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Pointwise observation of the state given by complex time lag parabolic system

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Various optimization problems for linear parabolic systems with multiple constant time lags are considered. In this paper, we consider an optimal distributed control problem for a linear complex parabolic system in which different multiple constant time lags appear both in the state equation and in the Neumann boundary condition. Sufficient conditions for the existence of a unique solution of the parabolic time lag equation with the Neumann boundary condition are proved. The time horizon T is fixed. Making use of the Lions scheme [13], necessary and sufficient conditions of optimality for the Neumann problem with the quadratic performance functional with pointwise observation of the state and constrained control are derived. The example of application is also provided.

Key words: distributed control, parabolic system, time lags, pointwise observation.

1. Introduction

Various optimization problems associated with the optimal control of distributed parabolic systems with lags appearing in the boundary conditions have been studied recently in Refs. [1] - [11] and [12], [16], [17].

In this paper, we consider an optimal distributed control problem for a linear complex parabolic system in which different multiple constant time lags appear both in the state equation and the Neumann boundary condition.

Such complex systems constitute in a linear approximation, a universal mathematical model for many diffusion processes.

Sufficient conditions for the existence of a unique solution of such time lag parabolic equations with the Neumann boundary conditions involving multiple time lags are proved.

In this paper, we restrict our considerations to the case of the distributed control for the Neumann problem. Consequently, we formulate the following optimal control problem. We assume that the performance functional has the quadratic form with pointwise

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observation of the state. Moreover, the time horizon is fixed in our optimization problem. Finally, we impose some constraints on the distributed control. Making use of the Lions framework [13] necessary and sufficient conditions of optimality with the quadratic performance functional with pointwise observation of the state and constrained control are derived for the Neumann problem. The example of application is also provided.

2. Existence and uniqueness of solutions

Consider now the distributed-parameter system described by the following parabolic lag equation

$$\frac{\partial y}{\partial t}+A(t)y+\sum_{i=1}^{m}y(x,t-h_i)=v \quad x\in\Omega,t\in(0,T) \tag{1}$$

$$y(x,t')=\Phi_0(x,t') \qquad x\in\Omega,t'\in[-\Upsilon,0) \tag{2}$$

$$y(x,0)=y_0(x) \qquad x\in\Omega \tag{3}$$

$$\frac{\partial y}{\partial \eta_A}=\sum_{s=1}^{l}y(x,t-k_s)+u \qquad x\in\Gamma,t\in(0,T) \tag{4}$$

$$y(x,t')=\Psi_0(x,t') \qquad x\in\Gamma,t'\in[-\Upsilon,0) \tag{5}$$

where: $\Omega\subset R^n$ is a bounded, open set with boundary Γ , which is a C^∞ - manifold of dimension $n-1$. Locally, Ω is totally on one side of Γ . $\dfrac{\partial y}{\partial \eta_A}$ is a normal derivative at Γ, directed towards the exterior of Ω,

$$y\equiv y(x,t;v), \qquad v\equiv v(x,t), \qquad u\equiv u(x,t),$$

$$Q=\Omega\times(0,T), \qquad \bar{Q}=\bar{\Omega}\times[0,T],$$

$$Q_0=\Omega\times[-\Upsilon,0), \qquad \Sigma=\Gamma\times(0,T),$$

$$\Sigma_0=\Gamma\times[-\Upsilon,0),$$

h_i,k_s are specified positive numbers representing time lags such that $0\leqslant h_1<h_2<\ldots<h_m$ for $i=1,\ldots,m$ and $0\leqslant k_1<k_2<\ldots<k_l$ for $s=1,\ldots,l$ respectively, Φ_0,Ψ_0 are initial functions defined on Q_0 and Σ_0 respectively. Moreover, $\Upsilon=\max\{h_m,k_l\}$.

