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Volatile ARMA Modelling of GARCH Squares

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Abstract

This paper points out that the ARMA models followed by GARCH squares are volatile and gives explicit and general forms of their dependent and volatile innovations. The volatility function of the ARMA innovations is shown to be the square of the corresponding GARCH volatility function. The prediction of GARCH squares is facilitated by the ARMA structure and predictive intervals are considered. Further, the developments suggest families of volatile ARMA processes.

Keywords: ARCH, ARMA, GARCH, prediction, time series, volatility

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1 Introduction

ARCH and GARCH time series models, following the ground-breaking work of Engle (1982) and Bollerslev (1986), are widely used in econometrics and finance to model volatile time series. Further, many papers with numerous developments and applications continue to be written. The basic models have conditional means of zero and their principal role is to capture locally non-constant conditional variances; they are often used in simulation and prediction of volatility. The present starting point goes back to Bollerslev (1986) who reports the observation by Pantula and an anonymous referee that the model structure of the squared-variable generated by a GARCH model is of ARMA form with uncorrelated innovations. Bollerslev(1986) gave no further development of this, although the idea has been subsequently mentioned many times in the research literature, and in several books, e.g. Tsay (2002) p.87&93, Fan and Yao (2003) p150 and Lai and Xing (2008) p.147. Also, variously noted in the literature is that the innovations of the equivalent nonlinear ARMA model are uncorrelated but dependent, and that the auto and partial autocorrelations of GARCH squares can be calculated from linear ARMA equations. None, however, indicates that the innovations of the nonlinear ARMA models are volatile themselves or gives explicitly their nonlinear volatile structure, as done this in paper. Two further aspects of the equivalence between GARCH and nonlinear ARMA models are explored. The first is that GARCH squares can be predicted in a similar way to that of linear ARMA variables, and the second is that the associated predictive intervals of the squares are of variable width. Baillie and Bollerslev (1992) developed the prediction aspect in a quite general econometric GARCH setting but did not explicitly focus on simple models. As a further and fortuitous consequence, the volatility structure of GARCH squares offers natural ways of extending linear ARMA models to include volatility. The extended ARMA models are doubly autoregressive, in both linear and innovation terms; their first-order case is somewhat similar to the DAR model of Ling (2004). An early, and yet still very informative survey of GARCH modelling, is that by Bera and Higgins (1993), and there are many others.

2 Squares from the ARCH(1) Model

The simplest ARCH(1) model in the variable X_t takes the form

$$X_t = \sigma_t \varepsilon_t, \quad \sigma_t = \sqrt{\alpha_0 + \alpha_1 X_{t-1}^2} \tag{1}$$

where σ_t^2 is the volatility function, being the conditional variance of X_t, and $\{\varepsilon_t\}$ is a sequence of independent and identically distributed innovations with $E(\varepsilon_t) = 0$ and $var(\varepsilon_t) = 1$.

Stationarity of (1) is required, and as a preliminary remark, the terminology 'full stationarity' will be used to signify the combination of strict (distributional) stationarity

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and second order (covariance) stationarity in both X_t and X_t^2.
For the model to be fully stationary, $\alpha_0 > 0$, $0 \leq \alpha_1 < 1$. In terms of the squared variable X_t^2, the ARCHs(1) model is

$$X_t^2 = \left(\alpha_0 + \alpha_1 X_{t-1}^2\right) \varepsilon_t^2. \tag{2}$$

This is seen to be autoregressive but with multiplicative form. To recast it in additive form, it is convenient, although not necessary, to use the centred squared variables

$$\widetilde{X}_t^2 = X_t^2 - \mu_{X^2}, \quad \widetilde{\varepsilon_t}^2 = \varepsilon_t^2 - 1 \tag{3}$$

where $\mu_{X^2} = E(X^2) = \frac{\alpha_0}{1-\alpha_1}$. Then after simple rearrangement, (2) is seen to be algebraicly equivalent to

$$\widetilde{X}_t^2 = \alpha_1 \widetilde{X}_{t-1}^2 + \widetilde{E}_t, \quad \widetilde{E}_t = \left(\mu_{X^2} + \alpha_1 \widetilde{X}_{t-1}^2\right) \widetilde{\varepsilon}_t^2. \tag{4}$$

