archives
of thermodynamics
Vol. **39**(2018), No. 1, 25–37
DOI: 10.1515/aoter-2018-0002

Stable solution to nonstationary inverse heat conduction equation

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Abstract The paper presents investigations related to solving of a direct and inverse problem of a non-stationary heat conduction equation for a cylinder. The solution of the inverse problem in the form of temperature distributions has been obtained through minimization of a functional being the measure of the difference between the values of measured and calculated temperatures in M points of the heated cylinder. The solution of the conduction equation was presented in the convolutional form and then numerically integrated approximating one of the integrand with a step function described with parameter $\Theta \in (0,1]$. The influence of the integration parameter Θ on the obtained solution of the inverse problem (including a number of temperature measurement points inside the heated body) has been analyzed. The influence of the parameter Θ on the sensitivity of the obtained temperature distributions has been investigated.

Keywords: Inverse problem; Nonstationary heat conduction equation; Solution sensitivity; Integral parameter

Nomenclature

c	–	specific heat, J/kgK
f	–	temperature at the cylinder edge
I	–	functional
J_0, J_1	–	Bessel functions of the first kind

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m	–	values calculated with the thermocouples installation error δr^* towards the cylinder axis
p	–	values calculated with the thermocouples installation error δr^* towards the cylinder edge
r	–	radius, m
ran	–	values calculated with the stochastic distortion of temperature measurement
t	–	time, s
T	–	temperature, °C
dp	–	values calculated with the direct problem
ip	–	values calculated with the inverse problem

Greek symbols

δ	–	absolute error
ϑ	–	dimensionless temperature
Θ	–	coefficient used during integration $\Theta \in (0, 1]$
λ	–	thermal conductivity, W/mK
ξ	–	radius in the dimensionless coordinates
ρ	–	density, kg/m^3
τ	–	dimensionless time (Fourier number)

Subscripts and superscripts

0	–	starting time, for $t = 0$
max	–	maximum during heating
c	–	calculated
m	–	measured
w	–	assuming constant temperature at the cylinder edge
z	–	cylinder outer surface
$*$	–	measurement

1 Introduction

In many high-temperature heating processes the measurement of temperature on the edge of a heated element may be difficult to perform with satisfactory accuracy [19]. The temperature distribution on the edge of the area can be obtained by measuring the temperature inside the heated object and solving the inverse problem [2,5,19,20]. Then, a very important issue is the analysis of the sensitivity of obtained solution to inaccurate installation of thermocouples, random error of temperature measurement [2,3,11,12] as well as the analysis of its stability [8,14,15]. The influence of the temperature measurement error and thermocouples installation error on the determination of temperature distribution on the edge of the object and on the heat transfer coefficient have been analyzed in [2,11,12,19]. Determination of the temperature distributions with the inverse problem method has been applied in analyses of boiler operation [9,19], heat exchangers [21], turbine blade [6] and thermal and thermal-chemical processing [1,10,16]. In [9], the exchange of heat in a cylinder was analyzed using the control volume method, energy equation and the inverse problem method. Heat transfer coefficient was determined in the investigations. Investigations were a basis for the analysis of heat exchange in the thermometer used in boilers. In [13] results of calculations related to the determination of temperature distributions in a steel pipe of a heat exchanger with mineral deposits have been presented. In [17], heat conduction of a material as a multinomial dependent on temperature was sought by solving the conduction equation for a 2D model in a stationary state using the inverse method. In [16], the flow of heat in high temperature industrial furnaces has been described.

This paper analyzes the direct and inverse problem using the conjugate gradients for the changes of phase of the solidifying metal. Thus far, investigations of phase transformations during tempering have been carried out using the inverse problem method [1]. Some of the methods of solving a one-dimensional inverse problem of temperature fields distribution for a cylinder have been shown in [4] and for the cylindrical layer in [3]. The inverse problem for the heat conduction equation has been solved with the sequential method, which was described in [3,10–12,22]. This paper describes the sequential solution of the inverse problem for a non-stationary heat conduction equation for a cylinder considering the temperature measurement at M points. The influence of thermocouples installation errors and stochastic temperature measurement errors on the obtained results have also been analyzed. The developed calculation methods will serve to analyze the heating in thermal processing.