The operator $A(t)$ has the form

$$A(t)y=-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial y(x,t)}{\partial x_j}\right) \tag{6}$$

and the functions $a_{ij}(x,t)$ satisfy the condition

$$\sum_{i,j=1}^{n}a_{ij}(x,t)\Phi_i\Phi_j\geqslant\alpha\sum_{i=1}^{n}\Phi_i^2 \qquad \alpha>0, \tag{7}$$

$$\forall (x,t) \in \bar{Q},\ \forall \Phi_i \in R$$

where: $a_{ij}(x,t)$ are real C^∞ functions defined on $\bar{Q}$ (closure of Q). The equations (1) - (5) constitute a Neumann problem.

First we shall prove sufficient conditions for the existence of a unique solution of the mixed initial-boundary value problem (1) - (5) for the case where $v \in L^2(Q)$.

For this purpose, for any pair of real numbers $r,s \geqslant 0$, we introduce the Sobolev space $H^{r,s}(Q)$ ([14], Vol. 2, p.6) defined by

$$\left.\begin{array}{l} H^{r,s}(Q) = H^0(0,T;H^r(\Omega)) \cap H^s(0,T;H^0(\Omega)) \\ \text{which is a Hilbert space normed by} \\ \left(\int\limits_0^T \| y(t) \|^2_{H^r(\Omega)}\, dt + \| y \|^2_{H^s(0,T;H^0(\Omega))} \right)^{\frac{1}{2}} \end{array}\right\} \tag{8}$$

where: the spaces $H^r(\Omega)$ and $H^s(0,T;H^0(\Omega))$ are defined in Chapter 1 ([14], Vol.1) respectively.

Consequently, some properties and central theorems for the functions $y \in H^{r,s}(Q)$ are given in [7], [10] and [14].

The existence of a unique solution for the mixed initial-boundary value problem (1) - (5) on the cylinder Q can be proved using a constructive method, i.e., first, solving (1) - (5) on the subcylinder Q_1 and in turn on Q_2, etc. until the procedure covers the whole cylinder Q. In this way the solution in the previous step determines the next one.

For simplicity, we introduce the following notations:

$$E_j \stackrel{\triangle}{=} ((j-1)\lambda, j\lambda) \text{ where } \lambda = \min\{h_1,k_1\}, \quad Q_j = \Omega \times E_j,$$

$$\Sigma_j = \Gamma \times E_j \ \text{ for } \ j = 1,\dots K\,.$$

Using the results of Section 14 ([13], pp. 182-185) we can prove the following result.

Lemma 1 *Let*

$$v \in L^2(Q) \tag{9}$$

$$f_j \in L^2(Q_j) \tag{10}$$

where

$$f_j(x,t) = v(x,t) - \sum_{i=1}^{m} y_{j-1}(x,t-h_i)$$

$$y_{j-1}(\cdot,(j-1)\lambda) \in L^2(\Omega) \tag{11}$$

$$q_j \in H^{1/2,1/4}(\Sigma_j) \tag{12}$$

where

$$q_j(x,t) = \sum_{s=1}^{l} y_{j-1}(x,t-k_s) + u(x,t).$$

Then, there exists a unique solution $y_j \in H^{2,1}(Q_j)$ *for the mixed initial-boundary value problem* (1), (4), (11).

Proof: We observe that for $j = 1$,

$$\sum_{i=1}^{m} y_{j-1}\big|_{Q_0}(x,t-h_i) = \sum_{i=1}^{m} \Phi_0(x,t-h_i) \quad \text{and}$$

$$\sum_{s=1}^{l} y_{j-1}\big|_{\Sigma_0}(x,t-k_s) = \sum_{s=1}^{l} \Psi_0(x,t-k_s).$$

