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Bayesian Analysis of Weak Form Reduced Rank Structure in VEC Models

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Abstract

The concept of cointegration that enables the proper statistical analysis of long-run comovements between unit root processes has been of great interest to numerous economic investigators since it was introduced. However, investigation of short-run comovement between economic time series seems equally important, especially for economic decision-makers. The concept of common features and based on it the idea of two additional reduced rank structure forms in a VEC model (the strong and the weak one) may be of some help. The strong form reduced rank structure (SF) takes place when at least one linear combination of the first differences of the variables exists, which is white noise. However, when this assumption seems too strong, the weaker case can be considered. The weak form appears when the linear combination of first differences adjusted for long-run efects exists, which is white noise.

The main focus of this paper is a Bayesian analysis of the VEC models involving the weak form of reduced rank restrictions.

After the introduction and discussion of the said Bayesian model, the presented methods will be illustrated by an empirical investigation of the price - wage spiral in the Polish economy.

Keywords: cointegration, Bayesian analysis, common cyclical features

JEL Classification: C11,C32, C53.

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1 Introduction and basic definitions

The concept of cointegration that enables the proper statistical analysis of long-run comovements between unit root processes has been of great interest to numerous economic investigators since it was introduced. However, investigation of short-run comovement between economic time series seems equally important, especially for economic decision-makers. The concept of common features, introduced by Engle and Kozicki (1993), may be of some help. The authors considered features satisfying three axioms:

1. If only one of two series has the feature, then the sum will also have it.

2. If none of the analysed series has the feature, then the sum will not have it either.

3. Whether a process has or does not have the feature can not be changed by multilpying it by a nonzero constant.

As common feature is defined as one, which is present in each of the analysed series, but there exists a nonzero linear combination of the series that does not have the feature. As pointed out by Ericsson (1993), common features can be treated as a generalization of the idea of cointegration. The concept of the serial correlation common feature (Engle, Kozicki 1993) is another example. In the case of the serial correlation commmon feature there exists at least one linear combination of the analysed series, which is an innovation.

As most of the macroeconomic time series can be treated as realisations of unit root processes it seems to be very important to analyse short-run comovements between first differences of the original processes together with long-run comovements between their levels. For this reason Vahid and Engle (1993) considered cointegration and serial correlation common feature together. Their idea was further developed by Hecq, Palm, Urbain (2006), who distinguish two additional reduced rank structure forms in a VEC model: the strong and the weak one. The strong form reduced rank structure (SF) appears when there exists at least one linear combination of the first differences of the series, which is white noise. This is the same as serial correlation common feature analysed by Vahid and Engle (1997). When, however, such assumption seems to be too strong, the weaker case may be considered. The weak form reduced rank structure (WF) appears when a linear combination of first differences adjusted for long-run efects exists, which is white noise.

Additionally, Hecq, Palm, Urbain (2006) introduced the mixed form reduced rank structure combining the weak form and the strong one.

In this paper we focus our attention on the Bayesian analysis of the weak form reduced rank structure.

Restrictions imposed on short-run parameters lead to a more parsimonious VEC model, which is of great importance in such "parametr consuming" models. If they are

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correctly imposed their occurrence may improve forecasting accuracy, especially that for short- and medium run periods. In such a case we could also expect more precise estimation of impulse response functions and variance decomposition caluculations.

2 Bayesian VEC model with weak form reduced rank structure

Let us consider the n-dimensional cointegrated process $\{x_t\}_{t=1,2,\ldots,T}$, where $x_t = (x_{t1}, x_{t2}, \ldots, x_{tn})'$, $t = 1, 2, \ldots, T$. According to the Granger representation theorem, any cointegrated process may by written in the error correction form (Strachan, van Dijk 2007):

$$\begin{aligned}\Delta x_t &= \alpha(\beta^{+\prime} x_{t-1} + \varphi_1' d_{1t}) + \Gamma_0 w_t + \sum_{i=1}^{k-1} \Gamma_i \Delta x_{t-i} + \varphi_2 d_{2t} + \varepsilon_t = \\ &= \alpha\beta' z_{1t} + \Gamma' z_{2t} + \Gamma_s' z_{3t} + \varepsilon_t = \Pi z_{1t} + \Gamma' z_{2t} + \Gamma_s' z_{3t} + \varepsilon_t \end{aligned} \quad (1)$$

where $z_{1t}' = (x_{t-1}', d_{1t}')$, $z_{2t}' = (\Delta x_{t-1}', \Delta x_{t-2}', \ldots, \Delta x_{t-k+1}')$, $z_{3t}' = (d_{2t}', w_t')$, $\beta = (\beta^{+\prime}, \varphi_1')'$, $\Gamma = (\Gamma_1, \ldots, \Gamma_{k-1})'$, $\Gamma_s = (\varphi_2, \Gamma_0)$, $\Pi = \alpha\beta'$, $\varepsilon_t \sim iiN^n(0, \Sigma)$ $t = 1, 2, \ldots, T$. Through d_{1t}, d_{2t} we introduce deterministic trends to the VEC form, w_t contains other non-random regressors, α is the $n \times r$ matrix of adjustment coefficients, β is the $m \times r$ matrix containing cointegrating vectors; $m \geq n$ and $m = n$ if there are no deterministic components in the cointegrating relations. Both matrices, α and β, are of rank r, where $0 \leq r \leq n$. For $r = n$ we assume $\alpha = I_n$.
Under the weak form reduced rank structure (WF), there exists a matrix $\tilde{\beta}_{n\times s}$, whose columns span the cofeature space, so that $\tilde{\beta}'(\Delta x_t - \alpha\beta' z_{1t} - \Gamma_s' z_{3t}) = \tilde{\beta}'\varepsilon_t$ is an s-dimensional vector mean innovation process with respect to the information available at time t (Hecq, Palm, Urbain 2006). The $\tilde{\beta}$ matrix must lie in the intersection of the left null spaces of the matrices describing the short-run dynamics, so in a VEC model with WF assumption the Γ matrix is of reduced rank $n - s = q$ and it can be treated in a very similar way to Π. In particular it can be written as a product of two full rank matrices: $\Gamma = \delta\gamma'$. This decomposition leads to the following model:

$$\Delta x_t = \alpha\beta' z_{1t} + \gamma\delta' z_{2t} + \Gamma_s' z_{3t} + \varepsilon_t. \quad (2)$$