2 Direct problem

The solution of the inverse problem is done based on the solution of the direct problem, which is most frequently expressed as a function dependent in an explicit form on the given and sought courses of temperature. The calculations have been made for a linear non-stationary heat conduction equation [2,7,18]:

$$\frac{\partial \vartheta}{\partial \tau} = \frac{\partial^2 \vartheta}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \vartheta}{\partial \xi}\,, \quad \xi \in (0,1)\,, \quad \tau > 0 \tag{1}$$

in the dimensionless coordinates

$$\xi = \frac{r}{r_z}, \quad \vartheta = \frac{T - T_0}{T_{\max}} , \quad \tau = \frac{\lambda}{\rho c} \frac{t}{r_z^2} , \tag{2}$$

with the following conditions

- initial condition

$$\vartheta(\xi, \tau = 0) = 0 , \tag{3}$$

- boundary conditions

$$\vartheta(\xi = 1, \tau) = \vartheta_z(\tau) , \quad \tau > 0 , \tag{4}$$

- condition of solution boundedness at point $\xi = 0$

$$|\vartheta(\xi = 0, \tau)| < \infty . \tag{5}$$

The solution of the direct problem for the temperature field in a cylinder can be expressed in the form of a function convolution [7,11,12,18], and for Eq. (1) it has the form

$$\vartheta(\xi, \tau) = \frac{\partial \vartheta(\xi = 1, \tau)}{\partial \tau} * \left[1 - 2\sum_{n=1}^{\infty} \frac{J_0(p_n \xi) e^{-p_n^2 \tau}}{p_n J_1(p_n)}\right] = \frac{\partial \vartheta(\xi = 1, \tau)}{\partial \tau} * \vartheta_w(\xi, \tau)$$

$$= \vartheta(\xi = 1, \tau) * \left[2\sum_{n=1}^{\infty} \frac{J_0(p_n \xi) p_n e^{-p_n^2 \tau}}{J_1(p_n)}\right] = \vartheta(\xi = 1, \tau) * \frac{\partial \vartheta_w(\xi, \tau)}{\partial \tau} , \tag{6}$$

where function $\vartheta_w(\xi, \tau)$ is the solution of Eq. (1) with the initial conditions (3) and boundary condition $\vartheta_z = 1$, symbol $*$ denotes convolution of functions.

3 Inverse problem for the measurement performed with M thermocouples

We shall seek an unknown temperature distribution on boundary $\xi = 1$ based on the temperature measurement at the interior points of the cylinder. For M thermocouples the measurement points $\xi_1^*, \xi_2^*, \xi_3^*, \ldots, \xi_{M-1}^*, \xi_M^*$ have been marked in Fig. 1. For the kth measuring point based on (6) we

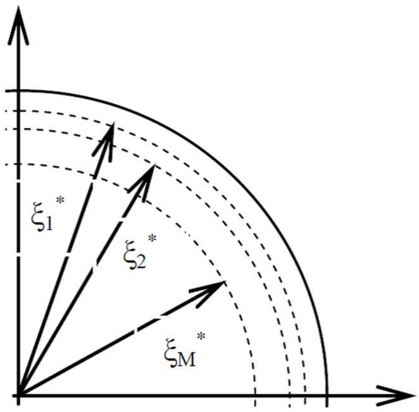

Figure 1: Measuring points.

can express the equality

$$\vartheta\left(\xi_k^*,\tau_i\right)=\int_0^{\tau_i}\vartheta'\left(\xi=1,u\right)\vartheta_w\left(\xi_k^*,\tau_i-u\right)du \tag{7}$$

and by discretizing the interval $\langle 0,\tau_i\rangle=\bigcup_{j=1}^{i}\langle\tau_{j-1},\tau_j\rangle$ we have

$$\vartheta\left(\xi_k^*,\tau_i\right)=\sum_{j=1}^{i}\int_{\tau_{j-1}}^{\tau_j}\vartheta'\left(\xi=1,u\right)\vartheta_w\left(\xi_k^*,\tau_i-u\right)du\ . \tag{8}$$

