(-\infty,0)}\left(x-z\right)dz=$$
$$=\eta_D\exp\left(\eta_D x_i-\mu^{'}\Delta\eta_D+\frac{1}{2}\sigma^2\Delta\eta_D^2\right)\cdot$$
$$\cdot\,\Phi\left(-\frac{x_i-\left(\mu^{'}\Delta-\sigma^2\Delta\eta_D\right)}{\sigma\sqrt{\Delta}};0,1\right)$$

and

$$P\left(\xi_i=-1|x_i,\theta\right)=\eta_D\exp\left(\eta_D x_i-\mu^{'}\Delta\eta_D+\frac{1}{2}\sigma^2\Delta\eta_D^2\right)\cdot$$
$$\cdot\,\Phi\left(-\frac{x_i-\left(\mu^{'}\Delta-\sigma^2\Delta\eta_D\right)}{\sigma\sqrt{\Delta}};0,1\right)\cdot\frac{L}{1+L}p_D\cdot\frac{p\left(\theta\right)}{p\left(x_i,\theta\right)}$$

c) The proof is based on the same idea as in the case of $l_i=-1$. *Formula (11) leads to the result:*

$$P\left(\xi_i=1|x_i,\theta\right)=\eta_U\exp\left(-\eta_U x_i+\mu^{'}\Delta\eta_U+\frac{1}{2}\sigma^2\Delta\eta_U^2\right)\cdot$$
$$\cdot\,\Phi\left(\frac{x_i-\left(\mu^{'}\Delta+\sigma^2\Delta\eta_U\right)}{\sigma\sqrt{\Delta}};0,1\right)$$
$$\cdot\,\frac{L}{1+L}p_U\frac{p\left(\theta\right)}{p\left(x_i,\theta\right)}.$$

Notice that

$$P\left(\xi_i=0|x_i,\theta\right)+P\left(\xi_i=-1|x_i,\theta\right)+P\left(\xi_i=1|x_i,\theta\right)=1,$$

so

$$G:=\frac{1}{\sigma\sqrt{\Delta}}\phi\left(\tfrac{x_i-\mu^{'}\Delta}{\sigma\sqrt{\Delta}};0,1\right)+$$
$$+\eta_D\exp\left(\eta_D x_i-\mu^{'}\Delta\eta_D+\tfrac{1}{2}\sigma^2\Delta\eta_D^2\right)\cdot\Phi\left(-\frac{x_i-\left(\mu^{'}\Delta-\sigma^2\Delta\eta_D\right)}{\sigma\sqrt{\Delta}};0,1\right)\cdot Lp_D$$
$$+\eta_U\exp\left(-\eta_U x_i+\mu^{'}\Delta\eta_U+\tfrac{1}{2}\sigma^2\Delta\eta_U^2\right)\Phi\left(\frac{x_i-\left(\mu^{'}\Delta+\sigma^2\Delta\eta_U\right)}{\sigma\sqrt{\Delta}};0,1\right)\cdot Lp_U.$$

d) Note that

$$p\left(J_i\left|x_i,\theta,\xi_i\right.\right)\propto p\left(x_i\left|\theta,\xi_i,J_i\right.\right)p\left(J_i\left|\theta,\xi_i\right.\right).$$

Applying (7) and (6) yields

$$p\left(J_i=j\left|x_i,\theta,\xi_i=0\right.\right)=\delta_0\left(j\right)$$

e) From (12) it follows that

$$\begin{aligned}p\left(J_i=j\left|x_i,\theta,\xi_i=-1\right.\right)&\propto\sqrt{\frac{h}{\Delta}}\phi\left(\sqrt{\frac{h}{\Delta}}\left(x_i-\mu^{'}\Delta-j\right);0,1\right)\cdot\\&\cdot\eta_D\exp\left(\eta_D j\right)\mathbb{I}_{(-\infty,0)}\left(j\right)\\&\propto\exp\left(-\frac{1}{2}\frac{h}{\Delta}\left(j-\left[\left(x_i-\mu^{'}\Delta\right)+\frac{\Delta}{h}\eta_D\right]\right)^2\right)\mathbb{I}_{(-\infty,0)}\left(j\right).\end{aligned}$$

Hence,

$$p\left(J_i=j\left|x_i,\theta,\xi_i=-1\right.\right)\propto\phi\left(j;x_i-\mu^{'}\Delta+\frac{\Delta}{h}\eta_D,\frac{\Delta}{h}\right)\mathbb{I}_{(-\infty,0)}\left(j\right)$$

f) The proof proceeds along the same line of reasoning as in the case of $l_i=-1$.