Thus, the ARCHs(1) model can be expressed as an AR(1) model in centred-squared variables with innovations $\widetilde{E}_t$ which are squares of the corresponding ARCH innovations and hence are volatile. It is doubly autoregressive, both in mean and innovation terms, and thus is similar to the doubly autoregressive DAR(1) model of Ling (2004),

$$X_t = \phi X_{t-1} + e_t, \quad e_t = \left(\kappa_0 + \kappa_1 X_{t-1}^2\right)^{\frac{1}{2}} \varepsilon_t. \tag{5}$$

However, in this model the innovation involves a square-rooted term.
Although the innovations $\widetilde{E}_t$ in (4) are dependent, $\widetilde{E}_t$, $\widetilde{E}_{t-k}$ for $k \geq 1$ are not autocorrelated, since $\mu_{X^2} + \alpha_1 \widetilde{X}_{t-k}^2$ is independent of $\widetilde{\varepsilon}_t^2$; moreover $\widetilde{E}_t$ is uncorrelated with $\widetilde{X}_{t-k}^2$ terms. The latter is sufficient to yield the geometric autocorrelations of linear AR(1) models and the cut-off property of its partial autocorrelations. In summary, it can be said that ARCH(1) squares follow the volatile AR(1) model (4). This is the result which is generalized in the rest of the paper, both in specificity and generality.

3 Squares from the ARCH(q) Model

The ARCHs(q) model is given by

$$X_t^2 = \sigma_t^2 \varepsilon_t^2, \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2 \tag{6}$$

and subject to the sufficient conditions for full stationarity

$$\alpha_0 > 0,\ \alpha_1 > 0,\ \alpha_2 > 0,\ \ldots,\ \alpha_q > 0,\ \ \alpha_1 + \alpha_2 + \cdots + \alpha_q < 1. \tag{7}$$

Very similar development to that for the ARCHs(1) model in Section 2 leads to the algebraicly equivalent volatile AR(q) model as

$$\widetilde{X}_t^2 = \sum_{i=1}^{q} \alpha_i \widetilde{X}_{t-i}^2 + \widetilde{E}_t, \quad \widetilde{E}_t = \left(\mu_{X^2} + \sum_{i=1}^{q} \alpha_i \widetilde{X}_{t-i}^2\right) \widetilde{\varepsilon}_t^2. \tag{8}$$

where $\mu_{X^2} = \frac{\alpha_0}{1-\alpha_1-\cdots-\alpha_q}$. In summary, the ARCHs(q) centred-squares follow a volatile AR(q) model with a volatility function similar to that of the innovations of an ARCH(q) model, a weighted addition of the q past centred-squared values.

4 Squares from the GARCH(1,1) Model

The GARCH(1,1) model generalizes the ARCH(1) model by involving the previous volatility as well as the previous squared value in its volatility function, and so its GARCHs(1,1) form is

$$X_t^2 = \sigma_t^2 \varepsilon_t^2, \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2. \tag{9}$$

Sufficient conditions for full stationarity are

$$\alpha_0 > 0,\ \alpha_1 \geq 0,\ \beta_1 \geq 0,\ \ \alpha_1 + \beta_1 < 1. \tag{10}$$

The GARCHs(1,1) model can be revealed to follow a volatile ARMA(1,1) model. First write

$$X_t^2 = \sigma_t^2 \varepsilon_t^2 = \sigma_t^2 + \sigma_t^2 \widetilde{\varepsilon}_t^2 = \sigma_t^2 + \widetilde{E}_t, \quad \widetilde{E}_t = \sigma_t^2 \widetilde{\varepsilon}_t^2 \tag{11}$$

and define

$$\widetilde{X}_t^2 = X_t^2 - \mu_{X^2}, \quad \mu_{X^2} = \frac{\alpha_0}{1-\alpha_1-\beta_1}, \quad \widetilde{\sigma}_t^2 = \sigma_t^2 - \mu_{X^2}. \tag{12}$$