Approximating the integrand function $\vartheta_w\left(\xi_k^*,\tau_i-u\right)$ with a step function $\vartheta_w\left(\xi_k^*,\tau_i-\tau_{j-1}\right)\Theta+\vartheta_w\left(\xi_k^*,\tau_i-\tau_j\right)(1-\Theta)$ with parameter $0<\Theta<1$ we obtain

$$\begin{aligned}\vartheta\left(\xi_k^*,\tau_i\right)=\sum_{j=1}^{i}\int_{\tau_{j-1}}^{\tau_j}\vartheta'\left(\xi=1,u\right)\Big[&\vartheta_w\left(\xi_k^*,\tau_i-\tau_{j-1}\right)\Theta\\&+\vartheta_w\left(\xi_k^*,\tau_i-\tau_j\right)(1-\Theta)\Big]du\ .\end{aligned} \tag{9}$$

Since $\vartheta_w\left(\xi_k^*,\tau_i-\tau_{j-1}\right)$, $\vartheta_w\left(\xi_k^*,\tau_i-\tau_j\right)$ and Θ are independent from u, hence

$$\vartheta\left(\xi_k^*,\tau_i\right)=\sum_{j=1}^{i}\left[\vartheta_w\left(\xi_k^*,\tau_i-\tau_{j-1}\right)\Theta+\vartheta_w\left(\xi_k^*,\tau_i-\tau_j\right)(1-\Theta)\right]$$

$$\times \int_{\tau_{j-1}}^{\tau_j} \vartheta' (\xi=1, u)\, du = \sum_{j=1}^{i} \Big[\vartheta_w (\xi_k^*, \tau_i - \tau_{j-1})\, \Theta + \vartheta_w (\xi_k^*, \tau_i - \tau_j)\,(1-\Theta)\Big]$$

$$\times \Big[\vartheta (\xi = 1, \tau_j) - \vartheta (\xi = 1, \tau_{j-1})\Big] (\tau_j - \tau_{j-1}) \tag{10}$$

and for the constant time step $\tau_j - \tau_{j-1} = \Delta\tau$

$$\vartheta (\xi_k^*, \tau_i) = \sum_{j=1}^{i} \Big[\vartheta_w (\xi_k^*, \tau_i - \tau_{j-1})\, \Theta + \vartheta_w (\xi_k^*, \tau_i - \tau_j)\,(1-\Theta)\Big]$$
$$\times \Big[\vartheta (\xi = 1, \tau_j) - \vartheta (\xi = 1, \tau_{j-1})\Big] \Delta\tau\,. \tag{11}$$

The above relation can be written in the form of a product of two vectors

$$\vartheta (\xi_k^*, \tau_i) = \begin{bmatrix} \Delta\tau \left[\vartheta_w (\xi_k^*, \tau_i - \tau_0)\, \Theta + \vartheta_w (\xi_k^*, \tau_i - \tau_1)\,(1-\Theta)\right] \\ \Delta\tau \left[\vartheta_w (\xi_k^*, \tau_i - \tau_1)\, \Theta + \vartheta_w (\xi_k^*, \tau_i - \tau_2)\,(1-\Theta)\right] \\ \vdots \\ \Delta\tau \left[\vartheta_w (\xi_k^*, \tau_i - \tau_{i-1})\, \Theta + \vartheta_w (\xi_k^*, \tau_i - \tau_i)\,(1-\Theta)\right] \end{bmatrix}^T$$
$$\times \begin{bmatrix} \vartheta (\xi = 1, \tau_1) - \vartheta (\xi = 1, \tau_0) \\ \vartheta (\xi = 1, \tau_2) - \vartheta (\xi = 1, \tau_1) \\ \vdots \\ \vartheta (\xi = 1, \tau_i) - \vartheta (\xi = 1, \tau_{i-1}) \end{bmatrix}. \tag{12}$$