Proposition 9 *1. The independent Metropolis-Hastings algorithm with the candidate-generating density $(2n+1)\,L\sim\chi^2_{2N+\nu_L}$ and the transition probability:*

$$\min\left\{\exp\left(n\left(L^{(m+1)}-L^{(m)}\right)\right)\left[1+L^{(m+1)}\right]^{-n}\left[1+L^{(m)}\right]^{n},1\right\},$$

from a state $L^{(m)}$ to $L^{(m+1)}$ can be used to sample from $p\left(L\left|x,\theta_{\setminus L},\xi,J\right.\right)$.

2. If $n-N-\frac{\nu_L}{2}>0$, then the acceptance-rejection sampling with a proposition density of the gamma-gamma distribution (Bernardo and Smith (2002)):

$$p\left(L\right)=p_{Gg}\left(L;n-N-\frac{\nu_L}{2},1,N+\frac{\nu_L}{2}\right),$$

and the acceptance probability $e^{-L/2}$ can be used to sample from $p\left(L\left|x,\theta_{\setminus L},\xi,J\right.\right)$, where

$$p_{Gg}\left(L;\alpha,\beta,n\right)=\frac{\beta^\alpha}{\Gamma\left(\alpha\right)}\frac{\Gamma\left(\alpha+n\right)}{\Gamma\left(n\right)}\frac{L^{n-1}}{\left(\beta+L\right)^{\alpha+n}}\mathbb{I}_{(0,\infty)}\left(L\right)$$

Γ is the gamma function, $\alpha>0,\beta>0,n\in\{1,2,...\}$.

In practice, the condition $n - N - \frac{\nu_L}{2} > 0$ is often satisfied.

Proof 10 *1. The proposition is based on the following approximation:*

$$L^{N+\frac{\nu_L}{2}-1} \exp\left(-\frac{L}{2}\right)(1+L)^{-n} \approx L^{\frac{2N+\nu_L}{2}-1} \exp\left(-(2n+1)\frac{L}{2}\right).$$

Note that $(2n+1)L \sim \chi^2_{2N+\nu_L}$. *Some simple calculations lead to the formula for the transition probability:*

$$\min\left\{\exp\left(n\left(L^{(m+1)} - L^{(m)}\right)\right)\left[1+L^{(m+1)}\right]^{-n}\left[1+L^{(m)}\right]^{n}, 1\right\}.$$

2. Note that

$$p\left(L\left|x, \theta_{\backslash L}, \xi = (l_1, ..., l_n), J\right.\right) \propto L^{N+\frac{\gamma_L}{2}-1} \exp\left(-\frac{L}{2}\right)\frac{1}{(1+L)^n}$$
$$< L^{N+\frac{\gamma_L}{2}-1}\frac{1}{(1+L)^n}.$$

If $n - N - \frac{\nu_L}{2} > 0$, *then* $s(L) = \frac{1}{\Gamma\left(n - N - \frac{\nu_L}{2}\right)}\frac{\Gamma(n)}{\Gamma\left(N + \frac{\nu_L}{2}\right)}\frac{L^{N+\frac{\nu_L}{2}-1}}{(1+L)^n}$ *is the density of the gamma-gamma distribution, and*

$$\frac{p\left(L\left|x, \theta_{\backslash L}, \xi = (l_1, ..., l_n), J\right.\right)}{s(L)} \propto \exp\left(-\frac{L}{2}\right)$$

is the acceptance probability.