Then the assumptions (10), (11) and (12) are fulfilled if we assume that $\Phi_0 \in H^{2,1}(Q_0)$, $y_0 \in L^2(\Omega)$, $u \in H^{1/2,1/4}(\Sigma)$ and $\Psi_0 \in H^{1/2,1/4}(\Sigma_0)$. These assumptions are sufficient to ensure the existence of a unique solution $y_1 \in H^{2,1}(Q_1)$. Next for $j = 2$ we have to verify that $f_2 \in L^2(Q_2)$, $y_1(\cdot,\lambda) \in L^2(\Omega)$ and $q_2 \in H^{1/2,1/4}(\Sigma_2)$. It is easy to notice that the condition (10) follows from the fact that $y_1 \in H^{2,1}(Q_1)$ and $v \in L^2(Q)$. Really, from the Theorem 3.1 ([14], Vol.1, p.19) we can prove that $y_1 \in H^{2,1}(Q_1)$ implies that the mapping $t \to y_1(\cdot,t)$ is continuous from $[0,\lambda] \to H^1(\Omega) \subset L^2(\Omega)$, hence $y_1(\cdot,\lambda) \in L^2(\Omega)$. Then using the Trace Theorem ([14], Vol. 2, p.9) we can verify that $y_1 \in H^{2,1}(Q_1)$ implies that $y_1 \to y_1\Big|_{\Sigma_1}$ is a linear, continuous mapping of $H^{2,1}(Q_1) \to H^{1/2,1/4}(\Sigma)$. Assuming that $v \in H^{1/2,1/4}(\Sigma)$, the condition $q_2 \in H^{1/2,1/4}(\Sigma_2)$ is fulfilled. Then, there exists a unique solution $y_2 \in H^{2,1}(Q_2)$. We shall now summarize the foregoing result for any $Q_j, j = 3,\ldots,K$.

Theorem 1 *Let* y_0, Φ_0, Ψ_0, u *and* v *be given with* $y_0 \in L^2(\Omega), \Phi_0 \in H^{2,1}(Q_0), \Psi_0 \in H^{1/2,1/4}(\Sigma_0)$, $u \in H^{1/2,1/4}(\Sigma)$ *and* $v \in L^2(Q)$. *Then, there exists a unique solution* $y \in H^{2,1}(Q)$ *for the mixed initial-boundary value problem* (1) – (5). *Moreover,* $y(\cdot, j\lambda) \in L^2(\Omega)$ *for* $j = 1,\ldots,K$.

3. Problem formulation. Optimization theorems

We shall now formulate the optimal distributed control problem for the Neumann problem. Let us denote by $U = L^2(Q)$ the space of controls. The time horizon T is fixed in our problem.

Let $x^1,\ldots,x^\mu$ be points of Ω. We assume that the observation is $\{y(x^j,t;v)\}$, $1 \leqslant j \leqslant \mu$ - provided we can attach a meaning to this.

If we now assume that the coefficients of the operator A in the equation (1) are sufficiently regular, then from the Theorem 1 it follows that

$$y(v) \in H^{2,1}(Q). \tag{13}$$

Hence, $y(v) \in L^2(0,T;H^2(\Omega))$ and $y(x^j,t)$ has meaning (and "$t \to y(x^j,t)$" $\in L^2(0,T)$) if

$$H^2(\Omega) \subset C^0(\Omega) \tag{14}$$

which is true if (and only if)

$$\frac{1}{2} - \frac{2}{n} < 0 \quad \text{i.e.} \quad n \leqslant 3\,.$$

Hence we make the standing hypothesis that the dimension is $n \leqslant 3$.

Then the observation

$$Cy(v) = \{y(x^j,t;v)\} \in (L^2(0,T))^\mu \tag{15}$$

The cost function is now given

$$I(v) = \lambda_1 \parallel Cy(v) - z_d \parallel^2_{(L^2(0,T))^\mu} + \lambda_2 \int_Q (Nv)v \, dxdt \tag{16}$$

If $z_d = \{z_{d1}, \ldots, z_\mu\}$,

$$I(v) = \lambda_1 \sum_{j=1}^{\mu} \int_0^T | \, y(x^j,t;v) - z_{dj}(t) \, |^2 \, dt + \lambda_2 \int_Q (Nv)v \, dxdt \tag{17}$$

where: $\lambda_i \geqslant 0$, $\lambda_1 + \lambda_2 > 0$; $z_{dj}(t)$ are given elements in $L^2(0,T)$ and N is a positive, linear operator on $L^2(Q)$ into $L^2(Q)$.