Introducing symbols $\vartheta (\xi = 1, \tau) = \vartheta_z (\tau)$, and considering the relation $\tau_j - \tau_k = \tau_{j-k}$ for a constant time step and notating the vector containing the temperature values on the cylinder boundary as a difference of vectors we obtain

$$\vartheta (\xi_k^*, \tau_i) = \{W\}^T \begin{bmatrix} \vartheta_z (\tau_1) \\ \vartheta_z (\tau_2) \\ \vdots \\ \vartheta_z (\tau_i) \end{bmatrix} - [W]^T \begin{bmatrix} \vartheta_z (\tau_0) \\ \vartheta_z (\tau_1) \\ \vdots \\ \vartheta_z (\tau_{i-1}) \end{bmatrix}, \tag{13}$$

where j – the element of vector $[W]$ is determined by the formula $W_j = \Delta\tau \left[\vartheta_w (\xi_k^*, \tau_{i-j+1})\, \Theta + \vartheta_w (\xi_k^*, \tau_{i-j})\,(1-\Theta)\right]$ for each $j = 1, 2, \mathrm{K}, i$. Hence, the dimensionless temperature at kth measuring point can be written as follows

$$\vartheta (\xi_k^*, \tau_i) = a_k \vartheta_z (\tau_i) + b_k\,, \tag{14}$$

where

$$a_k = \Delta\tau\Big[\vartheta_w\left(\xi_k^*, \tau_1\right)\Theta + \vartheta_w\left(\xi_k^*, \tau_0\right)(1-\Theta)\Big] , \tag{15}$$

$$b_k = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_{i-1} \end{bmatrix}^T \begin{bmatrix} \vartheta_z(\tau_1) \\ \vartheta_z(\tau_2) \\ \vdots \\ \vartheta_z(\tau_{i-1}) \end{bmatrix} - \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_i \end{bmatrix}^T \begin{bmatrix} \vartheta_z(\tau_0) \\ \vartheta_z(\tau_1) \\ \vdots \\ \vartheta_z(\tau_{i-1}) \end{bmatrix} . \tag{16}$$

In order to determine the temperature at the boundary for ith unit of time we minimize the functional

$$I\left(\vartheta_z(\tau_i)\right) = \sum_{k=1}^{M}\Big(\vartheta_m\Big(\xi_k^*, \tau_i\Big) - \vartheta_c\left(\xi_k^*, \tau_i\right)\Big)^2 = \min , \tag{17}$$

where ϑ_m denotes the measured temperature and ϑ_c the calculated temperature, described with the formula (14). The functional dependent on the temperature at the boundary $\vartheta_z(\tau_i)$ based on (14) can be written as follows:

$$I\left(\vartheta_z(\tau_i)\right) = \sum_{k=1}^{M}\Big[\vartheta_m\left(\xi_k^*, \tau_i\right) - \left(a_k\vartheta_z(\tau_i) + b_k\right)\Big]^2 = \min . \tag{18}$$

The functional reaches the minimum when its derivative with respect to temperature at the edge is zero. Thus,

$$\frac{\partial I\left(\vartheta_z(\tau_i)\right)}{\partial\vartheta_z(\tau_i)} = \sum_{k=1}^{M} 2\Big[\vartheta_m\left(\xi_k^*, \tau_i\right) - \left(a_k\vartheta_z(\tau_i) + b_k\right)\Big](-a_k) = 0 , \tag{19}$$

hence

$$\sum_{k=1}^{M}\Big[-a_k^2\vartheta_z(\tau_i) + a_k\left(\vartheta_m\left(\xi_k^*, \tau_i\right) - b_k\right)\Big] = 0 . \tag{20}$$

Expression (20) can be written as two sums, the first of which containing the sought for temperature at the boundary $\vartheta_z(\tau_i)$ and the other is dependent on the measurement values

$$\sum_{k=1}^{M}\Big(-a_k^2\vartheta_z(\tau_i)\Big) + \sum_{k=1}^{M}\Big[a_k\left(\vartheta_m\left(\xi_k^*, \tau_i\right) - b_k\right)\Big] = 0 . \tag{21}$$