As pointed out by Hecq, Palm, Urbain (2006), this definition of weak form reduced rank structure is not invariant to VEC models reparametrizations such as those where $\beta' x_{t-p}$ appears instead of $\beta' x_{t-1}$. We have decided to restrict our attention to VEC forms with $\beta' x_{t-1}$. Cubadda (2007) proposed another definition of a weak form. His definition is based on a first-order polynomial matrix and is invariant to such reparametrizations (see Cubadda 2007 for details).
In order to simplify the notation let us write the basic model (2) in a matrix form:

$$Z_0 = Z_1\Pi' + Z_2\Gamma + Z_3\Gamma_s + E = Z_1\beta\alpha' + Z_2\delta\gamma' + Z_3\Gamma_s + E, \quad (3)$$

where $Z_0 = (\Delta x_1, \Delta x_2, \ldots, \Delta x_T)'$, $Z_1 = (z_{11}, z_{12}, \ldots, z_{1T})'$, $z'_{1t} = (x'_{t-1}, d'_{1t})$, $Z_2 = (z_{21}, z_{22}, ..., z_{2T})'$, $z'_{2t} = (\Delta x'_{t-1}, \Delta x'_{t-2}, \ldots, \Delta x'_{t-k+1})$, $Z_3 = (z_{31}, z_{32}, \ldots, z_{3T})'$, $z'_{3t} = (d'_{2t}, w'_t)$, $\beta = (\beta^{+\prime}, \varphi'_1)'$, $\Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_{k-1})'$, $\Gamma_s = (\Gamma_0, \varphi_2)'$, $E = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_T)'$, $\varepsilon_t \sim iiN^n(0, \Sigma), t = 1, 2, \ldots, T$. In this model we have two reduced rank matrices (Π and Γ) and, as we have already said, they can be both written as products of two full rank matrices: $\Pi = \alpha\beta'$, $\Gamma = \gamma\delta'$. It is commonly known that such decomposition is not invariant, i.e. for any full rank matrices of adequate dimensions, C_Π and C_Γ, the following equalities are obtained: $\alpha\beta' = \alpha C_\Pi C_\Pi^{-1}\beta'$, $\gamma\delta' = \gamma C_\Gamma C_\Gamma^{-1}\delta'$. For this reason we should estimate spaces spanned by β and δ matrices rather than these matrices. These spaces are elements of the Grassman manifolds: $sp(\beta) \in G_{r,m-r}$ and $sp(\delta) \in G_{q,l-q}$, so while constructing a Bayesian model we have to choose priors for these matrices from the set of distributions defined on Grassmann manifolds. We have decided to use the scheme of estimation proposed by Koop, León-González and Strachan (Koop, León-González, Strachan 2010) for VEC models (see also Wróblewska 2010), which takes into account the curved geometry of the parameters and at the same time allows for the use of the parameter-augmented Gibbs sampling scheme to sample from the posterior distribution.

For Π and Γ two parametrisations will be used:

$$\alpha\beta' = (\alpha M_\Pi)(\beta M_\Pi^{-1})' \equiv AB', \tag{4}$$

where M_Π is an $r \times r$ symmetric positive definite matrix, and

$$\gamma\delta' = (\gamma M_\Gamma)(\delta M_\Gamma^{-1})' \equiv GD', \tag{5}$$

where M_Γ is a $q \times q$ symmetric positive definite matrix.

We assume that A, B, G and D are unrestricted matrices ($A \in \mathbb{R}^{nr}$,$B \in \mathbb{R}^{mr}$,$G \in \mathbb{R}^{nq}$,$D \in \mathbb{R}^{lq}$,), whilst β and δ have orthonormal columns, i.e. they are elements of the Stiefel manifolds: $\beta \in V_{r,m}$, $\delta \in V_{q,l}$. In this way, using the many-to-one relationships between Stiefel and Grassmann manifolds (of adequate dimesions), we will make inference about spaces through matrices with orthonormal columns.

Imposing on B the matrix Normal distribution $B|r,m,\tau_B \sim mN_{m\times r}(0,m^{-1}I_r,P_{\tau_B})$ leads us to the matrix angular central Gaussian (MACG) distribution for β: $\beta|r,m,\tau_B \sim MACG(P_{\tau_B})$ (see e.g. Chikuse 2002). Similary - matrix Normal distribution for D: $D|q,l,\tau_D \sim mN_{l\times q}(0,l^{-1}I_q,P_{\tau_D})$ leads to the MACG distribution for δ: $\delta|q,l,\tau_D \sim MACG(P_{\tau_D})$. Prior information for spaces spanned by β and δ may be incorporated into the model *via* matrices P_{τ_B} and P_{τ_D}, which are constructed as follows: $P_{\tau_B} = H_B H'_B + \tau_B H_B^\perp H_B^{\perp\prime}$, $P_{\tau_D} = H_D H'_D + \tau_D H_D^\perp H_D^{\perp\prime}$, where H_B, H_D are matrices with orthonormal columns containing prior information about cointegration space and the space of common dynamic factors. If we assume that the parameter $P_.$ in the MACG distribution is an identity matrix, we obtain the uniform distribution over the Stiefel manifold and so the uniform distribution over the Grassmann

manifold. For matrices A and G we also impose matrix Normal distributions: $A|\Sigma,\nu_A,r \sim mN(0,\nu_A I_r,\Sigma)$, $G|\Sigma,\nu_G,q \sim mN(0,\nu_G I_q,\Sigma)$. Parameters ν_A, ν_G, τ_B and τ_D control degrees of informativeness of the distributions stated above and may be set by the researcher or may be estimated. If they are to be estimated, inverted Gamma prior distributions may be used: $\nu_A \sim iG(s_{\nu_A},n_{\nu_A})$, $\nu_G \sim iG(s_{\nu_G},n_{\nu_G})$, $\tau_B \sim iG(s_{\tau_B},n_{\tau_B})$, $\tau_D \sim iG(s_{\tau_D},n_{\tau_D})$. For τ_B and τ_D it is recommended to settle such s_{τ_B}, n_{τ_B} and s_{τ_D}, n_{τ_D} that almost all the prior probability is allocated close to zero and is restricted to the [0,1] interval. For τ's close to zero we impose most of the prior probability to spaces close to those spanned by $H_{.}$, and for τ's equal one we get noninformative priors for the estimated spaces.
The priors for the remaining parameters are as follows:

1. inverted Wishart for Σ: $\Sigma \sim iW(S, q_\Sigma)$ (we opt for the informative prior for Σ, because, in order to estimate the marginal data density, we will use the Newton - Raftery method),
2. matrix Normal for Γ_s: $\Gamma_s|\Sigma, h \sim mN(0, \Sigma, hI)$,
3. inverted Gamma for h: $h \sim iG(s_h, n_h)$, if the researcher wants it to be estimated.

