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New discrete reactive power factor definition of the two-terminal network

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Abstract. This paper describes a new method of determining the reactive power factor. The reactive power factor herein is calculated on the basis of time samples and not with the Fourier transform of signals, like it was done previously. The new reactive power factor calculation results from the receiver admittance-operator decomposition into the product of self-adjoint and unitary operators. This is an alternative decomposition to another one, namely into a sum of the Hermitian and skew-Hermitian operators.

Key words: operators, digital filters, reactive power.

1. Introduction, function filters

A digital filter shall mean that the discrete-time signal processor $\{x_n\}_{n=-\infty}^{\infty}$ is operating according to the convolution operator:

$$(Ax)_n = \sum_m A_m x_{n-m}, \tag{1}$$

where n, $m \in (-\infty, \infty)$ are integers.

The digital filter identifier is a series of weights $\{A_n\}_{n=-\infty}^{\infty}$ or the so-called Z form:

$$A(z) = \sum_n A_n z^{-n}, \tag{2}$$

where:

$$A_n = \frac{1}{n!}\left[\frac{d^n A(1/p)}{dp^n}\right]_{p=0} \qquad \text{for n} \geq 0,$$

$$A_n = \frac{1}{2\pi j}\oint A(1/p)\, d(\ln p) \qquad \text{for n} < 0,$$

$p = 1/z$,

and the contour integral is the unit circle. This way, the digital filter has two equivalent identifiers:

$$A(z) \leftrightarrow \{A_n\}. \tag{3}$$

The Borel theorem of convolution can be applied to the two filters $A(z)$ and $B(z)$:

$$A(z)B(z) \leftrightarrow \sum_m A_{n-m}B_m \div \{A_n\} * \{B_n\}. \tag{4}$$

The filter is called causal (past dependent) if

$$A_n = 0 \text{ for n} < 0 \tag{5}$$

The filter A^* is an adjoint filter to A if for any two signals x and y it meets the following equation:

$$(Ax, y) = (x, A^*y), \tag{6}$$

where the dot product of signals or filters is defined as

$$(A, B) = \sum_{n=-\infty}^{\infty} A_n B_n. \tag{7}$$

For a digital convolution-type filter (1) it is held that

$$A^*_n = A_{-n} \leftrightarrow A^*(z) = A(z^{-1}).$$

A filter is called a self-adjoint (Hermitian) filter if $A^* = A$, i.e. when $A_{-n} = A_n \leftrightarrow A(z^{-1}) = A(z)$, and a skew-Hermitian filter if $A^* = -A$, i.e. when $A_{-n} = -A_n \leftrightarrow A(z^{-1}) = A(z)$.

A filter is stable when it converts a bounded signal into another bounded signal (with regard to its values); this can occur if and only if

$$\sum_{n=-\infty}^{\infty} |A_n| < \infty \leftrightarrow |A(z)| < \infty \ \text{ for } z\text{: } |1/z| \leq 1.$$

The functional filter is defined as a digital filter transformed from another digital filter A by the function $f: A \rightarrow f(A)$. If $\{A_n\}$ is a weight sequence identifier of filter A, the weight sequence of filter $f(A)$ will be denoted by $\{(f(A))_n\}$ and its Z form will be defined by $f(A)(z)$.

From the expression

$$\{(f(A))_n\} \leftrightarrow (f(A))(z) = f(\sum_n A_n z^{-n}) = \sum_n (f(A))_n z^{-n} \tag{8}$$

results that if f is an analytical function of a complex variable, then the causality of filter A determines the causality of filter $f(A)$. Furthermore, the adjoint filter function is also analytical: $f(A)(z) \ \ [f(A)]^* = f(A^*)$. It can also be proved that if $f(A)$ is an-

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alytic with respect to conjugation operator, so is its inverse. Indeed, this results from the diagram in Fig. 1.

$$B = f(A) \;\rightarrow B^* = f(A^*) \;\rightarrow A^* = f^{-1}(B^*)$$
$$\downarrow$$
$$A = f^{-1}(B) \longrightarrow A^* = [f^{-1}(B)]^* \;\rightarrow [f^{-1}(B)]^* = f^{-1}(B^*)$$