Then by substituting $\sigma_t^2 = \widetilde{X}_t^2 - \widetilde{E}_t$, $\sigma_{t-1}^2 = \widetilde{X}_{t-1}^2 - \widetilde{E}_{t-1}$ in σ_t^2 of the GARCHs(1,1) model (9), there is algebraicly

$$\widetilde{X}_t^2 = (\alpha_1 + \beta_1)\, \widetilde{X}_{t-1}^2 + \widetilde{E}_t - \beta_1 \widetilde{E}_{t-1}, \quad \widetilde{E}_t = \sigma_t^2 \widetilde{\varepsilon}_t^2 \tag{13}$$

which indicates AR(1) linear terms and MA(1) innovation terms. By (12), the volatility function in (9) is

$$\sigma_t^2 = \mu_{X^2} + \alpha_1 \widetilde{X}_{t-1}^2 + \beta_1 \widetilde{\sigma}_{t-1}^2 = \mu_{X^2} + \sum_{i=1}^{\infty} \alpha_1 \beta_1^{i-1} \widetilde{X}_{t-i}^2, \tag{14}$$

the latter equality by recursion, and so the innovations take the explicit form

$$\widetilde{E}_t = \left(\mu_{X^2} + \sum_{i=1}^{\infty} \alpha_1 \beta_1^{i-1} \widetilde{X}_{t-i}^2 \right) \widetilde{\varepsilon}_t^2. \tag{15}$$

They are dependent but not autocorrelated and are uncorrelated with earlier X_t^2 terms. Thus, the GARCHs(1,1) autocorrelations follow those of the corresponding volatile ARMA(1,1) model. The volatility function in (15) is seen to be an exponential smoothing of past centred-squared values.

5 Squares from GARCH(q,r) Models

Extensions to higher orders show the general similarities between models for GARCH squares and volatile ARMA models for squares. The GARCHs(q,r) model is

$$X_t^2 = \sigma_t^2 \varepsilon_t^2, \quad \sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i X_{t-i}^2 + \sum_{i=1}^{r} \beta_i \sigma_{t-i}^2 \tag{16}$$

with sufficient conditions for full stationarity, Giraitis, Kokoszka, Leipus (2000),

$$\alpha_0 > 0,\ \alpha_1 \geq 0,\ \ldots,\ \alpha_q \geq 0, \quad \beta_1 \geq 0, \ldots, \beta_r \geq 0, \quad \sum_{i=1}^{q} \alpha_i + \sum_{i=1}^{r} \beta_i < 1 \tag{17}$$

and with

$$\mu_{X^2} = E(X_t^2) = \frac{\alpha_0}{1 - \sum\limits_{i=1}^{q} \alpha_i - \sum\limits_{i=1}^{r} \beta_i}. \tag{18}$$

A rather intractable necessary and sufficient condition for strict stationarity has been given by Bougerol and Picard (1992).
By following the derivation in Section 4, the ARMA form of the GARCHs(q,r) model in centred-squared variables becomes

$$\widetilde{X}_t^2 = \sum_{i=1}^{q} \alpha_i \widetilde{X}_{t-i}^2 + \sum_{i=1}^{r} \beta_i \widetilde{X}_{t-i}^2 + \widetilde{E}_t - \sum_{i=1}^{r} \beta_i \widetilde{E}_{t-i} \tag{19}$$

where

$$\widetilde{E}_t = \sigma_t^2 \widetilde{\varepsilon}_t^2, \quad \sigma_t^2 = \mu_{X^2} + \sum_{i=1}^{q} \alpha_i \widetilde{X}_{t-i}^2 + \sum_{i=1}^{r} \beta_i \widetilde{\sigma}_{t-i}^2. \tag{20}$$