The temperature distribution on the boundary is thus

$$\vartheta_z(\tau_i) = \frac{\sum_{k=1}^{M}\Big[a_k\left(\vartheta_m\left(\xi_k^*, \tau_i\right) - b_k\right)\Big]}{\sum_{k=1}^{M} a_k^2} . \tag{22}$$

4 Sensitivity of the solution of the inverse problem to measurement errors for the measurement performed with M thermocouples

Analyzing the sensitivity of the inverse problem to a distortion of the measurement data, the thermocouples installation error by $\delta\xi_k$ and random temperature measurement errors $\delta\vartheta_{mk}$ for kth thermocouple ($k = 1, 2, K, M$) were considered. Then the sought temperature at the boundary is

$$\vartheta_z\left(\tau_i, [\delta\xi], [\delta\vartheta]\right) = \frac{\sum_{k=1}^{M} a_k \left[\vartheta_m\left(\xi_k^* + \delta\xi_k, \tau_i\right) + \delta\vartheta_{mk} - b_k\right]}{\sum_{k=1}^{M} a_k^2} . \tag{23}$$

Hence

$$\delta\vartheta_z = \vartheta_z\left(\tau_i, [\delta\xi], [\delta\vartheta]\right) - \vartheta_z\left(\tau_i, [0], [0]\right) = \frac{\sum_{k=1}^{M} a_k \left[\vartheta_m\left(\xi_k^* + \delta\xi_k, \tau_i\right) + \delta\vartheta_{mk} - \vartheta_m\left(\xi_k^*, \tau_i\right)\right]}{\sum_{k=1}^{M} a_k^2} . \tag{24}$$

5 Numerical example for $M = 2$ thermocouples

The analysis of the sensitivity of the inverse problem to the measurement errors was performed for a cylinder of the radius of $r_z = 50$ mm made from steel of the density of $\rho = 7841$ kg/m^3, specific heat $c = 456$ J/(kgK) and thermal conductivity $\lambda = 50.3$ W/(mK). The assumed temperatures distribution at the boundary in the direct problem is $f(\tau) = 1 - e^{-1.5\tau}$. Such a course of temperature corresponds to some processes of nitriding. The heating lasted for $t = 680$ s ($\tau = 3.826$) from 0 °C to 500 °C. It was assumed that the temperature measurement had been performed with two thermocouples placed on the radius $r_1^* = 48$ mm (2 mm from the edge of the cylinder) and on the radius $r_2^* = 46$ mm (4 mm from the edge of the cylinder). The measurement accuracy was 2.2 °C and the thermocouple installation error – $\delta\xi^* = \pm\, 0.5$ mm. The calculations were made for $\Theta =$ 0.5, 0.6, 0.7, 0.8, 0.9, and 1.0. For $\Theta < 0.5$ the solution of the inverse problem was unstable.

Temperature distribution assumed in the direct problem (dp) and that calculated with the inverse problem (ip) considering the thermocouple installation errors and the stochastic temperature error (ran) have been shown