Finally, we assume the following constraint on controls $v \in U_{ad}$, where

$$U_{ad} \text{ is a closed, convex subset of } U \tag{18}$$

Let $y(x,t;v)$ denote the solution of the mixed initial-boundary value problem (1)- (5) at (x,t) corresponding to a given control $v \in U_{ad}$. We note from the Theorem 1 that for any $v \in U_{ad}$ the performance functional (17) is well-defined since $y(v) \in H^{2,1}(Q)$. The solving of the formulated optimal control problem is equivalent to seeking a $v_0 \in U_{ad}$ such that $I(v_0) \leqslant I(v) \quad \forall v \in U_{ad}$. Then from the Theorem 1.3 ([13], p. 10) it follows that for $\lambda_2 > 0$ a unique optimal control v_0 exists; moreover, v_0 is characterized by the following condition

$$I'(v_0) \cdot (v - v_0) \geqslant 0 \qquad \forall v \in U_{\text{ad}} \tag{19}$$

Using the form of the cost function given by (17) we can express (19) in the following form

$$\lambda_1 \sum_{j=1}^{\mu} \int_0^T (y(x^j,t;v_0) - z_{dj}(t))(y(x^j,t;v) - y(x^j,t;v_0))dt +$$

$$+\lambda_2 \int_Q N v_0 (v - v_0) dxdt \geqslant 0 \qquad \forall v \in U_{ad} \tag{20}$$

To simplify (20), we introduce the adjoint equation and for every $v \in U_{ad}$, we define the adjoint variable $p = p(v) = p(x,t;v)$ as the solution of the equation

$$\begin{gathered} -\frac{\partial p(v)}{\partial t} + A^*(t)p(v) + \sum_{i=1}^{m} p(x,t+h_i;v) = \lambda_1 \sum_{j=1}^{\mu} (y(x^j,t;v) - z_{dj}(t)) \otimes \delta(x - x^j) \\ x \in \Omega, t \in (0, T - \Upsilon) \end{gathered} \tag{21}$$

$$\begin{gathered} -\frac{\partial p(v)}{\partial t} + A^*(t)p(v) = \lambda_1 \sum_{j=1}^{\mu} (y(x^j,t;v) - z_{dj}(t)) \otimes \delta(x - x^j) \\ x \in \Omega, t \in (T - \Upsilon, T) \end{gathered} \tag{22}$$

$$p(x,T;v) = 0 \qquad x \in \Omega \tag{23}$$

$$\frac{\partial p(v)}{\partial \eta_{A^*}}(x,t) = \sum_{s=1}^{l} p(x,t+k_s;v) \qquad x \in \Gamma, t \in (0, T - \Upsilon) \tag{24}$$

$$\frac{\partial p(v)}{\partial \eta_{A^*}}(x,t) = 0 \qquad x \in \Gamma, t \in (T - \Upsilon, T) \tag{25}$$

where

$$\left.\begin{aligned} &g(t) \otimes \delta(x - x^j) \text{ is the distribution,} \\ &\Psi \to \int_0^T g(t)\Psi(x^j,t)dt, \ \Psi \in \mathcal{D}(Q) \\ &A^*(t)p = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial p}{\partial x_i}\right) \end{aligned}\right\} \tag{26}$$

The existence of a unique solution for the problem (21) - (25) on the cylinder Q can be proved using a constructive method. It is easy to notice that for given z_d and v , problem (21) - (25) can be solved backwards in time starting from $t = T$, i.e., first, solving (21) - (25) on the subcylinder Q_K and in turn on Q_{K-1} , etc. until the procedure covers the whole cylinder Q . For this purpose, we may apply Theorem 1 (with an obvious change of variables) to problem (21) - (25) (with reversed sense of time, i.e., $t' = T - t$).