The joint prior distribution is truncated by the stability condition imposed on the parameters of the process:
$p(A,B,G,D,\Sigma,\Gamma_s,\nu_A,\nu_G,h) \propto f(A,B,G,D,\Sigma,\Gamma_s,\nu_A,\nu_G,h)I_{[0,1]}(|\lambda|_{max})$, where λ denotes the eigenvalue of the companion matrix.
The imposed joint prior distribution leads to posterior distribution proportional to:

$$\tau_B^{-n_{\tau_B}-1}|P_{\tau_B}|^{-\frac{r}{2}}\tau_D^{-n_{\tau_D}-1}|P_{\tau_D}|^{-\frac{q}{2}}\nu_A^{-n_{\nu_A}-\frac{nr}{2}-1}\nu_G^{-n_{\nu_G}-\frac{nq}{2}-1}h^{-n_h-\frac{nl_s}{2}-1}$$
$$\exp(-\frac{s_{\tau_B}}{\tau_B}-\frac{s_{\tau_D}}{\tau_D}-\frac{s_{\nu_A}}{\nu_A}-\frac{s_{\nu_G}}{\nu_G}-\frac{s_h}{h}-\frac{1}{2}tr(mB'P_{\tau_B}^{-1}B)-\frac{1}{2}tr(lD'P_{\tau_D}^{-1}D))$$
$$|\Sigma|^{-\dfrac{q_\Sigma+r+q+l_s+T+n+1}{2}}\exp\{-\frac{1}{2}tr[\Sigma^{-1}(S+\frac{AA'}{\nu_A}+\frac{GG'}{\nu_G}+\frac{1}{h}\Gamma_s'\Gamma_s+E'E)]\},$$

where $E = Z_0 - Z_1BA' - Z_2DG' - Z_3\Gamma_s$.
Thanks to the twofold parametrisations of the matrices Π and Γ given in (4) and (5) to samlpe from this posterior distribution we can use the parameter-augmented Gibbs sampler (Koop, León-González, Strachan 2010).
After setting initial values, the s-step in the proposed MCMC algorithm runs as follows:

1. draw Σ from the inverted Wishart:
   $iW(S+\frac{1}{h}\Gamma_s'\Gamma_s+\frac{1}{\nu_A}AA'+\frac{1}{\nu_G}GG'+E'E,q_\Sigma+l_s+r+q+T)$,
2. draw Γ_s form the matrix Normal:
   $mN(\mu_{\Gamma_s},\Sigma,\Omega_{\Gamma_s})$, where $\mu_{\Gamma_s} = (\frac{1}{h}I_{l_s}+z_3'Z_3)^{-1}Z_3'(Z_0-Z_1BA'-Z_2DG')$,
   $\Omega_{\Gamma_s} = (\frac{1}{h}I_{l_s}+z_3'Z_3)^{-1}$,

3. draw A from the matrix Normal:
$mN(\mu_A,(\frac{1}{\nu_A}I_r+B'Z_1'Z_1B)^{-1},\Sigma)$, where $\mu_A=(Z_0-Z_2DG'-Z_3\Gamma_s)'Z_1B(\frac{1}{\nu_G}I_r+$ $+B'Z_1'Z_1B)^{-1}$,

4. draw $vec(B)$ from the Normal with:
variance $\Omega_{vB}=([(A'\Sigma^{-1}A)\otimes(Z_1'Z_1)]+[mI_r\otimes P_{\frac{1}{\tau_B}}])^{-1}$
and mean $\mu_{vB}=\Omega_{vB}vec(Z_1'(Z_0-Z_2DG'-Z_3\Gamma_s)\Sigma^{-1}A)$,

5. obtain α and β as $\beta=B(B'B)^{-\frac{1}{2}}$, $\alpha=A(B'B)^{\frac{1}{2}}$,

6. draw G from the matrix Normal:
$mN(\mu_G,(\frac{1}{\nu_G}I_q+D'Z_2'Z_2D)^{-1},\Sigma)$, where $\mu_G=(Z_0-Z_1BA'-Z_3\Gamma_s)'Z_2D(\frac{1}{\nu_G}I_q+$ $+D'Z_2'Z_2D)^{-1}$,

7. draw $vec(D)$ from the Normal with variance
$\Omega_{vD}=([(G'\Sigma^{-1}G)\otimes(Z_2'Z_2)]+[lI_q\otimes P_{\frac{1}{\tau_D}}])^{-1}$ and mean $\mu_{vD}=\Omega_{vD}vec(Z_2'(Z_0+$ $-Z_1BA'-Z_3\Gamma_s)\Sigma^{-1}G)$,

8. obtain γ and δ as $\delta=D(D'D)^{-\frac{1}{2}}$, $\gamma=G(D'D)^{\frac{1}{2}}$,

9. additionaly ν_A may be drawn from the inverted Gamma
$iG(s_{\nu_A}+\frac{1}{2}tr(\Sigma^{-1}AA'),n_{\nu_A}+\frac{nr}{2})$, ν_G from $iG(s_{\nu_G}+\frac{1}{2}tr(\Sigma^{-1}GG'),n_{\nu_G}+\frac{nq}{2})$, h from $iG(s_h+\frac{1}{2}tr(\Sigma^{-1}\Gamma_s\Gamma_s'),n_h+\frac{nl_s}{2})$,

10. in the case of informative prior distribution for β draw τ_B form the distribution proportional to: $\tau_B^{-n_{\tau_B}-1}|P_{\tau_B}|^{-\frac{r}{2}}\exp(-\frac{1}{\tau_B}(s_{\tau_B}+\frac{m}{2}trB'H_B^{\perp}H_B^{\perp\prime}B))$ and, in the case of informative prior distribution for δ, τ_D - from the distribution proportional to: $\tau_D^{-n_{\tau_D}-1}|P_{\tau_D}|^{-\frac{q}{2}}\exp(-\frac{1}{\tau_D}(s_{\tau_D}+\frac{l}{2}trD'H_D^{\perp}H_D^{\perp\prime}D))$.

In the last step we may use e.g. the Metropolis-Hastings algorithm within the Gibbs sampler with proposal densities $iG(s_{\tau_B}+\frac{m}{2}trB'H_B^{\perp}H_B^{\perp\prime}B,n_{\tau_B})$ and $iG(s_{\tau_D}+$ $+\frac{l}{2}trD'H_D^{\perp}H_D^{\perp\prime}D,n_{\tau_D})$ for τ_B and τ_D respectively.
Having the sample from the posterior distribution the mean of β and δ can be computed with the method proposed by Villani (2006), i.e. by constructing the loss function, which takes the curved geometry of the Grassmann manifold into account, e.g. with the projective Frobenius distance between spaces.