The model is made as explicit as possible by solving (20) using the backward-shift operator B, which gives the result

$$\sigma_t^2 = \mu_{X^2} + \alpha_q(B)\,(1 - \beta_r(B))^{-1}\, \widetilde{X}_{t-1}^2, \quad \alpha_q(B) \equiv \sum_{i=1}^{q} \alpha_i B^{i-1}, \quad \beta_r(B) = \sum_{i=1}^{r} \beta_i B^i. \tag{21}$$

The autoregressive order in (19) is seen to be $q \vee r = \max(q, r)$ and the moving average order to be r. Thus, the ARMA $(q \vee r$,$r)$ form of the GARCHs(q,r) model is

$$\widetilde{X}_t^2 = \sum_{i=1}^{q \vee r} (\alpha_i + \beta_i)\, \widetilde{X}_{t-i}^2 + \widetilde{E}_t - \sum_{i=1}^{r} \beta_i \widetilde{E}_{t-i} \tag{22}$$

where any α's and β's of incomplete pairs are taken to be zero, and

$$\widetilde{E}_t = \left\{ \mu_{X^2} + \alpha_q(B)\,(1 - \beta_r(B))^{-1}\, \widetilde{X}_{t-1}^2 \right\} \widetilde{\varepsilon}_t^2. \tag{23}$$

As further specific illustrations, two cases of (22) and (23) are given next.

(i) For the GARCH(2,1) model in squared variables, the GARCHs(2,1) model, is

$$\widetilde{X}_t^2 = (\alpha_1 + \beta_1)\,\widetilde{X}_{t-1}^2 + \alpha_2 \widetilde{X}_{t-2}^2 + \widetilde{E}_t - \beta_1 \widetilde{E}_{t-1} \tag{24}$$

$$\widetilde{E}_t = \left(\mu_{X^2} + \alpha_1 \widetilde{X}_{t-1}^2 + (\alpha_1\beta_1 + \alpha_2)\sum_{i=2}^{\infty} \beta_1^{i-2} \widetilde{X}_{t-i}^2\right) \widetilde{\varepsilon}_t^2 \tag{25}$$

where $\mu_{X^2} = \frac{\alpha_0}{1-\alpha_1-\alpha_2-\beta_1}$. The innovation volatility function is seen to be a now slightly modified exponential smoothing of past centred-squared values.

(ii) The GARCH(1,2) model in squared variables, denoted as the GARCHs(1,2) model, is

$$\widetilde{X}_t^2 = (\alpha_1 + \beta_1)\,\widetilde{X}_{t-1}^2 + \beta_2 \widetilde{X}_{t-2}^2 + \widetilde{E}_t - \beta_1 \widetilde{E}_{t-1} - \beta_2 \widetilde{E}_{t-2}. \tag{26}$$

In this case, the innovation variables need the inverse roots r_1, r_2 of the $1-\beta_2(B)$ polynomial, and when real-valued,

$$\widetilde{E}_t = \left\{\mu_{X^2} + \alpha_1 \sum_{i=1}^{\infty}\left(\sum_{j=0}^{i-1} r_1^{i-j-1} r_2^{j}\right)\widetilde{X}_{t-i}^2\right\}\widetilde{\varepsilon}_t^2, \quad r_1 + r_2 = \beta_1,\ r_1 r_2 = -\beta_2 \tag{27}$$

where $\mu_{X^2} = \frac{\alpha_0}{1-\alpha_1-\beta_1-\beta_2}$; this is seen to be a smoothing of past centred-squared values which can alternatively be expressed as the linear combination of two exponential smoothings, of parameters r_1 and r_2.