Fig. 1. Diagram of analytic function with respect to conjugation operator

2. The additive and multiplicative decomposition of discrete-time admittance operator of a two-terminal network

Let sequence $\{y_n\}_{n=0}^{\infty}$ be a sample sequence of the impulse response of the admittance operator of an LSL-class electric two-terminal circuit (lumped time-invariant two-terminal linear circuit) [2]. This sequence is characteristic for a digital filter, which, for physical reasons, is causal and stable. Such filter has two equivalent decompositions, namely into a sum $G + B$, and product Ye^{ϕ}:

$$y = G + B = Ye^{\phi}, \tag{9}$$

where the component-filters meet the following conditions:

$$G^* = G; \;\; B^* = -B, \tag{10}$$

$$Y^* = Y; \;\; (e^{\phi})^* = (e^{\phi})^{-1}. \tag{11}$$

The G, B operators (filters) are self-adjoint and anti-self-adjoint, respectively, and Y is the self-adjoint filter, whereas e^{ϕ} is unitary. Since ϕ is regarded as a filter, it should be an anti-self-adjoint filter in order to satisfy the following exponential function equation:

$$(e^{\phi})^* = (e^{\phi^*}) = (e^{-\phi}) = (e^{\phi})^{-1} \tag{12}$$

Decompositions (10) and (11) mimic a much more elementary decomposition of complex numbers on real and imaginary parts, or on the module and argument (complex factor $e^{j\phi}$ meets the unitarity condition – it lies on the unit circle).

The G, B filter-components can be easily determined using formulas (9) and (10):

$$G = \tfrac{1}{2}(y + y^*); \;\; B = \tfrac{1}{2}(y + y^*), \tag{13}$$

or for the samples:

$$G_n = \tfrac{1}{2}\,(y_n + y_{-n}); \;\; B_n = \tfrac{1}{2}\,(y_n - y_{-n})$$

for $n \in \{0, \pm 1, \pm 2, \ldots\}$, and

$$G_n = \tfrac{1}{2}\, y_n = G_{-n}; \;\; B_n = \tfrac{1}{2}\, y_n = -B_n; \;\; n = 1, 2, 3, \ldots$$

$$G_0 = y_0; \;\; B_n = 0.$$

The calculation of operator-factor components Y and e^{ϕ} of "polar" decomposition in (9) is more complicated. It requires a square root filter. Formula (9) shows that:

$$Y^2 = yy^*, \;\; (e^{\phi})^2 = y(y^{-1})^*, \tag{14}$$

and thus,

$$Y = \sqrt{y}(\sqrt{y})^*, \;\; e^{\phi} = \sqrt{y}[(\sqrt{y})^{-1}]^*, \tag{15}$$

and $e^{-\phi} = (\sqrt{y})^{-1}(\sqrt{y})^*$.

The square root filter $\sqrt{y}$ is a function filter based on the y filter that satisfies [5–8]: $y = \sqrt{y}\sqrt{y}$, or using the convolution formula:

$$\left\{(\sqrt{y})_n\right\} * \left\{(\sqrt{y})_n\right\} = \{y_n\},$$

and using indexes:

$$\sum_{m=0}^{n} (\sqrt{y})_{n-m} (\sqrt{y})_m = y_n, \tag{16}$$

Therefore $2(\sqrt{y})_0(\sqrt{y})_n = y_n - \sum_{m=1}^{n-1} (\sqrt{y})_{n-m}(\sqrt{y})_m$, and finally, taking $(\sqrt{y})_0 = \sqrt{y_0}$ into account – see expression (8):

$$(\sqrt{y})_n = \frac{1}{2\sqrt{y}_0}\,(y_n - \sum_{m=1}^{n-1} (\sqrt{y})_{n-m} (\sqrt{y})_m \,) \tag{17}$$

The resulting formula (17) recursively calculates sequence values $\{(\sqrt{y})_n\}_{n=-\infty}^{\infty}$ for $n = 1, 2, 3, \ldots$

The resulting expressions (14) and (15) show that for a complete designation of filters Y and e^{ϕ}, the inversion algorithm is also needed. It may be obtained with the convolution formula:

$$\sum_{m=0}^{n} (y^{-1})_{n-m}\, y_m = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}.$$

Hence, it follows the recursive formula

$$(y^{-1})_n = -\frac{1}{y_0} \sum_{m=1}^{n} y_m (y^{-1})_{n-m}$$

from $n = 1, 2, 3, \ldots$; $(y^{-1})_0 = y_0^{-1}$.