3 An empirical example: the price - wage spiral in the Polish economy

The presented methods will be illustrated with the analysis of the price - wage spiral in the Polish economy. The seasonally unadjusted quarterly data represent five variables: average wages (current prices, W_t), price index of consumer goods

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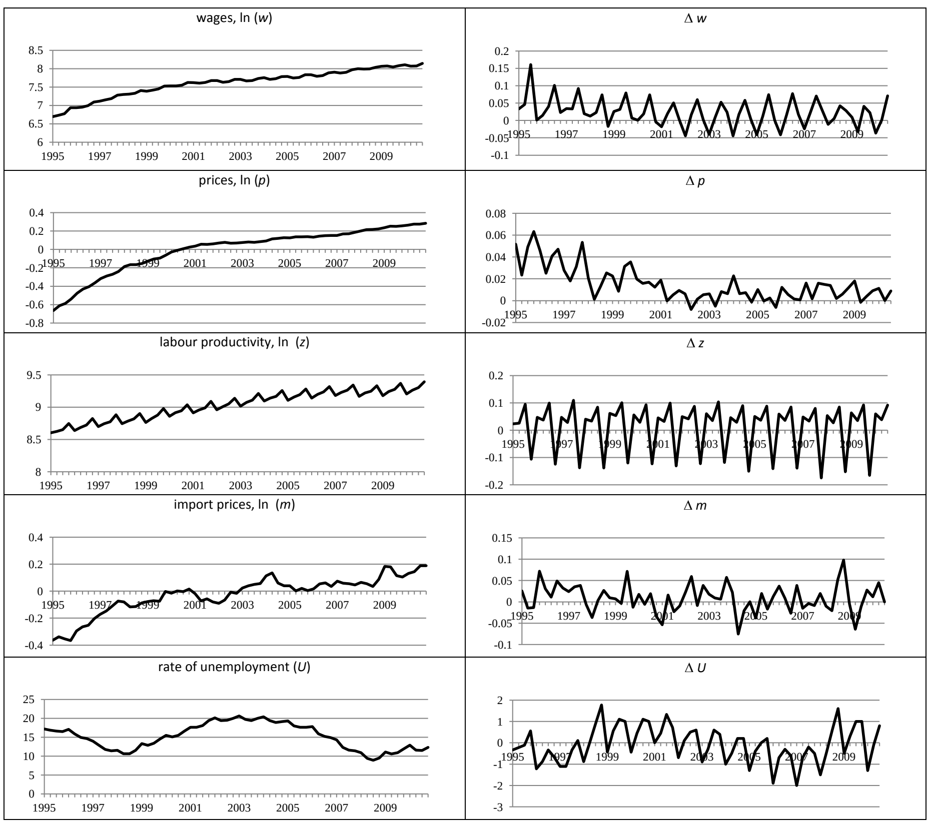
(P_t), labour productivity (constant prices, Z_t), price index of imported goods (M_t) and the unemployment rate (U_t). The analysed data cover the sixteen-year period ranging from 1995Q1 to 2010Q4. The data are plotted in Figure 1. The visual inspection of the analysed variables suggests that they may be realisations of the integrated processes, but they appear to move together in the long-run, so we can expect cointegration. The first differences of the series also seem to show a similar short-run behaviour, so it is reasonable to verify the hypothesis of the additional reduced rank restriction imposed on the short-run parameters of the VEC model. The seasonality of the analysed series will be modelled in the deterministic manner, i.e. via zero-mean seasonal dummies.

Figure 1: The analysed data



We will consider the set of models which differ in the number of lags $k \in \{2,3,4\}$, deterministic terms $d \in \{1,2\}$, where $d = 1$ stands for an unrestricted constant, $d = 2$ - a constant restricted to cointegrating relations (see e.g. Juselius 2007 for further details), the number of cointegrating relations $r \in \{1,2,3,4\}$ and the rank of Γ $q \in \{1,2,3,4,5\}$. We will compare 120 different specifications of VEC-WF model. As we want to treat them as equally possible we impose on each of them the same prior probability: $p(M_{(k,d,q,r)}) = \frac{1}{120} \approx 0.008$. The results are based on the following priors:

$$\Sigma \sim iW(S, 10+n+1),\ S = 10\begin{pmatrix} 0.05 & 0 & 0 & 0 & 0 \\ 0 & 0.01 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0.05 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$B|r,m \sim mN(0, m^{-1}I_r, I_m)$, which leads to $\beta|r \sim MACG(I_m)$,

$A|\nu_A, r, \Sigma \sim mN(0, \nu_A I_r, \Sigma)$,

$D|q,l \sim mN(0, l^{-1}I_q, I_l)$, which leads to $\delta|q \sim MACG(I_l)$,

$G|\nu_G, q, \Sigma \sim mN(0, \nu_G I_q, \Sigma)$,

$\Gamma_s|\Sigma, h \sim mN(0, \Sigma, hI)$,

$\nu_A \sim iG(2,3)$ $(E(\nu_A) = 1,\ Var(\nu_A) = 1)$,

$\nu_G \sim iG(2,3)$ $(E(\nu_G) = 1,\ Var(\nu_G) = 1)$,

$h \sim iG(20,3)$ $(E(h) = 10,\ Var(h) = 100)$,

$p(M_{(k,d,q,r)}) = 0.008$.

The joint prior resulting from this specification has been truncated by the stability condition imposed on the parameters of the cointegrated process.
The most probable models (i.e. with posterior probability higher than assumed prior probability) are presented in Table 2. The sum of posterior probabilities of the listed models equals 0.917.
The overall posterior probability of the models without additional rank reduction equals around 0.028, i.e. much less than assumed 0.2 prior probability (see Table 1), so it seems that the results of the model comparison confirm the empirical relevance of the imposed short-run restrictions.

Table 1: Marginal prior and posterior probabilities of q

q	1	2	3	4	5
$p(q)$	0.2	0.2	0.2	0.2	0.2
$p(q\|x)$	0.460	0.198	0.133	0.181	0.028

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As was pointed out in the introduction, the correctly imposed short-run restrictions should improve forecast accuracy. Figure 2 and Table 3 present predictive means and standard deviations obtained in models $M_{(4,2,1,3)}$ and $M_{(3,2,5,2)}$ for the four of five analysed variables. As it does not seem to be reasonable to forecast import prices in such a model, we decided not to present the obtained results for it, especially that the conclusions for this variable are similar to that presented for the remaining variables.