6 Prediction of Volatility

Two relevant aspects of volatility prediction can be identified for GARCH models, that of future conditional variances $\sigma_{t+k}^2|\sigma_t^2, \sigma_{t-1}^2, \ldots,$ and less usually, that of squared-variables $X_{t+k}^2|X_t, X_{t-1}, \ldots$; the focus here will be on the latter, although for ARCH models they are much related. The nonlinear ARMA model equivalents of GARCH models will first be used to give predictions and prediction intervals of ARCH squared-variables. The enabling point is that the ARMA model structure is convenient and that although its innovations are dependent, they are uncorrelated. This is enough for the standard approach to linear ARMA prediction, e.g. Tsay (2002, Section 2.6), to be used as a starting point for predicting $X_{t+k}^2|X_t, X_{t-1}, \ldots$. Let

$$\chi_{t+k|t}^2 = E\left(\widetilde{X}_{t+k}^2|\widetilde{X}_t, \widetilde{X}_{t-1}, \ldots\right) \tag{28}$$

be the predictor, with its associated predictive variance $\upsilon_t^2(k)$ given by

$$\upsilon_{t+k|t}^2(k) = var\left(\widetilde{X}_{t+k}^2|\widetilde{X}_t, \widetilde{X}_{t-1}, \ldots\right). \tag{29}$$

Both quantities are thus based on the conditional distribution of $\widetilde{X}^2_{t+k}|\widetilde{X}_t, \widetilde{X}_{t-1}, \ldots$. After calculation of these quantities, the ± 2 standard deviation prediction intervals are available as functions of past values and thus not of constant width.
A brief indicative illustration is given for the ARCHs(1) model via its volatile AR(1) counterpart. First iterate the $t+k$ case of the volatile AR(1) equation (4) in the usual way, to obtain

$$\begin{gathered}\widetilde{X}^2_{t+k} = \alpha_1^k \widetilde{X}^2_t + \alpha_1^{k-1}\widetilde{E}_{t+1} + \alpha_1^{k-2}\widetilde{E}_{t+2} + \cdots + \alpha_1 \widetilde{E}_{t+k-1} + \widetilde{E}_{t+k}, \\ \widetilde{E}_t = \left(\mu_{X^2} + \alpha_1 \widetilde{X}^2_{t-1}\right)\widetilde{\varepsilon}^2_t = \alpha_1^k \widetilde{X}^2_t + \alpha_1^{k-1}\left(\mu_{X^2} + \alpha_1 \widetilde{X}^2_t\right)\widetilde{\varepsilon}^2_{t+1} + \\ \alpha_1^{k-2}\left(\mu_{X^2} + \alpha_1 \widetilde{X}^2_{t+1}\right)\widetilde{\varepsilon}^2_{t+2} + \cdots + \alpha_1\left(\mu_{X^2} + \alpha_1 \widetilde{X}^2_{t+k-2}\right)\widetilde{\varepsilon}^2_{t+k-1} + \\ \left(\mu_{X^2} + \alpha_1 \widetilde{X}^2_{t+k-1}\right)\widetilde{\varepsilon}^2_{t+k}.\end{gathered} \tag{30}$$

Next take expectations of (30) conditional on $X_t, X_{t-1}, \ldots$, noting that the future ARCH(1) innovations $\widetilde{\varepsilon}^2_{t+1}, \ldots, \widetilde{\varepsilon}^2_{t+k}$ still have zero means, and so giving the linear AR(1) k-period predictor

$$\chi^2_{t+k|t} = E\left(\widetilde{X}^2_{t+k}|\widetilde{X}_t, \widetilde{X}_{t-1}, \ldots\right) = \alpha_1^k \widetilde{X}^2_t. \tag{31}$$

The associated predictive variance $\upsilon^2_{t+k|t}$ is less straightforward except for the one-period horizon for which

$$\upsilon^2_{t+1|t} = \left(\mu_{X^2} + \alpha_1 \widetilde{X}^2_t\right) var\left(\widetilde{\varepsilon}^2_{t+1}\right). \tag{32}$$