Lemma 2 *Let the hypothesis of Theorem 1 be satisfied. Then, for given $z_{dj}(t) \in L^2(0,T)$ and any $v \in L^2(Q)$, there exists a unique solution $p(v) \in H^{2,1}(Q)$ for the problem* (21) - (25) *defined by transposition*

$$\int_Q p(v_0)\left(\frac{\partial \Psi}{\partial t} + A\Psi\right) dxdt = \sum_{j=1}^{\mu} \int_0^T \left(y(x^j,t;v_0) - z_{dj}(t)\right)\Psi(x^j,t)dt \tag{27}$$

$$\forall \Psi \in H^{2,1}(Q), \quad \Psi\Big|_{\Sigma} = 0, \quad x \in \Gamma,\ t \in (0,T) \quad \text{and} \quad \Psi(x,T) = 0.$$

Remark 1 *The right hand side of* (27) *is a continuous linear form on* $H^{2,1}(Q)$ *if* $n \leqslant 3$.

Consequently, after transformations the first component on the left-hand side of (20) can be rewriten as

$$\lambda_1 \sum_{j=1}^{\mu} \int_0^T (y(x^j,t;v_0) - z_{dj}(t))(y(x^j,t;v) - y(x^j,t;v_0))dt = \int_Q p(v_0)(v - v_0)\, dxdt \tag{28}$$

Substituting (28) into (20) we obtain

$$\int_Q (p(v_0) + \lambda_2 N v_0)(v - v_0)\, dxdt \geqslant 0, \quad \forall v \in U_{ad} \tag{29}$$

Theorem 2 *For the problem* (1) - (5) *with the performance functional* (17) *with* $z_{dj}(t) \in L^2(0,T)$ *and* $\lambda_2 > 0$ *and with constraints on controls* (18)*, there exists a unique optimal control* v_0 *which satisfies the maximum condition (29).*

Consider now the particular case where $U_{ad} = L^2(Q)$. Thus the maximum condition (29) is satisfied when

$$v_0 = -\lambda_2^{-1} N^{-1} p(v_0) \tag{30}$$

We must notice that the conditions of optimality derived above (Theorem 2) allow us to obtain an analytical formula for the optimal control in particular cases only (e.g. there are no constraints on controls). This results from the following: the determining of the function $p(v_0)$ in the maximum condition from the adjoint equation is possible if and only if we know y_0 which corresponds to the control v_0. These mutual connections make the practical use of the derived optimization formulas difficult. Therefore we resign from the exact determining of the optimal control and we use approximation methods.

In the case of performance functional (17) with $\lambda_1 > 0$ and $\lambda_2 = 0$, the optimal control problem reduces to the minimizing of the functional on a closed and convex subset in a Hilbert space. Then, the optimization problem is equivalent to a quadratic programming one (Ref. [10]) which can be solved by the use of the well-known algorithms, e.g. Gilbert's (Ref. [10]).

The practical application of Gilbert's algorithm to an optimal control problem for a parabolic system with boundary condition involving a time lag is presented in [12]. Using Gilbert's algorithm, a one-dimensional numerical example of the plasma control process is solved.

4. Example

Making use of the results of [13], we shall express the optimal control (30) in the feedback form.