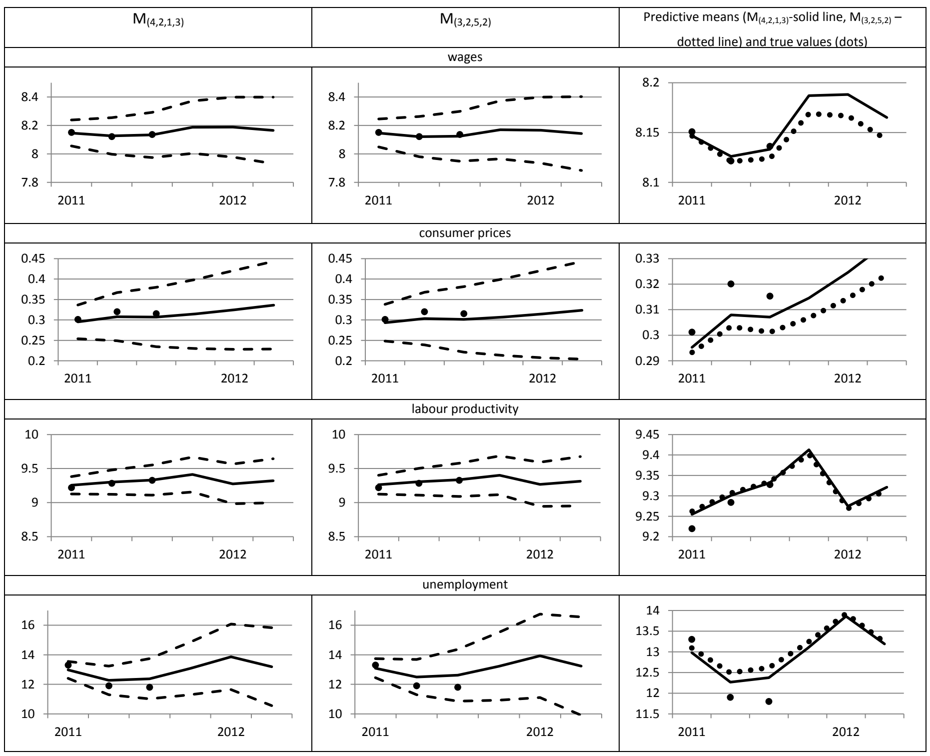
Table 2: The most probable models, $p(M_{(k,d,r,o,e)}|x) > p(M_{(k,d,r,q)})$

k	d	q	r	$log_{10}(\hat{p}(x\|M_{(k,d,q,r)}))$	$p(M_{(k,d,r,o,e)}\|x)$
4	2	1	3	129.502	0.146
2	1	3	1	129.321	0.097
2	1	1	3	129.291	0.090
4	2	4	2	129.164	0.067
3	2	1	4	129.129	0.062
2	1	4	1	128.991	0.045
4	2	2	3	128.987	0.045
2	1	2	2	128.961	0.042
3	2	1	3	128.953	0.041
2	2	1	4	128.800	0.029
3	2	5	2	128.779	0.028
2	2	4	1	128.734	0.025
4	2	2	2	128.690	0.023
2	2	2	2	128.676	0.022
2	2	1	2	128.674	0.022
3	2	4	1	128.623	0.019
3	2	3	2	128.607	0.019
2	2	2	4	128.587	0.018
2	2	4	2	128.583	0.018
3	1	1	1	128.580	0.018
4	1	1	2	128.545	0.016
2	1	2	4	128.459	0.013
3	2	2	3	128.447	0.013

The obtained forecast values are compared to the observed ones. This forecast exercise confirms the claim that in the case of more restricted models we can improve the precision of forecasting. Visual analysis of the third column of Figure 2 leads to the conclusion that in this empirical study the imposed restrictions increase the accuracy of out-of-sample forecast.

As these additional restrictions lead to more parsimonious models, we could also expect more efficient estimates, e.g. of impulse responses. Now we present a comparison of chosen impulse response functions obtained in the most probable models with and without this restriction, i.e. in models $M_{(4,2,1,3)}$ and $M_{(3,2,5,2)}$, respectively. As an example, impulse responses of the analysed variables to shock in unemployment are presented (see Figure 3). It can be easily noticed that the posterior distributions of impulse responses in the restricted VEC (model $M_{(4,2,1,3)}$), are less diffused than those obtained in the unrestritced VEC (model $M_{(3,2,5,2)}$). Shapes of the functions of impulse responses obtained in these two specifications may differ in the short-run whilst in the very long run they should be the same. And in fact, in Figure 4 we can observe some differences in point estimates of chosen impulse responses. The analysis of the obtained impulse responses to shocks in other variables included in the model leads to similar conclusions. They may be presented upon request.

Figure 2: Predictive means (solid lines) and standard deviations (dashed lines) in models $M_{(4,2,1,3)}$ and $M_{(3,2,5,2)}$ compared to true values of forecasted variables (dots).



The performed forecast error variance decomposition of import prices/ unemployment/ prices/ productivity/ wages system, which is presented in Figure 5 (see also Table 4), further illustrates the impact of additional short-run restrictions on the analysis of relationships between the variables in question.

The main conclusion following from the performed forecast error variance decompositions is that in the unrestricted VEC model ($M_{(3,2,5,2)}$) the percentage share of own innovations in the forecast error variance (for all variables) drops much faster than in the restricted VEC model ($M_{(4,2,1,3)}$).

Table 3: Predictive means and standard deviations (in brackets) in models $M_{(4,2,1,3)}$ and $M_{(3,2,5,2)}$ compared to true values of forecasted variables.

variable	model	2011Q1	2011Q2	2011Q3	2011Q4	2012Q1	2012Q2
wages	$M_{(4,2,1,3)}$	8.147 (0.091)	8.126 (0.129)	8.133 (0.158)	8.187 (0.184)	8.188 (0.210)	8.165 (0.233)
	$M_{(3,2,5,2)}$	8.147 (0.098)	8.121 (0.141)	8.124 (0.175)	8.169 (0.204)	8.167 (0.232)	8.143 (0.259)
	observed	8.151	8.122	8.136			
prices	$M_{(4,2,1,3)}$	0.295 (0.041)	0.308 (0.059)	0.307 (0.073)	0.315 (0.084)	0.325 (0.096)	0.336 (0.107)
	$M_{(3,2,5,2)}$	0.293 (0.045)	0.303 (0.064)	0.301 (0.080)	0.307 (0.094)	0.314 (0.107)	0.324 (0.119)
	observed	0.301	0.320	0.315			
productivity	$M_{(4,2,1,3)}$	9.255 (0.127)	9.301 (0.179)	9.332 (0.221)	9.413 (0.256)	9.275 (0.292)	9.321 (0.324)
	$M_{(3,2,5,2)}$	9.262 (0.139)	9.307 (0.197)	9.335 (0.244)	9.401 (0.285)	9.269 (0.325)	9.314 (0.361)
	observed	9.220	9.284	9.327			
import prices	$M_{(4,2,1,3)}$	0.203 (0.093)	0.201 (0.131)	0.199 (0.161)	0.203 (0.186)	0.217 (0.211)	0.213 (0.234)
	$M_{(3,2,5,2)}$	0.201 (0.101)	0.197 (0.143)	0.194 (0.176)	0.198 (0.204)	0.210 (0.232)	0.206 (0.258)
	observed	0.203	0.216	0.254			
unemployment	$M_{(4,2,1,3)}$	12.977 0.569	12.266 0.971	12.374 1.358	13.081 1.779	13.856 2.215	13.192 2.635
	$M_{(3,2,5,2)}$	13.091 (0.644)	12.494 (1.192)	12.615 (1.752)	13.222 (2.290)	13.925 (2.822)	13.237 (3.326)
	observed	13.300	11.900	11.800			