3. The operator-factors of polar decomposition of admittance causal operator calculus

Individual factors, i.e. self-adjoint Y and unitary e^{ϕ}, appearing in the polar decomposition (9), are determined by the use of general convolution, composition of two digital filters A and B:

$$(AB^*)_n = \sum_m A_{n-m} B_{-m} = \sum_m A_{n+m} B_m \tag{18}$$

where the summation index m is over the integer range from $-\infty$ to $+\infty$.

Applying the formula of cross-correlation (18) to expression (15) yields [5–8]:

$$Y_n = \sum_{m=0}^{n} (\sqrt{y})_{n+m} (\sqrt{y})_m, \tag{19}$$

$$(e^{\phi})_n = \sum_{m=0}^{n} (\sqrt{y})_{n+m} \left((\sqrt{y})^{-1}\right)_m, \tag{20}$$

$$\text{and } (e^{-\phi})_n = \sum_{m=0}^{n} \left((\sqrt{y})^{-1}\right)_{n+m} (\sqrt{y})_m, \tag{21}$$

following $e_n^{-\phi} = e_{-n}^{\phi}$.

4. Scalar products, norms and properties of operator-factor polar decomposition of a causal operator

In order to the make the relation between components of orthogonal G, B and polar Y, e^{ϕ} decomposition clearer, causal hyperbolic operator-filters of y are introduced:

$$\mathrm{sh}(\phi) = \frac{1}{2}(e^{\phi} - e^{-\phi}) = \frac{1}{2}\left(e^{\phi} - \left(e^{\phi}\right)^*\right) \tag{22}$$

$$\mathrm{ch}(\phi) = \frac{1}{2}(e^{\phi} + e^{-\phi}) = \frac{1}{2}\left(e^{\phi} + \left(e^{\phi}\right)^*\right), \tag{23}$$

where:

$(\mathrm{ch}(\phi))^* = \mathrm{ch}(\phi)$ – self-adjoint filter;
$(\mathrm{sh}(\phi))^* = -\mathrm{sh}(\phi)$ – anti self-adjoint filter;

thus, the relationship between the pair of filters G, B and Y, e^{ϕ} is obtained:

$G = Y\mathrm{ch}(\phi)$; $B = Y\mathrm{sh}(\phi)$
because $e^{\phi} = \mathrm{ch}(\phi) + \mathrm{sh}(\phi)$
and $(e^{\phi})^* = \mathrm{ch}(\phi) - \mathrm{sh}(\phi)$.

Hence, its result is the sequential-convolution relation for hyperbolic filters:
$(\mathrm{ch}(\phi) + \mathrm{sh}(\phi))(\mathrm{ch}(\phi) - \mathrm{sh}(\phi)) = \mathrm{ch}(\phi)^2 - \mathrm{sh}(\phi)^2 = e^{\phi}(e^{\phi})^* = I$,
where I is a unit-filter:

$$I_n = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}.$$

It is also easy to obtain the following formulas for scalar products and norms:

$$\begin{aligned} &\left(e^{\phi}, e^{\phi}\right) = \left(e^{\phi}(e^{\phi})^*, I\right) = \left(I, I\right) = 1 \\ &\|\mathrm{ch}(\phi)\|^2 = \left(\mathrm{ch}(\phi), \mathrm{ch}(\phi)\right) = \\ &= \tfrac{1}{4}(e^{\phi} + e^{-\phi}, e^{\phi} + e^{-\phi}) = \tfrac{1}{2}\left(1 + \left((e^{\phi})^2, I\right)\right), \end{aligned} \tag{24}$$

$$\begin{aligned} &\|\mathrm{sh}(\phi)\|^2 = \left(\mathrm{sh}(\phi), \mathrm{sh}(\phi)\right) = \\ &= \tfrac{1}{4}(e^{\phi} - e^{-\phi}, e^{\phi} - e^{-\phi}) = \tfrac{1}{2}\left(1 - \left((e^{\phi})^2, I\right)\right). \end{aligned} \tag{25}$$