For this purpose we consider the following set of equations with $\varepsilon \in (0,T)$:

$$\begin{cases} \dfrac{\partial y}{\partial t} + A(t)y + \sum\limits_{i=1}^{m} y(x,t-h_i) + \lambda_2^{-1}N^{-1}p = 0 & \text{for } t-h_i \geqslant \varepsilon \\ \dfrac{\partial y}{\partial t} + A(t)y + \sum\limits_{i=1}^{m} \Phi_\varepsilon(x,t-h_i) + \lambda_2^{-1}N^{-1}p = 0 & \text{for } t-h_i < \varepsilon \end{cases} \tag{31}$$

$$(x,t) \in \Omega \times (\varepsilon, T)$$

$$\begin{cases} -\dfrac{\partial p}{\partial t} + A^*(t)p + \sum\limits_{i=1}^{m} p(x,t+h_i) \;-\; \lambda_1 y = -\lambda_1 z_d & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } (x,t) \in \Omega \times (\varepsilon, T-\Upsilon) & \\ -\dfrac{\partial p}{\partial t} + A^*(t)p - \lambda_1 y = -\lambda_1 z_d \qquad \text{for } (x,t) \in \Omega \times (T-\Upsilon, T) & \end{cases} \tag{32}$$

with boundary conditions

$$\frac{\partial y}{\partial \eta_A}(x,t) = \begin{cases} \sum\limits_{s=1}^{l} y(x,t-k_s) + u(x,t) & \text{for } t-k_s \geqslant \varepsilon \\ \sum\limits_{s=1}^{l} \Psi_\varepsilon(x,t-k_s) + u(x,t) & \text{for } t-k_s < \varepsilon \end{cases} \tag{33}$$

$$(x,t) \in \Gamma \times (\varepsilon, T)$$

$$\frac{\partial p}{\partial \eta_{A^*}}(x,t) = \begin{cases} \sum\limits_{s=1}^{l} p(x,t+k_s) & \text{for } (x,t) \in \Gamma \times (\varepsilon, T-\Upsilon) \\ 0 & \text{for } (x,t) \in \Gamma \times (T-\Upsilon, T) \end{cases} \tag{34}$$

and with initial and final conditions

$$\begin{cases} y(x,\varepsilon) = y_\varepsilon(x) & x \in \Omega \\ p(x,T) = 0 & x \in \Omega \end{cases} \tag{35}$$

where: $y_\varepsilon \in H^1(\Omega)$, Φ_ε and Ψ_ε are given function defined on $\Omega \times [\varepsilon - \Upsilon, \varepsilon)$ and $\Gamma \times [\varepsilon - \Upsilon, \varepsilon)$ respectively, that is $\Phi_\varepsilon \in H^{2,1}(\Omega \times [\varepsilon - \Upsilon, \varepsilon))$ and $\Psi_\varepsilon \in H^{1/2,1/4}(\Gamma \times [\varepsilon - \Upsilon, \varepsilon))$.

We shall consider problem (31) - (35) subject to (1) for $t \in (\varepsilon, T)$ and $U_{ad} = L^2(Q)$. The performance functional is given by

$$I_\varepsilon(v) = \lambda_1 \int_\varepsilon^T \int_\Omega \mid y(x,t;v) - z_d \mid^2 dxdt + \lambda_2 \int_\varepsilon^T \int_\Omega (Nv)v \, dxdt \tag{36}$$

Then the problem (31)-(36) with $\lambda_2 > 0$ has a unique optimal control in the form (30). Also it is easy to verify that (31) - (35) has a unique solution

$$\{y,p\} \in H^{2,1}(\Omega \times (\varepsilon,T)).$$

Proposition 1 *Let* $\{y,p\}$ *be solution of* (31) - (35) *with* $\varepsilon = 0$. *We define* σ_ε, *the system "state" at time* ε, *by the triplet* $(y(\cdot,\varepsilon),\Phi_\varepsilon,\Psi_\varepsilon)$, *where*

$$\Phi_\varepsilon(\cdot,t') = \begin{cases} \Phi_0(\cdot,t') & \text{for } t' \in \hat{E}_\varepsilon = [-\Upsilon,0) \cap [\varepsilon-\Upsilon,\varepsilon) \\ y(\cdot,t')|_\Omega & \text{for } t' \in [\varepsilon-\Upsilon,\varepsilon) - \hat{E}_\varepsilon \end{cases} \tag{37}$$

$$\Psi_\varepsilon(\cdot,t') = \begin{cases} \Psi_0(\cdot,t') & \text{for } t' \in \hat{E}_\varepsilon = [-\Upsilon,0) \cap [\varepsilon-\Upsilon,\varepsilon) \\ y(\cdot,t')|_\Gamma & \text{for } t' \in [\varepsilon-\Upsilon,\varepsilon) - \hat{E}_\varepsilon \end{cases} \tag{38}$$