This result shows that the width of the prediction interval is a function of $\widetilde{X}^2_t$ which is available at the prediction origin. Continuing with the two-period predictor, taking the conditional variance of (30) with $k = 2$ gives

$$\upsilon^2_{t+2|t} = \alpha_1^2\left(\mu_{X^2} + \alpha_1 \widetilde{X}^2_t\right) var\left(\widetilde{\varepsilon}^2_{t+1}\right) + E_{|X_t}\left(\mu_{X^2} + \alpha_1 \widetilde{X}^2_{t+1}\right)^2 var\left(\widetilde{\varepsilon}^2_{t+2}\right) \tag{33}$$

where the expectation, conditional on X_t, is tractable but not neat. This result extends with further terms to any prediction horizon and suggests that the prediction interval should eventually be of constant width, although still be a function of X_t. However, from a practical financial point of view, these two short horizon cases may be the most important.
It is interesting to note, as pointed out by Bera and Higgins (1993) in the econometric context of using GARCH models for error assumptions in linear models, that prediction intervals need not monotonically increase in width as the horizon increases. Their behaviour will depend on the behaviour of the series up to the prediction origin; such a result is plausible from comparing (32) and (33).
Continuing to the widely used GARCH(1,1) case, considered valuable for its flexibility

and parsimony, a similar prediction approach may be used. For the equation basis of the one-period predictor, from (11) in the $t+1$ case and using (14), there is

$$X_{t+1}^2 = \sigma_{t+1}^2 + \sigma_{t+1}^2 \widetilde{\varepsilon}_{t+1}^2 \tag{34}$$

and thus

$$\widetilde{X}_{t+1}^2 = \sum_{i=1}^{\infty} \alpha_1 \beta_1^{i-1} \widetilde{X}_{t-i+1}^2 + \left(\mu_{X^2} + \sum_{i=1}^{\infty} \alpha_1 \beta_1^{i-1} \widetilde{X}_{t-i+1}^2 \right) \widetilde{\varepsilon}_{t+1}^2 \tag{35}$$

in terms of past $\widetilde{X}_t$'s. The required results then follow as

$$\chi_{t+1|t}^2 = E\left(\widetilde{X}_{t+1}^2 | \widetilde{X}_t, \widetilde{X}_{t-1}, \dots \right) = \sum_{i=1}^{\infty} \alpha_1 \beta_1^{i-1} \widetilde{X}_{t-i+1}^2 \tag{36}$$

and

$$\upsilon_{t+1|t}^2 = var\left(\widetilde{X}_{t+1}^2 | \widetilde{X}_t, \widetilde{X}_{t-1}, \dots \right) = \left(\mu_{X^2} + \sum_{i=1}^{\infty} \alpha_1 \beta_1^{i-1} \widetilde{X}_{t-i+1}^2 \right)^2 var\left(\widetilde{\varepsilon}_{t+1}^2 \right). \tag{37}$$

The predictor is noted as an exponential smooth of all past $\widetilde{X}_t$'s, intuitively appealing and still practical given a sufficient history at the prediction origin.

7 Mean and Volatility Models

The ARMA volatility structure of the squares in GARCH models suggests that this volatility structure can be introduced into linear ARMA models. Such models then have a predictive ability based on their linear autocorrelation and a predictive uncertainty calibrated by GARCH volatility. Thus a general family of volatile ARMA models according to the GARCH link could be

$$X_t = \phi_0 + \sum_{i=1}^{p} \phi_i X_{t-i} + E_t - \sum_{i=1}^{q} \psi_i E_{t-i}, \quad E_t = \left(\alpha_0 + \sum_{i=1}^{r} \alpha_i |X_{t-i}| \right) \varepsilon_t. \tag{38}$$

Another mean and volatility general model is given by extending the first-order DAR(1) model of Ling (2004) as

$$X_t = \phi_0 + \sum_{i=1}^{p} \phi_i X_{t-i} + E_t - \sum_{i=1}^{q} \psi_i E_{t-i}, \quad E_t = \left(\alpha_0 + \sum_{i=1}^{r} \alpha_i X_{t-i}^2 \right)^{\frac{1}{2}} \varepsilon_t. \tag{39}$$

The two models differ somewhat in their volatility structures. Both are open for further investigation.

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