It follows the filter orthogonality:

$$\mathrm{ch}(\phi) \text{ i } \mathrm{sh}(\phi) : (\mathrm{ch}(\phi), \mathrm{sh}(\phi)) = 0,$$

and the "Pythagorean theorem" for filters:

$$\|\mathrm{ch}(\phi)\|^2 + \|\mathrm{sh}(\phi)\|^2 = 1$$

On the other hand, by introducing the impedance operator z as an inversion to admittance y operator, formula (14) can be expressed in the form of:

$$(e^{\phi})^2 = yz^*. \tag{26}$$

From (26) it results that:

$$\left((e^{\phi})^2, I\right) = (z^*y, I) = (y, z). \tag{27}$$

Combining formulas (25) and (27) yields:

$$\|\mathrm{sh}(\phi)\|^2 = \tfrac{1}{2}\left(1 - (y, z)\right),$$

and taking into account the fact that the admittance and impedance samples of 0 are its mutual inverses, i.e. $z_0 y_0 = 1$, it finally yields:

$$\|\mathrm{sh}(\phi)\|^2 = -\frac{1}{2}\sum_{n=1}^{\infty} y_n z_n \tag{28}$$

The norm $\|\mathrm{sh}(\phi)\|^2$ has a special meaning in power theory, because it can be used to estimate the so-called reactive power of a two-terminal circuit. It informs us about a harmful increase of current norm of two-terminal circuit, whereas the active power factor remains unchanged. The reactive power factor $\|\mathrm{sh}(\phi)\|^2$ has a comparative meaning, i.e. it must be applied to more than one receiver.

5. Calculation examples

5.1. Example 1. Calculation of the norm $\|\mathrm{sh}\phi\|^2$ for a simple admittance operator:

$$y(z) = \tfrac{a - z}{b - z}; \qquad a > 1,\ b > 1.$$

Its inversion, i.e. the impedance operator has the form of:

$$\hat{z}(z) = \tfrac{b - z}{a - z}.$$

These operators meet the operator-equation

$$y = b^{-1}(a - z)\delta + b^{-1}zy; \quad \hat{z} = a^{-1}(b - z)\delta + a^{-1}z\hat{z},$$

where δ is the Kronecker delta, or equivalent recursive equations:

$$y_n = b^{-1}a\delta_n - b^{-1}\delta_{n-1} + b^{-1}y_{n-1},$$
$$\hat{z}_n = a^{-1}b\delta_n - a^{-1}\delta_{n-1} + a^{-1}\hat{z}_{n-1}.$$

Their solutions have the form of:

$$y_0 = b^{-1}a; \quad y_n = b^{-n}(y_0 - 1) \ \ n = 1, 2, \ldots$$

$$\hat{z}_0 = a^{-1}b; \quad \hat{z}_n = a^{-n}(\hat{z}_0 - 1) \ \ n = 1, 2, \ldots$$

thus,

$$\|\sinh\phi\|^2 = \frac{1}{2}\sum_{m=1}^{\infty} y_m \hat{z}_m$$
$$= -\frac{1}{2}\sum_{m=1}^{\infty}(y_0 - 1)(\hat{z}_0 - 1)(ab)^{-m}$$
$$= -\frac{1}{2}\left(\frac{a}{b} - 1\right)\left(\frac{b}{a} - 1\right)\frac{(ab)^{-1}}{1-(ab)^{-1}}$$
$$= \frac{1}{2}\frac{1}{1-(ab)^{-1}}\left(\frac{a-b}{ab}\right)^2.$$