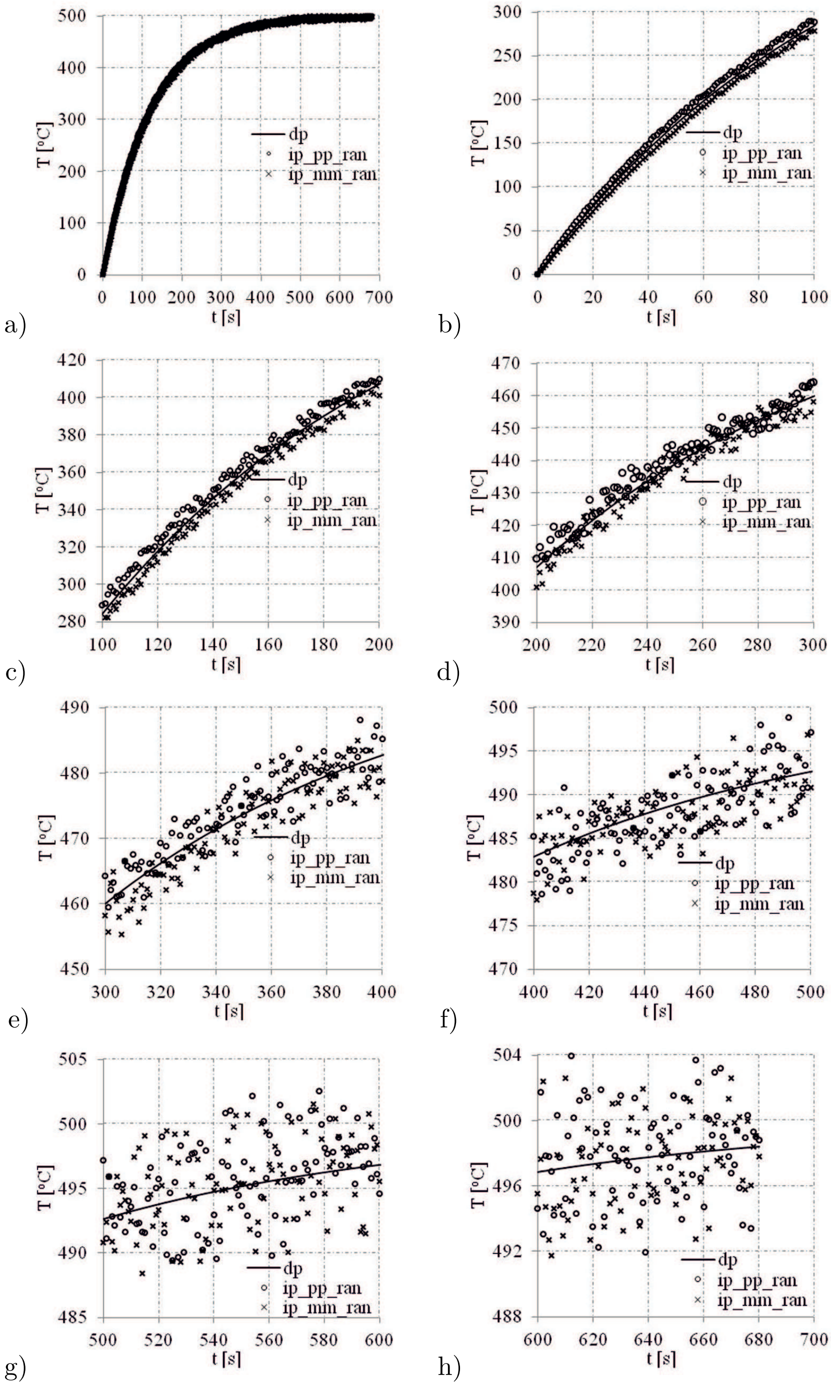

Figure 2: Temperature at the boundary of the cylinder assumed in the direct problem (dp) and calculated using the inverse problem (ip) considering the thermocouples installation errors (pp, mm) and stochastic temperature measurement error (ran) for $\Theta = 0.5$ and time: a) 0–680 s, b) 0–100 s, c) 100–200 s, d) 200–300 s, e) 300–400 s, f) 400–500 s, g) 500–600 s, h) 600–680 s.

in Fig. 2. The symbol pp denotes that both thermocouples are placed closer to the edge of the cylinder, i.e., $r_1^* = 48.5$ mm and $r_2^* = 46.5$ mm while mm denotes that both thermocouples are closer to the axis of the cylinder, i.e., $r_1^* = 47.5$ mm and $r_2^* = 45.5$ mm. Symbol pm indicates that the first thermocouple is placed closer to the edge of the cylinder with the thermocouple installation error $\partial r^* = 0.5$ mm and second thermocouple is closer to the axis of the cylinder with the thermocouple installation error $\partial r^* = -0.5$ mm ($r_1^* = 48.5$ mm, $r_2^* = 45.5$ mm). Symbol mp indicates that the first thermocouple is placed closer to the axis of the cylinder with the thermocouple installation error $\partial r^* = -0.5$ mm and second thermocouple is closer to the edge of the cylinder with the thermocouple installation error $\partial r^* = 0.5$ mm ($r_1^* = 47.5$ mm and $r_2^* = 46.5$ mm).