Table 4: Posterior mean forecast error variance decomposition of all five analysed variables in models $M_{(4,2,1,3)}$ and $M_{(3,2,5,2)}$

forecast horizon	m	U	p	z	w	m	U	p	z	w
forecast error in import prices										
1	1	0	0	0	0	1	0	0	0	0
2	0.997	0.002	0.000	0.000	0.000	0.904	0.013	0.013	0.046	0.024
3	0.992	0.006	0.000	0.001	0.001	0.811	0.020	0.033	0.084	0.052
4	0.984	0.012	0.001	0.002	0.002	0.750	0.028	0.049	0.099	0.074
5	0.974	0.018	0.001	0.004	0.003	0.708	0.035	0.061	0.106	0.091
6	0.964	0.025	0.001	0.006	0.004	0.670	0.042	0.072	0.111	0.104
7	0.953	0.032	0.002	0.008	0.005	0.639	0.048	0.082	0.115	0.116
8	0.942	0.039	0.002	0.010	0.006	0.612	0.054	0.090	0.118	0.125
9	0.931	0.046	0.003	0.012	0.008	0.590	0.060	0.097	0.120	0.133
10	0.920	0.053	0.003	0.015	0.009	0.570	0.066	0.103	0.121	0.140
11	0.908	0.059	0.004	0.017	0.011	0.553	0.071	0.108	0.122	0.145
12	0.897	0.066	0.005	0.020	0.013	0.538	0.075	0.113	0.123	0.150
13	0.886	0.072	0.006	0.022	0.014	0.525	0.080	0.117	0.124	0.154
14	0.875	0.078	0.006	0.025	0.016	0.513	0.084	0.120	0.125	0.158
15	0.863	0.085	0.007	0.027	0.018	0.502	0.088	0.123	0.125	0.161
16	0.853	0.091	0.008	0.030	0.019	0.493	0.092	0.126	0.126	0.164
17	0.842	0.096	0.009	0.032	0.021	0.484	0.095	0.128	0.126	0.167
18	0.831	0.102	0.009	0.035	0.023	0.475	0.099	0.130	0.126	0.169
19	0.821	0.108	0.010	0.037	0.024	0.468	0.102	0.132	0.127	0.171
20	0.811	0.113	0.011	0.039	0.026	0.461	0.105	0.134	0.127	0.173
21	0.802	0.118	0.012	0.041	0.027	0.454	0.108	0.136	0.127	0.175
forecast error in unemployment										
1	0.014	0.986	0	0	0	0.014	0.986	0	0	0
2	0.015	0.981	0.000	0.002	0.001	0.026	0.896	0.006	0.058	0.015
3	0.017	0.975	0.001	0.005	0.002	0.040	0.822	0.012	0.092	0.034
4	0.018	0.970	0.001	0.007	0.004	0.051	0.757	0.020	0.113	0.058
5	0.020	0.963	0.002	0.009	0.005	0.061	0.704	0.030	0.124	0.081
6	0.023	0.956	0.002	0.012	0.007	0.071	0.659	0.039	0.130	0.100
7	0.025	0.949	0.003	0.015	0.009	0.079	0.621	0.049	0.134	0.116
8	0.028	0.941	0.003	0.017	0.010	0.087	0.589	0.059	0.137	0.129
9	0.031	0.932	0.004	0.020	0.012	0.093	0.561	0.069	0.138	0.139
10	0.034	0.923	0.005	0.023	0.014	0.100	0.536	0.077	0.139	0.147
11	0.038	0.913	0.005	0.026	0.017	0.105	0.515	0.086	0.140	0.154
12	0.042	0.903	0.006	0.030	0.019	0.110	0.496	0.093	0.141	0.160
13	0.046	0.892	0.007	0.033	0.021	0.115	0.480	0.100	0.141	0.165
14	0.050	0.881	0.008	0.037	0.024	0.119	0.465	0.106	0.141	0.169
15	0.055	0.870	0.009	0.040	0.026	0.123	0.452	0.112	0.141	0.172
16	0.059	0.859	0.010	0.044	0.029	0.126	0.440	0.117	0.141	0.176
17	0.063	0.847	0.011	0.047	0.031	0.130	0.429	0.121	0.142	0.179
18	0.068	0.836	0.012	0.050	0.034	0.132	0.419	0.125	0.142	0.181
19	0.072	0.825	0.013	0.054	0.036	0.135	0.411	0.129	0.142	0.184
20	0.076	0.815	0.014	0.057	0.038	0.138	0.403	0.132	0.142	0.186
21	0.080	0.804	0.015	0.060	0.041	0.140	0.395	0.135	0.142	0.188
forecast error in prices										
1	0.014	0.014	0.972	0	0	0.016	0.015	0.970	0	0
2	0.015	0.016	0.968	0.001	0.000	0.066	0.022	0.858	0.035	0.019
3	0.016	0.020	0.960	0.002	0.001	0.134	0.023	0.750	0.056	0.037
4	0.018	0.026	0.948	0.005	0.003	0.161	0.028	0.685	0.072	0.054
5	0.021	0.034	0.934	0.007	0.004	0.174	0.033	0.639	0.084	0.071
6	0.024	0.041	0.918	0.011	0.006	0.182	0.038	0.602	0.093	0.086
7	0.028	0.049	0.901	0.014	0.008	0.186	0.043	0.572	0.100	0.099
8	0.032	0.057	0.883	0.018	0.010	0.188	0.049	0.547	0.105	0.111
9	0.036	0.065	0.865	0.022	0.013	0.189	0.055	0.527	0.109	0.121
10	0.040	0.072	0.847	0.026	0.015	0.189	0.060	0.510	0.112	0.129
11	0.045	0.080	0.828	0.030	0.017	0.188	0.065	0.495	0.114	0.137
12	0.049	0.088	0.810	0.034	0.020	0.188	0.070	0.482	0.116	0.143
13	0.053	0.095	0.792	0.037	0.022	0.187	0.075	0.471	0.118	0.149
14	0.058	0.103	0.774	0.041	0.024	0.186	0.080	0.461	0.119	0.154
15	0.062	0.110	0.756	0.045	0.027	0.186	0.084	0.451	0.121	0.158
16	0.066	0.117	0.740	0.048	0.029	0.185	0.089	0.443	0.121	0.162
17	0.071	0.123	0.723	0.052	0.031	0.185	0.093	0.435	0.122	0.165
18	0.075	0.129	0.707	0.055	0.033	0.184	0.097	0.428	0.123	0.168
19	0.079	0.135	0.692	0.059	0.035	0.184	0.100	0.421	0.124	0.171
20	0.083	0.141	0.677	0.062	0.037	0.183	0.104	0.415	0.124	0.174
21	0.087	0.146	0.663	0.065	0.039	0.183	0.107	0.409	0.125	0.176