Then, for all triplets $\varepsilon \leqslant t$ *in* $(0,T)$,

$$p(\cdot,t) = P(t,\varepsilon)\sigma_\varepsilon + r_\varepsilon(\cdot,t) \tag{39}$$

where $P(t,\varepsilon)$ *and* $r_\varepsilon(\cdot,t)$ *are determined by the following procedure*

First we solve the set of equations

$$\begin{cases} \dfrac{\partial\alpha}{\partial t} + A(t)\alpha + \displaystyle\sum_{i=1}^{m} \alpha(x,t-h_i) + \lambda_2^{-1}N^{-1}\beta = 0 \\ \text{for } \ t-h_i \geqslant \varepsilon \\ \\ \dfrac{\partial\alpha}{\partial t} + A(t)\alpha + \displaystyle\sum_{i=1}^{m} \Phi_\varepsilon(x,t-h_i) + \lambda_2^{-1}N^{-1}\beta = 0 \\ \text{for } \ t-h_i < \varepsilon \end{cases} \tag{40}$$

$$(x,t) \in \Omega \times (\varepsilon,T)$$

$$\begin{cases} -\dfrac{\partial\beta}{\partial t} + A^*(t)\beta + \displaystyle\sum_{i=1}^{m} \beta(x,t+h_i) \ -\lambda_1\alpha = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } \ (x,t) \in \Omega \times (\varepsilon, T-\Upsilon) \\ -\dfrac{\partial\beta}{\partial t} + A^*(t)\beta - \lambda_1\alpha = 0 \qquad\quad \text{for } \ (x,t) \in \Omega \times (T-\Upsilon, T) \end{cases} \tag{41}$$

with boundary conditions

$$\frac{\partial\alpha}{\partial\eta_A}(x,t) = \begin{cases} \displaystyle\sum_{s=1}^{l} \alpha(x,t-k_s) & \text{for } t-k_s \geqslant \varepsilon \\ \displaystyle\sum_{s=1}^{l} \Psi_\varepsilon(x,t-k_s) & \text{for } t-k_s < \varepsilon \end{cases} \tag{42}$$

$$(x,t) \in \Gamma \times (\varepsilon, T)$$

$$\frac{\partial \beta}{\partial \eta_{A^*}}(x,t) = \begin{cases} \sum_{s=1}^{l} \beta(x,t+k_s) & \text{for } (x,t) \in \Gamma \times (\varepsilon, T-\Upsilon) \\ 0 & \text{for } (x,t) \in \Gamma \times (T-\Upsilon, T) \end{cases} \tag{43}$$

and with initial and final conditions

$$\begin{cases} \alpha(x,\varepsilon) = y(x,\varepsilon) & x \in \Omega \\ \beta(x,T) = 0 & x \in \Omega \end{cases} \tag{44}$$

then

$$P(t,\varepsilon)\sigma_\varepsilon = \beta(\cdot,t) \tag{45}$$

Next we solve the set of equations

$$\begin{cases} \dfrac{\partial \kappa}{\partial t} + A(t)\kappa + \sum_{i=1}^{m} \kappa(x,t-h_i) + \lambda_2^{-1} N^{-1} \delta = 0 \\ \qquad \text{for } t - h_i \geqslant \varepsilon \\ \dfrac{\partial \kappa}{\partial t} + A(t)\kappa + \lambda_2^{-1} N^{-1} \delta = 0 \quad \text{for } t - h_i < \varepsilon \end{cases} \tag{46}$$