The resulting formula can be square rooted, giving:

$$\|\sinh\phi\| = \frac{1}{\sqrt{2}}\left(1-(ab)^{-1}\right)^{-1/2}\frac{|a-b|}{ab}.$$

5.2. Example 3. Calculation of the reactive power factor $\|\mathrm{sh}\phi\|^2$ for the wave impedance operator [1, 4, 9]:

$$\hat{z} = \sqrt{\frac{a-z}{b-z}}; \quad a > 1, \ b > 1.$$

It is the integral-derivative operator of order ½ and its inversion has the form of:

$$y = \sqrt{\frac{b-z}{a-z}}.$$

The time samples of these operators are given by power series expansion with respect to the variable z [1, 2, 4, 5]:

$$\hat{z}_n = \sqrt{\frac{a}{b}}\, a^{-n}\alpha_n; \ y_n = \sqrt{\frac{b}{a}}\, a^{-n}\beta_n; \ n = 0, 1, 2 \ldots$$

$$\alpha_n = \sum_{m=0}^{n}\left(\frac{a}{b}\right)^m k_{n-m} h_m; \ \beta_n = \sum_{m=0}^{n}\left(\frac{a}{b}\right)^m h_{n-m} k_m$$

where $\{k_n\}$ and $\{h_n\}$ are the universal sequences determined by the formulas:

$$k_m = -\frac{1}{2}\frac{1}{4}\frac{3}{6}\frac{5}{8}\ldots\frac{2m-3}{2m} \qquad \text{derivative}$$
$$h_m = -\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{7}{8}\ldots\frac{2m-1}{2m} \qquad \text{integral}$$

while $k_0 = h_0 = 1$.
Therefore, the reactive power factor is obtained from:

$$\|\sinh\phi\|^2 = -\frac{1}{2}\sum_{m=1}^{\infty} y_m \hat{z}_m = -\frac{1}{2}\sum_{m=1}^{\infty} a^{-2m}\alpha_m\beta_m.$$

When taking into account the first two terms of this expansion and setting $x = a/b$, the first component is positive:

$$\frac{1}{8}a^{-2}(x-1)^2 = \frac{1}{8}\left(\frac{a-b}{ab}\right)^2,$$

and the second is:

$$\frac{1}{2}a^{-4}\frac{1}{64}(3x^2 - \overset{(1)}{2x} - 1)(x^2 + \overset{(2)}{2x} - 3).$$

Figure 2 shows that the second term of the series expansion (as a product of 1 and 2) is also positive, and therefore the reactive power factor for wave impedance is a positive function.

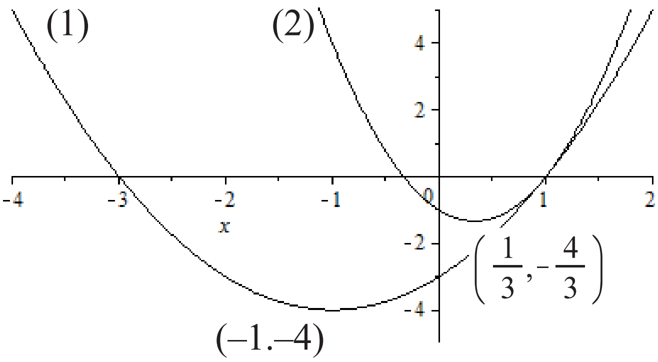

Fig. 2. Graph of the two factors of the second-order series expansion term of the reactive power factor $\|\mathrm{sh}\phi\|^2$

6. Conclusions

The new reactive power factor of electric two-terminal circuit, defined by formula (28) acts as a comparator. This means that it can be useful when comparing a number of receivers according to the rule "the bigger, the worse".

Its major advantage is that it is calculated using samples of mutually inverse operators of impedance and admittance. It also appears to be universal, because it is independent of the frequency of supply voltage. Frequency dependence is a drawback of the commonly-used reactive power factor definitions.

These results may be the thesis (the content) of the following theorem:

$$\|\sinh\phi\| = 0 \ \leftrightarrow \ \begin{matrix}(y, z) = 1\\ yz = I\end{matrix}.$$

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