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Table 5: Posterior mean forecast error variance decomposition of all five analysed variables in models $M_{(4,2,1,3)}$ and $M_{(3,2,5,2)}$

forecast horizon	m	U	p	z	w	m	U	p	z	w
forecast error in producti										
1	0.014	0.013	0.013	0.960	0	0.013	0.013	0.014	0.959	0
2	0.014	0.015	0.013	0.957	0.000	0.035	0.025	0.021	0.888	0.031
3	0.015	0.019	0.013	0.952	0.001	0.053	0.030	0.030	0.815	0.071
4	0.016	0.024	0.013	0.945	0.001	0.065	0.037	0.040	0.755	0.103
5	0.017	0.030	0.013	0.936	0.002	0.074	0.043	0.050	0.709	0.124
6	0.019	0.037	0.014	0.927	0.003	0.082	0.050	0.061	0.668	0.140
7	0.021	0.043	0.014	0.917	0.005	0.089	0.056	0.070	0.634	0.151
8	0.023	0.049	0.014	0.907	0.006	0.096	0.062	0.079	0.604	0.159
9	0.026	0.055	0.015	0.897	0.008	0.101	0.067	0.087	0.580	0.165
10	0.028	0.062	0.015	0.886	0.009	0.106	0.072	0.095	0.559	0.169
11	0.031	0.068	0.015	0.876	0.011	0.110	0.076	0.101	0.540	0.173
12	0.033	0.074	0.016	0.865	0.013	0.114	0.081	0.106	0.524	0.175
13	0.036	0.079	0.016	0.854	0.014	0.117	0.085	0.111	0.510	0.178
14	0.038	0.085	0.017	0.844	0.016	0.119	0.088	0.116	0.497	0.179
15	0.041	0.090	0.017	0.833	0.018	0.122	0.092	0.119	0.486	0.181
16	0.044	0.096	0.018	0.823	0.020	0.124	0.095	0.123	0.475	0.182
17	0.046	0.101	0.018	0.813	0.022	0.126	0.098	0.126	0.466	0.184
18	0.049	0.106	0.019	0.803	0.023	0.128	0.101	0.129	0.457	0.185
19	0.051	0.111	0.019	0.793	0.025	0.130	0.104	0.131	0.450	0.186
20	0.054	0.116	0.020	0.784	0.027	0.131	0.107	0.133	0.442	0.186
21	0.056	0.120	0.020	0.775	0.029	0.133	0.109	0.136	0.436	0.187
forecast error in wages										
1	0.014	0.014	0.014	0.013	0.946	0.014	0.013	0.014	0.012	0.947
2	0.014	0.016	0.014	0.013	0.943	0.030	0.024	0.020	0.048	0.878
3	0.015	0.019	0.014	0.015	0.937	0.043	0.027	0.026	0.076	0.828
4	0.017	0.025	0.014	0.017	0.927	0.054	0.033	0.036	0.096	0.781
5	0.019	0.032	0.014	0.019	0.916	0.064	0.039	0.046	0.110	0.742
6	0.021	0.039	0.014	0.022	0.903	0.073	0.045	0.057	0.119	0.706
7	0.024	0.047	0.015	0.026	0.889	0.081	0.051	0.068	0.125	0.675
8	0.027	0.054	0.015	0.029	0.875	0.088	0.058	0.077	0.130	0.647
9	0.030	0.061	0.015	0.033	0.861	0.095	0.063	0.086	0.133	0.623
10	0.033	0.068	0.016	0.037	0.846	0.101	0.069	0.094	0.136	0.601
11	0.036	0.075	0.016	0.041	0.831	0.106	0.074	0.101	0.138	0.581
12	0.039	0.082	0.017	0.045	0.817	0.110	0.079	0.108	0.139	0.564
13	0.043	0.088	0.018	0.050	0.802	0.114	0.083	0.113	0.141	0.549
14	0.046	0.094	0.018	0.054	0.788	0.118	0.087	0.118	0.142	0.535
15	0.049	0.100	0.019	0.058	0.774	0.121	0.091	0.123	0.143	0.522
16	0.053	0.106	0.019	0.062	0.760	0.124	0.095	0.127	0.144	0.510
17	0.056	0.111	0.020	0.066	0.747	0.127	0.099	0.130	0.145	0.500
18	0.059	0.116	0.021	0.070	0.734	0.129	0.102	0.133	0.145	0.490
19	0.062	0.121	0.021	0.074	0.721	0.131	0.105	0.136	0.146	0.481
20	0.065	0.126	0.022	0.078	0.709	0.134	0.108	0.139	0.147	0.473
21	0.068	0.131	0.023	0.081	0.697	0.135	0.111	0.141	0.147	0.465

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Figure 3: Posterior results of impulse responses of analysed variables to shock in unemployment. The solid line represents the posterior median and the dotted lines are the 10^{th} and 90^{th} percentiles.

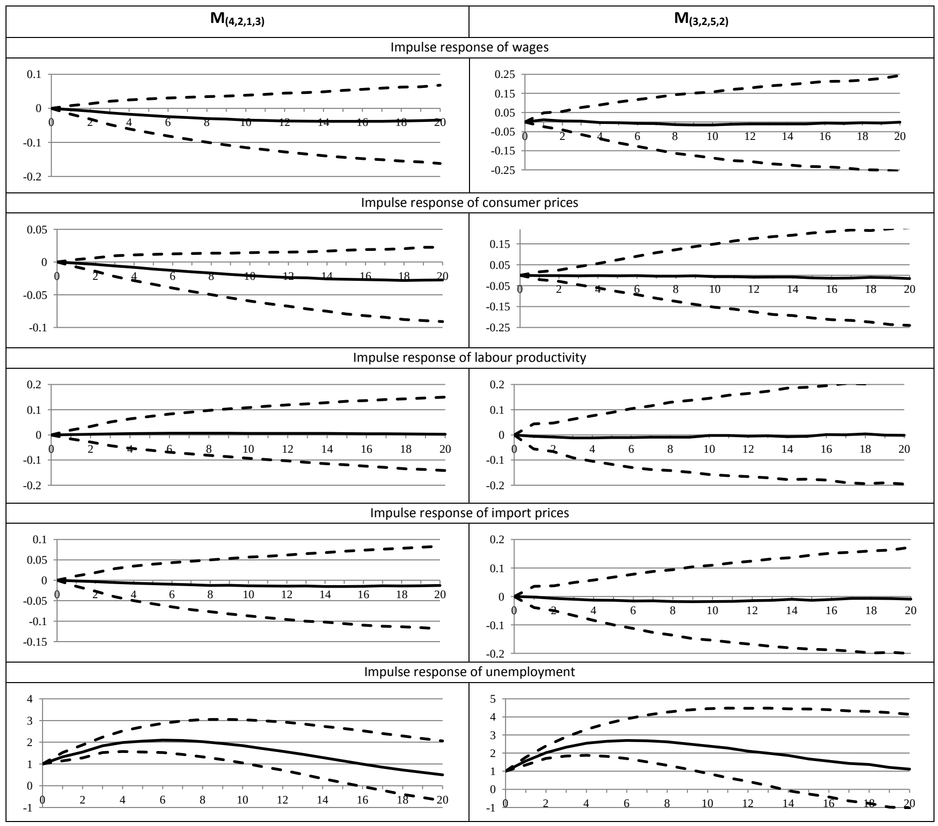

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Figure 4: Posterior medians of impulse responses of analysed variables to shock in unemployment.