$$(x,t) \in \Omega \times (\varepsilon, T)$$

$$\begin{cases} -\dfrac{\partial \delta}{\partial t} + A^*(t)\delta + \sum_{i=1}^{m} \delta(x,t+h_i) - \lambda_1 \kappa = -\lambda_1 z_d \\ \qquad \text{for } (x,t) \in \Omega \times (\varepsilon, T-\Upsilon) \\ -\dfrac{\partial \delta}{\partial t} + A^*(t)\delta - \lambda_1 \kappa = -\lambda_1 z_d \quad \text{for } (x,t) \in \Omega \times (T-\Upsilon, T) \end{cases} \tag{47}$$

with boundary conditions

$$\frac{\partial \kappa}{\partial \eta_A}(x,t) = \begin{cases} \sum_{s=1}^{l} \kappa(x,t-k_s) + v(x,t) & \text{for } t - k_s \geqslant \varepsilon \\ v(x,t) & \text{for } t - k_s < \varepsilon, \end{cases} \tag{48}$$

$$(x,t) \in \Gamma \times (\varepsilon, T)$$

$$\frac{\partial \delta}{\partial \eta_{A^*}}(x,t) = \begin{cases} \sum_{s=1}^{l} \delta(x,t+k_s) & \text{for } (x,t) \in \Gamma \times (\varepsilon, T-\Upsilon) \\ 0 & \text{for } (x,t) \in \Gamma \times (T-\Upsilon, T) \end{cases} \tag{49}$$

and with initial and final conditions

$$\begin{cases} \kappa(x,\varepsilon) = 0 & x \in \Omega \\ \delta(x,T) = 0 & x \in \Omega \end{cases} \tag{50}$$

then

$$r_\varepsilon(x,t) = \delta(x,t) \tag{51}$$

Setting $\varepsilon = t$ in (39) and substituting the result into (30) we obtain

$$v_0(\cdot,t) = -\lambda_2^{-1} N^{-1}\left(P(t,t), \sigma_t + r_t(\cdot,t)\right), \qquad t \in (0,T) \tag{52}$$

Let us assume that N is the identity operator on $L^2(Q)$. Then using of Schwartz's Kernel Theorem [15], it is easy to prove that the optimal feedback control (52) can be expressed in the following form

$$\begin{aligned} v_0(x,t) = -\lambda_2^{-1} \Bigg\{ & \int_\Omega K_0(x,x',t) y(x',t) dx' + \\ & + \int_{t-\Upsilon}^{t} \int_\Omega K_1(x,x',t,t') \Phi_t(x',t') dx' dt' + \\ & + \int_{t-\Upsilon}^{t} \int_\Gamma K_2(x,x',t,t') \Psi_t(x',t') d\Gamma dt' + r_t(x,t) \Bigg\} \end{aligned} \tag{53}$$

where $\{K_0, K_1, K_2\}$ is the kernel of $P(t,t)$.

5. Conclusions

The results presented in the paper can be treated as a generalization of the results concerning pointwise observation of state given by parabolic systems with the Neumann boundary conditions involving multiple time delays obtained in [11] and by the parabolic equations with the homogeneous Dirichlet boundary conditions obtained in [13] onto the case of different multiple constant time lags appearing both in the parabolic state equations and in the Neumann boundary conditions.

Sufficient conditions for the existence of a unique solution of such time lag parabolic equations with the Neumann boundary conditions involving multiple constant time lags are proved (Lemma 1 and Theorem 1). The optimal control is characterized by using the adjoint equation (Lemma 2). The necessary and sufficient conditions of optimality are derived for a linear quadratic problem (1)-(5), (17), (18) (Theorem 2). The optimal control is obtained in the feedback form (Example).

We can also obtain estimates and a sufficient condition for the boundedness of solutions for such parabolic time lag systems with specified forms of feedback control.

Finally, we can consider optimal control problems of time lag hyperbolic systems with pointwise observation of the state.

The ideas mentioned above will be developed in forthcoming papers.

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