For the analyzed heating process, the maximum differences between the temperature assumed at the boundary of the cylinder and that calculated with the inverse problem have been determined, considering the temperature measurement error for $\Theta = 0.5$, 0.6, 0.7, 0.8, 0.9, and 1.0 (Fig. 3). These differences decrease as the Θ parameter grows and they fall in the temperature range from 9 °C to 3 °C. The greatest temperature distribution errors at the boundary of the cylinder occur if we have a stochastic error of temperature measurement and both thermocouples installation negative error ($r_1^* = 47.5$ mm and 45.5 mm). These range from 5.5 °C to 9 °C. When both thermocouples are closer to the edge than assumed, the maximum error varies from 3.5 °C to 9 °C. If one of the thermocouples is shifted towards the edge and the other towards the axis of the cylinder the error at the edge is 7 °C for $\Theta = 0.5$ and as Θ grows it decreases to reach 3.5 °C for $\Theta = 1$. The selection of parameter Θ for the two measurement thermocouples influences the sensitivity of the solution of the inverse problem, yet, the errors occurring in the solution are much smaller than in the case when the measurement is carried out with just one thermocouple [11]. The solution for $\Theta < 0.5$, similarly to the measurement with only one thermocouple, is unstable [11]. The smallest values of temperature distribution error at the boundary of the cylinder were obtained for $\Theta = 1$ (this denotes excess integration).

6 Conclusions

In the paper the inverse problem has been solved for a transient heat conduction equation, allowing for calculation of the temperature in the cross-

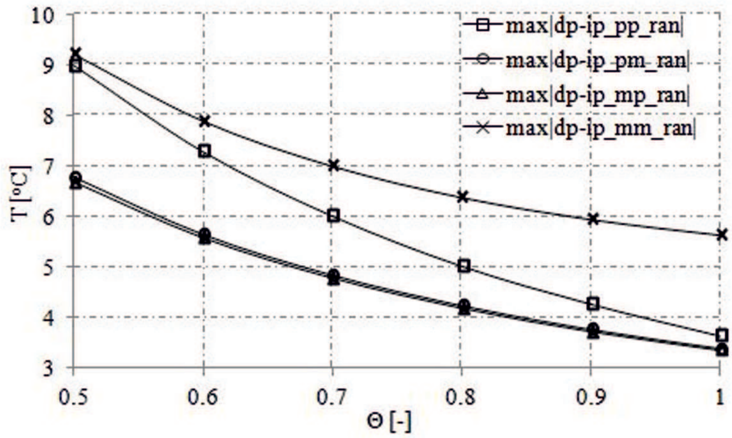

Figure 3: Maximum from the absolute value of difference of temperature at the boundary of the cylinder assumed in the direct problem and that calculated with the inverse problem (ip) considering the thermocouples installation error (pp, pm, mp, mm) and stochastic temperature measurement error (ran) for $\Theta \in [0, 5, 1]$.

section of the cylinder. For this research, a calculation model based on the inverse problem was developed. The sensitivity of the solution of the inverse problem to temperature measurement errors and thermocouple installation errors has been analyzed. In these calculations, the thermocouple installation error of 0.5 mm and random temperature measurement error of maximum 2.2 °C have been taken into account. The sensitivity tests have been performed for the exponential function $f(\tau) = 1 - e^{-1.5\tau}$. The sensitivity of the solution of the inverse problem has been analyzed by changing the integral parameter Θ. For two thermocouples a maximum difference between the assumed temperature at the edge and the temperature calculated below 6 °C (for $\Theta = 1$) was obtained using the inverse problem. It has been ascertained that a proper selection of Θ during numerical integration leads to a solution that is much less sensitive to measurement data distortions. The calculation model provides convergent solutions of low sensitivity to distortions for $\Theta \geq 0.5$. Parameter Θ has the properties that regularize the inverse problem. Taking into account the non-linearity of the thermal conductivity, specific heat and density (as a function of temperature) leads to a numerical solution of the conduction equation.

Received 30 December 2015, in revised form 25 May 2017

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