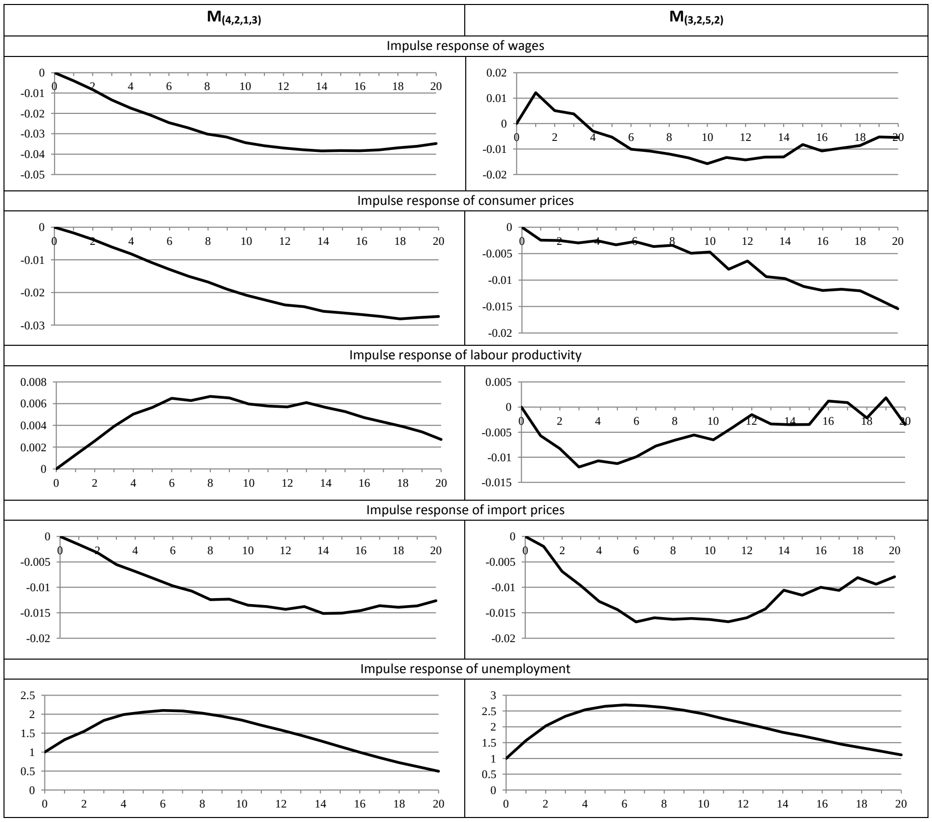
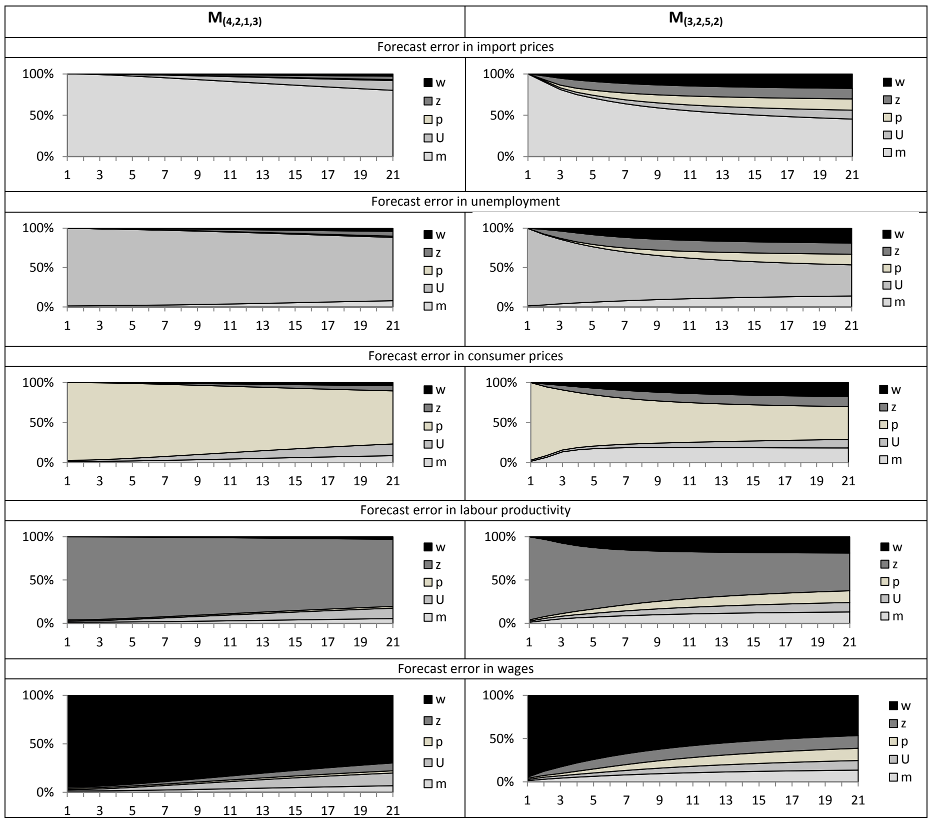


Figure 5: Posterior means of forecast error variance decomposition of all five analysed variables in models $M_{(4,2,1,3)}$ and $M_{(3,2,5,2)}$.



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4 Conclusions

In this paper we proposed a Bayesian treatment (i.e. estimation and comparison) of VEC models with the additional weak form reduced rank restriction imposed on the short-run parameters of such models. In the empirical example we used the proposed method to analyse the price - wage spiral in the Polish economy. The Bayesian comparison of the models confirmed the hypothesis of the presence of long-run and short-run relations among the analysed variables.
Additionally, we showed the consequences of such restrictions for forecasting and for further analysis of the VEC-WF system.
In conclusion it is worth noting that Hecq, Palm and Urbain (2006) showed that the existence of s weak form common feature vectors with $s > r$, implies the existence of $s - r$ strong form common features. In the presented empirical example the posterior probability of models fulfilling this assumption equals 0.574 and is higher than the assumed 0.3 prior probability, so in the future it will be useful to extend this analysis for the strong case.

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